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ON PULLBACK AND INDUCED CROSSED MODULES OF R-ALGEBROIDS

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Abstract. In this paper we study the pullback and induced crossed modules of R-algebroids, prove that the related induced crossed module functor is the left adjoint of the related pullback crossed module functor and give some consequences of the adjunction.

1. Introduction

Crossed modules, algebraic models of two types, were firstly invented by Whitehead [22, 23] in his study on homotopy groups. Various studies on crossed modules of groups and groupoids can be found in papers and books such as [8, 9, 19], and those of algebras in [4, 5, 18, 20, 21] and in [11, 12, 13] with different names. G.H. Mosa [17] has studied crossed modules of R-algebroids and double R-algebroids.

Pullback crossed modules of groups is introduced in [8, 10] and induced crossed modules of groups in [7, 8, 10]. It's proved in [8] that, in the category of crossed modules of groups, the induced crossed module functor is the left adjoint of the pullback crossed module functor.

R-algebroids were especially studied by B. Mitchell, [14, 15, 16], and by S. M. Amgott, [3]. B. Mitchell has given a categorical definition of R-algebroids. G.H. Mosa has defined crossed modules of R-algebroids and proved the equivalence of crossed modules of algebroids and special double algebroids with connections in [17]. M. Alp has constructed the pullback and pushout crossed modules of algebroids in [1] and [2], respectively.

After the introduction, in the second section of this study we give some basic data on R-algebroids, modules over R-algebroids and (pre)crossed modules of R-algebroids.

In the third section, first we study the pullback crossed modules of R-algebroids, whose construction is done by M. Alp in [1]. Then we prove the 'naturality property' of this construction (Proposition 2).

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In the fourth section first we give a construction of induced crossed modules of R-algebroids using the construction of free precrossed modules of R-algebroids in [6]. Then we prove the 'naturality property' of this construction (Proposition 4). Finally, as the basic goal of this paper, we prove in the category of crossed modules of R-algebroids that the induced crossed module functor is the left adjoint of the pullback crossed module functor (Theorem 1).

In the fifth section we explore some consequences of the adjunction given in Theorem 1.

Throughout the paper R is a commutative ring.

2. Preliminaries

The following data can be found in [3, 14, 15, 16, 17]:

Definition 1. *A category of which each homset has an R-module structure and of which composition is R-bilinear is called an* R-category. *A small R-category is called an* R-algebroid. *Moreover if we omit the axiom of the existence of identities from an R-algebroid structure then the remaining structure is called a* pre-R-algebroid.

Remark 1. *If* A *is an R-algebroid then:*

1. A *has an object set* $\mathrm{Ob}\,(\mathrm{A}) = \mathrm{A}_0$*, a morphism set* $\mathrm{Mor}\,(\mathrm{A})$ *and two functions* $s, t : \mathrm{Mor}\,(\mathrm{A}) \longrightarrow \mathrm{Ob}\,(\mathrm{A})$*, the source and target functions, respectively.*
2. *For any* $a \in \mathrm{Mor}\,(\mathrm{A})$ *if* $sa = x$ *and* $ta = y$ *then* x *and* y *are called as the source and target of* a*, respectively, and* a *is said to be from* x *to* y*.*
3. *For all* $x, y \in \mathrm{A}_0$ *the set of all morphisms from* x *to* y*, which is denoted by* $\mathrm{A}\,(x, y)$ *and called a* homset*, is an R-module.*

Throughout this paper, for any R-algebroid A, $a \in \mathrm{A}$ will mean that a is a morphism of A and the composition of any $a, b \in \mathrm{A}$ with $ta = sb$ will be denoted by ab. Moreover the identity morphism on any $x \in \mathrm{A}_0$ will be denoted by 1_x or only by 1 if there is no ambiguity.

Definition 2. *An R-linear functor between two R-categories is called an* R-functor *and an R-functor between two R-algebroids is called an* R-algebroid morphism. *Moreover a map between two pre-R-algebroids obtained by omitting the axiom of identity preservation from an R-algebroid morphism is called a* pre-R-algebroid morphism.

Note from the Definitions 1 and 2 that an R-algebroid is a pre-R-algebroid and an R-algebroid morphism is a pre-R-algebroid morphism.

Definition 3. *Let* A *be an R-algebroid and* $\mathrm{I} = \{\mathrm{I}\,(x, y) \subseteq \mathrm{A}\,(x, y) \mid x, y \in \mathrm{A}_0\}$ *be a family of R-submodules. For all* $w, x, y, z \in \mathrm{A}_0$*,* $a' \in \mathrm{A}\,(w, x)$ *,* $a'' \in \mathrm{A}\,(y, z)$ *and* $a \in \mathrm{I}\,(x, y)$ *if* $a'a \in \mathrm{I}\,(w, y)$ *and* $aa'' \in \mathrm{I}\,(x, z)$ *then* I *is said to be a* two sided ideal *of* A*.*

Definition 4. *Let* A *be an-R-algebroid and* M *be a pre-R-algebroid with the same object set* A_0 *as* A*. A family of maps defined for all* $x, y, z \in \mathrm{A}_0$ *as*

$$\begin{array}{ccc} \mathrm{M}(x,y) \times \mathrm{A}(y,z) & \longrightarrow & \mathrm{M}(x,z) \\ (m,a) & \longmapsto & m^a \end{array}$$

is called a right action *of* A *on* M*, if the conditions*

$$\begin{array}{llll} 1. & (m^a)^{a'} = m^{aa'}, & 4. & (m_1+m_2)^a = m_1^a + m_2^a, \\ 2. & m^{a_1+a_2} = m^{a_1} + m^{a_2}, & 5. & (r\cdot m)^a = r \cdot m^a = m^{r\cdot a}, \\ 3. & (m'm)^a = m'm^a, & 6. & m^{1_{tm}} = m \end{array}$$

are satisfied for all $r \in R$, $a, a', a_1, a_2 \in \mathrm{A}$, $m, m', m_1, m_2 \in \mathrm{M}$ *with* $tm' = sm$, $tm = tm_1 = tm_2 = sa = sa_1 = sa_2$, $ta = sa'$.

A left action *of* A *on* M *is defined in the same manner.*

Definition 5. *Let* A *be an R-algebroid and* M *be a pre-R-algebroid with the same object set* A_0 *as* A*. If* A *has a right and a left action on* M *and if the condition*

$$({}^a m)^{a'} = {}^a\left(m^{a'}\right)$$

is satisfied for all $m \in \mathrm{M}$, $a, a' \in \mathrm{A}$ *with* $ta = sm$, $tm = sa'$ *then* A *is said to have an* associative action *on* M*.*

Definition 6. *Let* A *be an R-algebroid and* M *be a pre-R-algebroid with the same object set* A_0 *as* A*. If* A *has an associative action on* M *then* M *is called an* A-module*. In this case we write* (M, A) *and call this pair an (*A-)module over R-algebroids*. Moreover, for any two modules* (M, A) *and* (N, B) *over R-algebroids, a pair* $(f, g) : (\mathrm{M}, \mathrm{A}) \longrightarrow (\mathrm{N}, \mathrm{B})$ *is called a* module morphism over R-algebroids *if* $f : \mathrm{M} \longrightarrow \mathrm{N}$ *is a pre-R-algebroid morphism,* $g : \mathrm{A} \longrightarrow \mathrm{B}$ *is an R-algebroid morphism and the conditions*

$$\begin{array}{ll} 1. & fm \in \mathrm{N}\left(g\left(sm\right), g\left(tm\right)\right), \\ 2. & f\left({}^a m\right) = {}^{ga}\left(fm\right) \text{ and } f\left(m^{a'}\right) = \left(fm\right)^{ga'} \end{array}$$

are satisfied for all $m \in \mathrm{M}$, $a, a' \in \mathrm{A}$ *with* $ta = sm$, $tm = sa'$.

All modules over R-algebroids and their morphisms form a category denoted by $\mathrm{ModAlg}(R)$. Moreover, all A-modules with the identity morphism on A form a subcategory $\mathrm{ModAlg}(R)/\mathrm{A}$ of $\mathrm{ModAlg}(R)$.

Definition 7. *Let* A *be an R-algebroid,* M *be a pre-R-algebroid with the same object set* A_0 *as* A *and* A *have an associative action on* M*. A pre-R-algebroid morphism* $\mu : \mathrm{M} \longrightarrow \mathrm{A}$ *is called a* precrossed *(*A-*)*module of R-algebroids *if the condition*

$$\text{CM1)} \quad \mu\left({}^a m\right) = a\left(\mu m\right) \quad \text{and} \quad \mu\left(m^{a'}\right) = \left(\mu m\right) a'$$

*is satisfied and a precrossed (*A-*)module* $\mu : \mathrm{M} \longrightarrow \mathrm{A}$ *of R-algebroids is called a* crossed *(*A-*)*module of R-algebroids *if the condition*

$$\text{CM2)} \quad m^{\mu m'} = mm' = {}^{\mu m}m'$$

is satisfied for all $a, a' \in \mathrm{A}$ *and* $m, m' \in \mathrm{M}$ *with* $ta = sm$, $tm = sa' = sm'$.

Let $\mathcal{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ *and* $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{B})$ *be two (pre)crossed modules of* R*-algebroids,* $f : \mathrm{M} \longrightarrow \mathrm{N}$ *be a pre-*R*-algebroid morphism and* $g : \mathrm{A} \longrightarrow \mathrm{B}$ *be an* R*-algebroid morphism. The pair* $(f, g) : \mathcal{M} \longrightarrow \mathcal{N}$ *is called a (*pre*)*crossed module morphism *if the conditions*

$$\begin{array}{lrcl} 1. & f\left({}^{a}m\right) & = & {}^{ga}\left(fm\right) \quad \textit{and} \quad f\left(m^{a'}\right) = \left(fm\right)^{ga'}, \\ 2. & \eta f & = & g\mu \end{array}$$

are satisfied for all $a, a' \in \mathrm{A}$, $m \in \mathrm{M}$ *with* $ta = sm$, $tm = sa'$. *Note that if* $\mu : \mathrm{M} \longrightarrow \mathrm{A}$ *is a (pre)crossed module then* M *is an* A*-module and a (pre)crossed module morphism is a module morphism satisfying the second condition.*

All precrossed modules of R-algebroids and their morphisms form a category denoted by $\mathrm{PXAlg}(R)$. Moreover, all precrossed A-modules of R-algebroids with the identity morphism on A form a subcategory, $\mathrm{PXAlg}(R)/\mathrm{A}$, of $\mathrm{PXAlg}(R)$. Similarly, all crossed modules of R-algebroids form the category $\mathrm{XAlg}(R)$ and all crossed A-modules of R-algebroids form the subcategory $\mathrm{XAlg}(R)/\mathrm{A}$ of $\mathrm{XAlg}(R)$. Obviously, $\mathrm{XAlg}(R)$ is a full subcategory of $\mathrm{PXAlg}(R)$ and $\mathrm{XAlg}(R)/\mathrm{A}$ is a full subcategory of $\mathrm{PXAlg}(R)/\mathrm{A}$.

Example 1. *If* A *is an* R*-algebroid and* I *is a two sided ideal of* A*, then the inclusion morphism*

$$i : \mathrm{I} \longrightarrow \mathrm{A}$$

is a crossed module with the action of A *on* I *defined by*

$${}^{a}b = ab \quad \textit{and} \quad b^{a'} = ba'$$

for all $a, a' \in \mathrm{A}$, $b \in \mathrm{I}$ *with* $ta = sb$, $tb = sa'$.

3. Pullback crossed modules of R-algebroids

M. Alp has given a construction of the pullback crossed modules of R-algebroids in [1]. In this section, in the proof of the following proposition after giving a brief summary of his construction in part 1 we show that the pullback crossed module satisfies the related universal property in part 2. Moreover we also specify the pullback crossed module functor.

Proposition 1. *Let* A *and* B *be two* R*-algebroids,* $f : \mathrm{A} \longrightarrow \mathrm{B}$ *be an* R*-algebroid morphism and* $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{B})$ *be a crossed* B*-module of* R*-algebroids. There exists a crossed* A*-module* $f^{*}\mathcal{N} = \left(\eta_{f^{*}} : f^{*}\mathrm{N} \longrightarrow \mathrm{A}\right)$ *of* R*-algebroids and a crossed module morphism* $\left(\widehat{f}, f\right) : f^{*}\mathcal{N} \longrightarrow \mathcal{N}$ *such that for any crossed* A*-module* $\mathcal{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ *of* R*-algebroids and crossed module morphism* $(\nu, f) : \mathcal{M} \longrightarrow \mathcal{N}$ *there exists a unique crossed* A*-module morphism* $(h, id_{\mathrm{A}}) : \mathcal{M} \longrightarrow f^{*}\mathcal{N}$ *such that* $\nu = \widehat{f}h$*, i.e. the universal diagram in Figure 1 is commutative:*

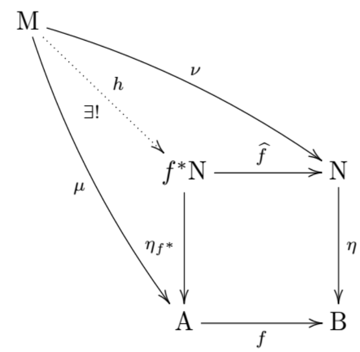

Figure 1

$f^*\mathcal{N}$, with the morphism $\left(\widehat{f}, f\right)$, is called the pullback crossed module of $\mathcal{N}$ along *f. The pullback crossed module is unique up to isomorphism.*

Proof. 1. *i*. Define the source and target of each $(n,a) \in \mathrm{N} \times \mathrm{A}$ as $s(n,a) = sa$ and $t(n,a) = ta$, respectively, and for all $x, y \in \mathrm{A}_0$ form the subset $f^*\mathrm{N}(x,y) = \{(n,a) \mid s(n,a) = x,\, t(n,a) = y \text{ and } fa = \eta n\}$ of $\mathrm{N} \times \mathrm{A}$.

ii. For all $x, y \in \mathrm{A}_0$ the set $f^*\mathrm{N}(x,y)$ has an R-module structure with the addition defined as $(n_1, a_1) + (n_2, a_2) = (n_1 + n_2, a_1 + a_2)$ and the R-action defined as $r \cdot (n,a) = (r \cdot n, r \cdot a)$.

iii. The family $f^*\mathrm{N} = \{f^*\mathrm{N}(x,y) \mid x, y \in \mathrm{A}_0\}$ is a pre-R-algebroid with the composition defined as $(n,a)(n',a') = (nn', aa')$ and an A-module with the associative A-action defined as ${}^{a''}(n,a) = \left({}^{fa''}n, a''a\right)$ and $(n,a)^{a'} = \left(n^{fa'}, aa'\right)$ under the conditions $ta'' = s(n,a)$, $t(n,a) = sa'$.

iv. The map $\eta_{f^*} : f^*\mathrm{N} \longrightarrow \mathrm{A}$ defined as $\eta_{f^*}(n,a) = a$ is a crossed module. (for details see [1])

2. Define $\widehat{f} : f^*\mathrm{N} \longrightarrow \mathrm{A}$ as $\widehat{f}(n,a) = n$. Clearly $\widehat{f}$ is a pre-R-algebroid morphism and $\left(\widehat{f}, f\right)$ is a crossed module morphism since

$$\begin{aligned} \widehat{f}(n,a) &= n \in \mathrm{N}(sn, tn) = \mathrm{N}(s(\eta n), t(\eta n)) = \mathrm{N}(s(fa), t(fa)) \\ &= \mathrm{N}(f(sa), f(ta)) = \mathrm{N}(f(s(n,a)), f(t(n,a))) \end{aligned}$$

and

$$\begin{aligned} \widehat{f}\left((n,a)^{a'}\right) &= \widehat{f}\left(n^{fa'}, aa'\right) = n^{fa'} = \left(\widehat{f}(n,a)\right)^{fa'}, \\ \widehat{f}\left({}^{a''}(n,a)\right) &= \widehat{f}\left({}^{fa''}n, a''a\right) = {}^{fa''}n = {}^{fa''}\left(\widehat{f}(n,a)\right) \end{aligned}$$

Now, for any crossed A-module $\mathcal{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ and crossed module morphism $(\nu, f) : \mathcal{M} \longrightarrow \mathcal{N}$, define $h : \mathrm{M} \longrightarrow f^*\mathrm{N}$ as $h(m) = (\nu m, \mu m)$. h is well defined since $\eta\nu = f\mu$ and so $\eta(\nu m) = f(\mu m)$ for all $m \in \mathrm{M}$. By a direct calculation, it can be shown that $(h, id_{\mathrm{A}}) : \mathcal{M} \longrightarrow f^{*}\mathcal{N}$ is a crossed A-module morphism. Moreover $\nu m = \widehat{f}\,(\nu m, \mu m) = \widehat{f} h m$ for all $m \in \mathrm{M}$, which means $\nu = \widehat{f} h$, as required.

Let $(h', id_{\mathrm{A}}) : \mathcal{M} \longrightarrow f^{*}\mathcal{N}$ be a crossed A-module morphism satisfying $\nu = \widehat{f} h'$. Then h' must be defined as $h'm = (h'_1 m, h'_2 m)$ where $h'_1 m \in \mathrm{N}$ and $h'_2 m \in \mathrm{A}$. But, in this case, $h'_1 m = \widehat{f}\,(h'_1 m, h'_2 m) = \widehat{f} h' m = \nu m$ and $h'_2 m = \eta_{f^{*}}\,(h'_1 m, h'_2 m) = \eta_{f^{*}} h' m = \mu m$ since $\eta_{f^{*}} h' = \mu$. So $h'm = (h'_1 m, h'_2 m) = (\nu m, \mu m) = hm$ for all $m \in \mathrm{M}$, which means h is unique.

Finally, assume that $\widetilde{f}^{*}\mathcal{N} = \left(\eta_{\widetilde{f}^{*}} : \widetilde{f}^{*}\mathrm{N} \longrightarrow \mathrm{A}\right)$ is a crossed module of R-algebroids and $\left(\widetilde{f}, f\right) : \widetilde{f}^{*}\mathcal{N} \longrightarrow \mathcal{N}$ is a crossed module morphism which together satisfy the same conditions as $f^{*}\mathcal{N}$ and $\left(\widehat{f}, f\right)$. Then there exists unique morphisms $\left(\widetilde{h}, id_{\mathrm{A}}\right) : f^{*}\mathcal{N} \longrightarrow \widetilde{f}^{*}\mathcal{N}$ and $(h, id_{\mathrm{A}}) : \widetilde{f}^{*}\mathcal{N} \longrightarrow f^{*}\mathcal{N}$ making related universal diagrams commutative. So $\eta_{f^{*}} = \eta_{\widetilde{f}^{*}} \widetilde{h} = \eta_{f^{*}} h \widetilde{h}$ and $\eta_{\widetilde{f}^{*}} = \eta_{f^{*}} h = \eta_{\widetilde{f}^{*}} \widetilde{h} h$ which together requires $h\widetilde{h} = id_{f^{*}\mathrm{N}}$ and $\widetilde{h} h = id_{\widetilde{f}^{*}\mathrm{N}}$. Thus $\widetilde{h}$ is an isomorphism and the pullback crossed module $f^{*}\mathcal{N}$, with the morphism $\left(\widehat{f}, f\right)$, is unique up to isomorphism. □

So, we get a pullback crossed module functor $f^{*} : \mathrm{XAlg}(R)/\mathrm{B} \longrightarrow \mathrm{XAlg}(R)/\mathrm{A}$ which gives a crossed A-module $f^{*}\mathcal{N}$ for any crossed B-module $\mathcal{N}$ and is defined as $f^{*}\,(g, id_{\mathrm{B}}) = (f^{*} g, id_{\mathrm{A}})$ on morphisms such that $(f^{*} g)\,(n, a) = (gn, a)$.

Now we prove an important property of pullback crossed module:

Proposition 2. *If* A, B, C *are R-algebroids and* $f : \mathrm{A} \longrightarrow \mathrm{B}$, $f' : \mathrm{B} \longrightarrow \mathrm{C}$ *are R-algebroid morphisms then the functor $f^{*} f'^{*}$ is naturally isomorphic to $(f'f)^{*}$.*

Proof. For any $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{C}) \in \mathrm{XAlg}(R)/\mathrm{C}$, the B-module $f'^{*}\mathrm{N}$, the A-module $(f^{*} f'^{*})\,\mathrm{N} = f^{*}\,(f'^{*}\mathrm{N})$ and the A-module $(f'f)^{*}\,\mathrm{N}$ is formed by the pairs (n, b), $((n, b), a)$ and (n, a), respectively, under the conditions $\eta n = f'b$, $\eta_{f'^{*}}\,(n, b) = fa$ and $\eta n = (f'f)\,a$, the second of which means $b = fa$ since $\eta_{f'^{*}}\,(n, b) = b$ by definition. So, any element $((n, b), a)$ of $(f^{*} f'^{*})\,\mathrm{N}$ is, in fact, of the form $((n, fa), a)$.

It can easily be shown that, for all crossed modules $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{C})$, the map $\alpha_{\mathrm{N}} : (f^{*} f'^{*})\,\mathrm{N} \longrightarrow (f'f)^{*}\,\mathrm{N}$, defined as $\alpha_{\mathrm{N}}\,((n, fa), a) = (n, a)$, is an isomorphism. Moreover, for all $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{C}), \mathcal{N}' = (\eta' : \mathrm{N}' \longrightarrow \mathrm{C}) \in \mathrm{XAlg}(R)/\mathrm{C}$, for all crossed module morphisms $(g, id_{\mathrm{C}}) : \mathcal{N} \longrightarrow \mathcal{N}'$ and for all $((n, fa), a) \in (f^* f'^*)\,\mathrm{N}$

$$\begin{aligned}
\left(\left((f'f)^* g\right) \alpha_{\mathrm{N}}\right)((n, fa), a) &= \left((f'f)^* g\right)\left((\alpha_{\mathrm{N}})((n, fa), a)\right) \\
&= \left((f'f)^* g\right)(n, a) \\
&= (gn, a) \\
&= \alpha_{\mathrm{N}'}((gn, fa), a) \\
&= \alpha_{\mathrm{N}'}((f'^* g)(n, fa), a) \\
&= \alpha_{\mathrm{N}'}((f^* (f'^* g))((n, fa), a)) \\
&= \alpha_{\mathrm{N}'}(((f^* f'^*) g)((n, fa), a)) \\
&= (\alpha_{\mathrm{N}'}((f^* f'^*) g))((n, fa), a),
\end{aligned}$$

i.e. the diagram in Figure 2 is commutative:

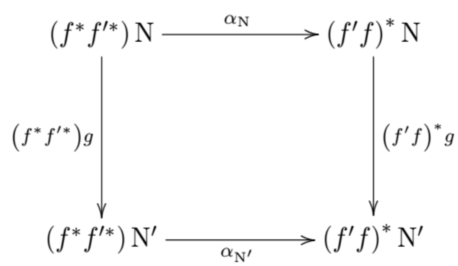

Figure 2

That means $\left((f'f)^* g\right) \alpha_{\mathrm{N}} = \alpha_{\mathrm{N}'}((f^* f'^*) g)$ and the family

$$\left\{(\alpha_{\mathrm{N}}, id_{\mathrm{A}}) : (f^* f'^*)\,\mathcal{N} \longrightarrow (f'f)^*\,\mathcal{N} \mid \mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{C}) \in \mathrm{XAlg}(R)/\mathrm{C}\right\}$$

is a natural isomorphism between $f^* f'^*$ and $(f'f)^*$. □

4. Induced crossed modules of R-algebroids

Although a similar crossed module construction might be possible to that in [2] given by M. Alp, for the construction of the induced crossed module, we prefer to use the free precrossed modules of R-algebroids constructed in [6], to provide an application. The summary of the construction, in [6], of the free precrossed A-module $F_P(\omega) = (\omega_P : \mathrm{F}_P(\omega) \longrightarrow \mathrm{A})$ of R-algebroids determined by the function $\omega : \mathrm{K} \longrightarrow \mathrm{A}$ where K is a set and A is an R-algebroid such that ωk is a morphism of A for all $k \in \mathrm{K}$, is as follows:

1. The building blocks are elements of the form aka' with $ta = s(\omega k)$ and $t(\omega k) = sa'$, and words of the form $a_1 k_1 a_1' a_2 k_2 a_2' ... a_n k_n a_n'$ with $ta_1' = sa_2, ..., ta_{n-1}' = sa_n$, where $n \in \mathbb{N}^+$, $a, a_1, ..., a_n, a', a_1', ..., a_n' \in \mathrm{A}$ and $k, k_1, ..., k_n \in \mathrm{K}$. The source and the target of any word $p = a_1 k_1 a_1' a_2 k_2 a_2' ... a_n k_n a_n'$ are $sp = sa_1$ and $tp = ta_n'$, respectively.

2. For all $x, y \in \mathrm{A}_0$, $\mathrm{F}_P(\omega)(x, y)$ is the quotient group obtained by dividing the free additive abelian group generated by all words with source x and target y by

its normal subgroup generated by all elements of the form

$$\begin{gathered}
a_1k_1a_1'...\left(a_i+a_i''\right)k_ia_i'...a_nk_na_n' - a_1k_1a_1'...a_ik_ia_i'...a_nk_na_n' - a_1k_1a_1'...a_i''k_ia_i'...a_nk_na_n',\\
a_1k_1a_1'...a_ik_i\left(a_i'+a_i'''\right)...a_nk_na_n' - a_1k_1a_1'...a_ik_ia_i'...a_nk_na_n' - a_1k_1a_1'...a_ik_ia_i'''...a_nk_na_n',\\
\left(r\cdot a_1\right)k_1a_1'...a_ik_ia_i'...a_nk_na_n' - a_1k_1a_1'...\left(r\cdot a_i\right)k_ia_i'...a_nk_na_n',\\
\left(r\cdot a_1\right)k_1a_1'...a_ik_ia_i'...a_nk_na_n' - a_1k_1a_1'...a_ik_i\left(r\cdot a_i'\right)...a_nk_na_n'.
\end{gathered}$$

So, the elements of $\mathrm{F}_P\left(\omega\right)\left(x,y\right)$ are of the form $\sum\limits_i\left[p_i\right]$ where p_i is a word with $sp_i=x$ and $tp_i=y$, and $\left[p_i\right]$ is the coset of p_i.

3. $\mathrm{F}_P\left(\omega\right)\left(x,y\right)$ has an R-module structure with the R-action defined as $r\cdot\left[p_i\right]=\left[\left(r\cdot a_{i_1}\right)k_{i_1}a_{i_1}'...a_{i_n}k_{i_n}a_{i_n}'\right]$ and as $r\cdot\left(\sum\limits_i\left[p_i\right]\right)=\sum\limits_i r\cdot\left[p_i\right]$.

4. $\mathrm{F}_P\left(\omega\right)=\left\{\mathrm{F}_P\left(\omega\right)\left(x,y\right)\ \mid x,y\in \mathrm{A}_0\right\}$ is a pre-R-algebroid on A_0 with the composition defined as $\sum\limits_i\left[p_i\right]\sum\limits_j\left[p_j\right]=\sum\limits_{i,j}\left[p_ip_j\right]$ where if $p_i=a_{i_1}k_{i_1}a_{i_1}'...a_{i_n}k_{i_n}a_{i_n}'$ and $p_j=a_{j_1}k_{j_1}a_{j_1}'...a_{j_{n'}}k_{j_{n'}}a_{j_{n'}}'$ then $p_ip_j=a_{i_1}k_{i_1}a_{i_1}'...a_{i_n}k_{i_n}a_{i_n}'a_{j_1}k_{j_1}a_{j_1}'...a_{j_{n'}}k_{j_{n'}}a_{j_{n'}}'$ under the condition $t\left[p_i\right]=ta_{i_n}'=sa_{j_1}=s\left[p_j\right]$.

5. $\mathrm{F}_P\left(\omega\right)$ is an A-module with the associative A-action defined as ${}^a\left(\sum\limits_i\left[p_i\right]\right)=\sum\limits_i\left[{}^ap_i\right]$ and $\left(\sum\limits_i\left[p_i\right]\right)^{a'}=\sum\limits_i\left[p_i^{a'}\right]$ where ${}^ap_i=\left(aa_{i_1}\right)k_{i_1}a_{i_1}'...a_{i_n}k_{i_n}a_{i_n}'$ and $p_i^{a'}=a_{i_1}k_{i_1}a_{i_1}'...a_{i_n}k_{i_n}\left(a_{i_n}'a'\right)$ with the condition that $ta=sp_i$, $tp_i=sa'$.

6. $\omega_{_P}:\mathrm{F}_P\left(\omega\right)\longrightarrow \mathrm{A}$ is defined as $\omega_{_P}\left[aka'\right]=a\left(\omega k\right)a'$ on generators and as $\omega_{_P}\sum\limits_i\left[p_i\right]=\sum\limits_i\omega_{_P}\left[p_i\right]$ on elements where $\omega_{_P}\left[p_i\right]=\omega_{_P}\left[a_{i_1}k_{i_1}a_{i_1}'...a_{i_n}k_{i_n}a_{i_n}'\right]=\omega_{_P}\left[a_{i_1}k_{i_1}a_{i_1}'\right]...\omega_{_P}\left[a_{i_n}k_{i_n}a_{i_n}'\right]$.

Proposition 3. *Let* A *and* B *be two R-algebroids, $f:\mathrm{A}\longrightarrow\mathrm{B}$ be an R-algebroid morphism and $\mathcal{N}=\left(\eta:\mathrm{N}\longrightarrow\mathrm{A}\right)$ be a crossed* A*-module of R-algebroids. There exists a crossed* B*-module $f_*\mathcal{N}=\left(\eta_{f_*}:f_*\mathrm{N}\longrightarrow\mathrm{B}\right)$ of R-algebroids and a crossed module morphism $\left(\overline{f},f\right):\mathcal{N}\longrightarrow f_*\mathcal{N}$ such that for any crossed* B*-module $\mathcal{M}=\left(\mu:\mathrm{M}\longrightarrow\mathrm{B}\right)$ of R-algebroids and crossed module morphism $\left(\nu,f\right):\mathcal{N}\longrightarrow\mathcal{M}$ there exists a unique crossed* B*-module morphism $\left(h,id_{\mathrm{B}}\right):f_*\mathcal{N}\longrightarrow\mathcal{M}$ such that $\nu=h\overline{f}$, i.e. the universal diagram in Figure 3 is commutative:*

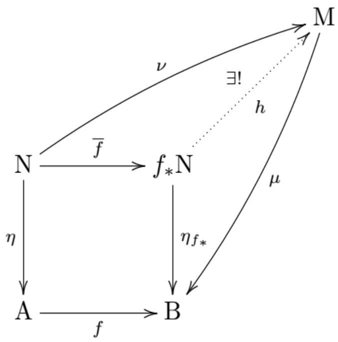

Figure 3

$f_*\mathcal{N}$, with the morphism $\left(\overline{f}, f\right)$, is called the crossed module induced from $\mathcal{N}$ by f. *The induced crossed module is unique up to isomorphism.*

Proof. As summarised above, there exists a free precrossed B-module $F_P\left(f\eta_{_m}\right) = \left(\left(f\eta_{_m}\right)_{_P} : \mathrm{F}_P\left(f\eta_{_m}\right) \longrightarrow \mathrm{B}\right)$ determined by $f\eta_{_m}$ where $\eta_{_m} : \mathrm{Mor}\left(\mathrm{N}\right) \longrightarrow \mathrm{A}$ is defined as $\eta_{_m}\left(n\right) = \eta n$. Let I be the two sided ideal of $\mathrm{F}_P\left(f\eta_{_m}\right)$ generated by all elements of the form

$$\begin{gathered}
\left[b_1 n_1 b_1' b_2 n_2 b_2'\right] - \left[\left(b_1\left(f\eta n_1\right) b_1' b_2\right) n_2 b_2'\right], \\
\left[b_1 n_1 b_1' b_2 n_2 b_2'\right] - \left[b_1 n_1\left(b_1' b_2\left(f\eta n_2\right) b_2'\right)\right], \\
\left[bnb'\right] + \left[bn_1 b'\right] - \left[b\left(n + n_1\right) b'\right], \\
\left[b\left({}^a n\right) b'\right] - \left[\left(b\left(fa\right)\right) nb'\right], \\
\left[b\left(n^{a'}\right) b'\right] - \left[bn\left(\left(fa'\right) b'\right)\right], \\
\left[\left(r \cdot b\right) nb'\right] - \left[b\left(r \cdot n\right) b'\right], \\
\left[bn\left(r \cdot b'\right)\right] - \left[b\left(r \cdot n\right) b'\right].
\end{gathered}$$

Obviously, I is closed under the actions of R and B. Now, construct the family $f_*\mathrm{N} = \frac{\mathrm{F}_P\left(f\eta_{_m}\right)}{\mathrm{I}} = \left\{f_*\mathrm{N}\left(x,y\right) = \frac{\mathrm{F}_P\left(f\eta_{_m}\right)\left(x,y\right)}{\mathrm{I}\left(x,y\right)} \mid x, y \in \mathrm{B}_0\right\}$ of quotient R-modules. For any word $b_1 n_1 b_1' ... b_t n_t b_t'$ let's denote the coset of $\left[b_1 n_1 b_1' ... b_t n_t b_t'\right]$ by $\overline{b_1 n_1 b_1' ... b_t n_t b_t'}$. Then, note that, any coset $\overline{b_1 n_1 b_1' ... b_t n_t b_t'}$ is, in fact, of the form $\overline{b_1 n_1 b_1''}$, where $b_1'' = b_1' b_2\left(f\eta n_2\right) b_2' ... b_t\left(f\eta n_t\right) b_t' \in \mathrm{B}$. Thus each element of $f_*\mathrm{N}$ is of the form $\sum\limits_i \overline{b_i n_i b_i'}$ for some $b_i, b_i' \in \mathrm{B}$, $n_i \in \mathrm{N}$. Clearly, $f_*\mathrm{N}$ is an R-algebroid B-module thanks to the addition, composition, R-action and associative B-action induced by those defined on $\mathrm{F}_P\left(f\eta_{_m}\right)$.

Moreover $\left(f\eta_{_m}\right)_{_P}$ induces a precrossed module $f_*\mathcal{N} = \left(\eta_{f_*} : f_*\mathrm{N} \longrightarrow \mathrm{B}\right)$ defined as $\eta_{f_*}\left(\overline{bnb'}\right) = \left(f\eta_{_m}\right)_{_P}\left[bnb'\right] = b\left(f\eta_{_m} n\right) b' = b\left(f\eta n\right) b'$ on generators and the precrossed module $f_*\mathcal{N}$ is also a crossed module thanks to the first two generators of I.

Define the function $\overline{f} : \mathrm{N} \longrightarrow f_*\mathrm{N}$ as $\overline{f}n = \overline{1n1}$ where $\overline{1n1} = \overline{1_{s(f\eta n)}n1_{t(f\eta n)}}$. By this definition the pair $\left(\overline{f}, f\right)$ is a crossed module morphism since

$$
\begin{array}{llcl}
1. & \overline{f}\left(n_1+n_2\right) & = & \overline{1\left(n_1+n_2\right)1} = \overline{1n_11+1n_21} = \overline{1n_11}+\overline{1n_21} = \overline{f}n_1+\overline{f}n_2, \\
2. & \overline{f}\left(nn'\right) & = & \overline{1\left(nn'\right)1} = \overline{1\left(n^{\eta n'}\right)1} = \overline{1n\left(\left(f\eta n'\right)1\right)} = \overline{1n\left(11\left(f\eta n'\right)1\right)} \\
 & & = & \overline{1n11n'1} = \overline{1n1}\;\overline{1n'1} = \overline{f}\left(n\right)\overline{f}\left(n'\right), \\
3. & \overline{f}\left(r\cdot n\right) & = & \overline{1\left(r\cdot n\right)1} = \overline{\left(r\cdot 1\right)n1} = r\cdot\overline{1n1} = r\cdot\overline{f}n, \\
4. & \overline{f}\left({}^{a}n\right) & = & \overline{1\left({}^{a}n\right)1} = \overline{\left(1\left(fa\right)\right)n1} = \overline{\left(\left(fa\right)1\right)n1} \\
 & & = & \overline{{}^{fa}\left(1n1\right)} = {}^{fa}\overline{1n1} = {}^{fa}\left(\overline{f}n\right), \\
5. & \overline{f}\left(n^{a'}\right) & = & \overline{1\left(n^{a'}\right)1} = \overline{1n\left(\left(fa'\right)1\right)} = \overline{1n\left(1\left(fa'\right)\right)} \\
 & & = & \overline{\left(1n1\right)^{fa'}} = \overline{1n1}^{fa'} = \left(\overline{f}n\right)^{fa'}, \\
6. & \eta_{f_*}\left(\overline{f}n\right) & = & \eta_{f_*}\left(\overline{1n1}\right) = 1\left(f\eta n\right)1 = f\eta n
\end{array}
$$

for all $n, n_1, n_2, n' \in \mathrm{N}$, $a, a' \in \mathrm{A}$ and $r \in R$ such that $sn_1 = sn_2$, $tn_1 = tn_2$, $ta = sn, tn = sn' = sa'$.

Now for all crossed B-module $\mathcal{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{B})$ of R-algebroids and for all crossed module morphism $(\nu, f) : \mathcal{N} \longrightarrow \mathcal{M}$ define the function $h : f_*\mathrm{N} \longrightarrow \mathrm{M}$ as $h\left(\overline{bnb'}\right) = {}^{b}\left(\nu n\right)^{b'}$ on generators. It can easily be shown that h preserves the addition, R-action and B-action. Moreover

$$
\begin{aligned}
h\left(\overline{b_1n_1b_1'}\;\overline{b_2n_2b_2'}\right) &= h\left(\overline{b_1n_1\left(b_1'b_2\left(f\eta n_2\right)b_2'\right)}\right) = {}^{b_1}\left(\nu n_1\right)^{b_1'b_2\left(\mu\nu n_2\right)b_2'} \\
&= {}^{b_1}\left(\nu n_1\right)^{b_1'\left(b_2\left(\mu\left(\nu n_2\right)\right)b_2'\right)} = \left({}^{b_1}\left(\nu n_1\right)^{b_1'}\right)^{\mu\left({}^{b_2}\left(\nu n_2\right)^{b_2'}\right)} \\
&= \left({}^{b_1}\left(\nu n_1\right)^{b_1'}\right)\left({}^{b_2}\left(\nu n_2\right)^{b_2'}\right) = h\left(\overline{b_1n_1b_1'}\right)h\left(\overline{b_2n_2b_2'}\right)
\end{aligned}
$$

for all generators $\overline{b_1n_1b_1'}$, $\overline{b_2n_2b_2'}$ of $f_*\mathrm{N}$ with $tb_1' = sb_2$, which means h preserves the composition. Besides,

$$
\begin{aligned}
\left(\mu h\right)\left(\overline{bnb'}\right) &= \mu\left(h\left(\overline{bnb'}\right)\right) = \mu\left({}^{b}\left(\nu n\right)^{b'}\right) \\
&= b\left(\mu\left(\nu n\right)\right)b' = b\left(\left(\mu\nu\right)\left(n\right)\right)b' \\
&= b\left(\left(f\eta\right)\left(n\right)\right)b' = \eta_{f_*}\left(\overline{bnb'}\right) \\
&= \left(id_{\mathrm{B}}\eta_{f_*}\right)\left(\overline{bnb'}\right)
\end{aligned}
$$

on generators. That is, (h, id_{B}) is a crossed B-module morphism. Finally

$$\left(h\overline{f}\right)(n) = h\left(\overline{1n1}\right) = {}^{1}\left(\nu n\right)^{1} = \nu n$$

for all $n \in \mathrm{N}$, i.e. h makes the universal diagram in Figure 3 commutative. It can also directly be shown that (h, id_{B}) is the unique morphism satisfying $\nu = h\overline{f}$, and $f_*\mathcal{N}$, with the morphism $\left(\overline{f}, f\right)$, is unique up to isomorphism. □

Thus we get an induced crossed module functor $f_* : \mathrm{XAlg}(R)/\mathrm{A} \longrightarrow \mathrm{XAlg}(R)/\mathrm{B}$ which gives a crossed B-module $f_*\mathcal{N}$ for any crossed A-module $\mathcal{N}$ and is defined as

$f_*(g, id_{\mathrm{A}}) = (f_*g, id_{\mathrm{B}})$ on morphisms such that $(f_*g)\left(\overline{bnb'}\right) = \overline{b\,(gn)\,b'}$ on generators.

Proposition 4. *If* A, B, C *are R-algebroids and* $f : \mathrm{A} \longrightarrow \mathrm{B}$, $f' : \mathrm{B} \longrightarrow \mathrm{C}$ *are R-algebroid morphisms then the functor* $f'_* f_*$ *is naturally isomorphic to* $(f'f)_*$.

Proof. For any $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A}) \in \mathrm{XAlg}(R)/\mathrm{A}$, generators of the B-module $f_*\mathrm{N}$, the C-module $(f'_*f_*)\,\mathrm{N} = f'_*(f_*\mathrm{N})$ and the C-module $(f'f)_*\,\mathrm{N}$ are of the forms $\overline{bnb'}$, $\overline{c\left(\overline{bnb'}\right)c'}$ and $\overline{cnc'}$, respectively.

For all crossed module $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A})$ define $\alpha_{\mathrm{N}} : (f'_*f_*)\,\mathrm{N} \longrightarrow (f'f)_*\,\mathrm{N}$ as $\alpha_{\mathrm{N}}\left(\overline{c\left(\overline{bnb'}\right)c'}\right) = \overline{(c\,(f'b))\,n\,((f'b')\,c')}$ on generators. Obviously α_{N} preserves the addition, R-action and C-action. It also preserves the composition since

$$\begin{aligned}
&\alpha_{\mathrm{N}}\left(\overline{c_1\left(\overline{b_1n_1b'_1}\right)c'_1}\;\overline{c_2\left(\overline{b_2n_2b'_2}\right)c'_2}\right)\\
=\;&\alpha_{\mathrm{N}}\left(\overline{c_1\left(\overline{b_1n_1b'_1}\right)\left(c'_1c_2\left(\left(f'\eta_{f_*}\right)\left(\overline{b_2n_2b'_2}\right)\right)c'_2\right)}\right)\\
=\;&\overline{(c_1\,(f'b_1))\,n_1(\,(f'b'_1)\,(c'_1c_2\,(f'\,(b_2\,((f\eta)\,(n_2))\,b'_2))\,c'_2)\,)}\\
=\;&\overline{(c_1\,(f'b_1))\,n_1(\,(f'b'_1)\,c'_1c_2\,((f'b_2)\,((f'f\eta)\,(n_2))\,(f'b'_2))\,c'_2)}\\
=\;&\overline{(c_1\,(f'b_1))\,n_1(\,((f'b'_1)\,c'_1)\,(c_2\,(f'b_2))\,(((f'f)\,\eta)\,(n_2))\,((f'b'_2)\,c'_2)\,)}\\
=\;&\overline{(c_1\,(f'b_1))\,n_1\,((f'b'_1)\,c'_1)}\;\overline{(c_2\,(f'b_2))\,n_2\,((f'b'_2)\,c'_2)}\\
=\;&\alpha_{\mathrm{N}}\left(\overline{c_1\left(\overline{b_1n_1b'_1}\right)c'_1}\right)\alpha_{\mathrm{N}}\left(\overline{c_2\left(\overline{b_2n_2b'_2}\right)c'_2}\right)
\end{aligned}$$

for all generators $\overline{c_1\left(\overline{b_1n_1b'_1}\right)c'_1}$, $\overline{c_2\left(\overline{b_2n_2b'_2}\right)c'_2}$ of $(f'_*f_*)\,\mathrm{N}$ with $tc'_1 = sc_2$.

Moreover

$$\begin{aligned}
\left(\eta_{(f'f)_*}\alpha_{\mathrm{N}}\right)\left(\overline{c\left(\overline{bnb'}\right)c'}\right) &= \eta_{(f'f)_*}\left(\alpha_{\mathrm{N}}\left(\overline{c\left(\overline{bnb'}\right)c'}\right)\right)\\
&= \eta_{(f'f)_*}\left(\overline{(c\,(f'b))\,n\,((f'b')\,c')}\right)\\
&= (c\,(f'b))\,((f'f)\,\eta n)\,((f'b')\,c')\\
&= c\,((f'b)\,(f'\,((f\eta)\,(n)))\,(f'b'))\,c'\\
&= c\,(f'\,(b\,((f\eta)\,(n))\,b'))\,c'\\
&= c\left(f'\left(\eta_{f_*}\left(\overline{bnb'}\right)\right)\right)c'\\
&= c\left(\left(f'\eta_{f_*}\right)\left(\overline{bnb'}\right)\right)c'\\
&= \left(\eta_{f_*}\right)_{f'_*}\left(\overline{c\left(\overline{bnb'}\right)c'}\right)\\
&= \left(id_{\mathrm{C}}\left(\eta_{f_*}\right)_{f'_*}\right)\left(\overline{c\left(\overline{bnb'}\right)c'}\right)
\end{aligned}$$

on generators, which means $(\alpha_{\mathrm{N}}, id_{\mathrm{C}}) : (f'_*f_*)\,\mathcal{N} \longrightarrow (f'f)_*\,\mathcal{N}$ is a crossed C-module morphism.

Now, for all crossed modules $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A})$ define $\beta_{\mathrm{N}} : (f'f)_*\,\mathrm{N} \longrightarrow (f'_*f_*)\,\mathrm{N}$ as $\beta_{\mathrm{N}}\left(\overline{cnc'}\right) = \overline{c\left(\overline{1n1}\right)c'}$ on generators, in the reverse direction. Obviously

β_{N} preserve the addition, R-action and C-action. It also preserves the composition since

$$\begin{aligned}
\beta_{\mathrm{N}}\left(\overline{c_1 n_1 c_1'}\ \overline{c_2 n_2 c_2'}\right) &= \beta_{\mathrm{N}}\left(\overline{c_1 n_1 \left(c_1' c_2 \left(\left(\left(f' f\right) \eta\right)\left(n_2\right)\right) c_2'\right)}\right) \\
&= \overline{c_1 \overline{1 n_1 1} \left(c_1' c_2 \left(f' \left(\left(f \eta\right)\left(n_2\right)\right)\right) c_2'\right)} \\
&= \overline{c_1 \overline{1 n_1 1} \left(c_1' c_2 \left(f' \left(1 \left(\left(f \eta\right)\left(n_2\right)\right) 1\right)\right) c_2'\right)} \\
&= \overline{c_1 \overline{1 n_1 1} \left(c_1' c_2 \left(f' \left(\eta_{f_*} \overline{1 n_2 1}\right)\right) c_2'\right)} \\
&= \overline{c_1 \overline{1 n_1 1} \left(c_1' c_2 \left(\left(f' \eta_{f_*}\right)\left(\overline{1 n_2 1}\right)\right) c_2'\right)} \\
&= \overline{c_1 \overline{1 n_1 1} c_1'}\ \overline{c_2 \overline{1 n_2 1} c_2'} \\
&= \beta_{\mathrm{N}}\left(\overline{c_1 n_1 c_1'}\right) \beta_{\mathrm{N}}\left(\overline{c_2 n_2 c_2'}\right)
\end{aligned}$$

for all generators $\overline{c_1 n_1 c_1'}$, $\overline{c_2 n_2 c_2'}$ of $\left(f' f\right)_* \mathrm{N}$ with $t c_1' = s c_2$. Moreover

$$\begin{aligned}
\left(\left(\eta_{f_*}\right)_{f_*'} \beta_{\mathrm{N}}\right)\left(\overline{c n c'}\right) &= \left(\eta_{f_*}\right)_{f_*'}\left(\beta_{\mathrm{N}}\left(\overline{c n c'}\right)\right) \\
&= \left(\eta_{f_*}\right)_{f_*'}\left(\overline{c \overline{1 n 1} c'}\right) \\
&= c\left(\left(f' \eta_{f_*}\right) \overline{1 n 1}\right) c' \\
&= c\left(f' \left(\eta_{f_*}\left(\overline{1 n 1}\right)\right)\right) c' \\
&= c\left(f' \left(1 \left(\left(f \eta\right)\left(n\right)\right) 1\right)\right) c' \\
&= c\left(f' \left(f \eta n\right)\right) c' \\
&= c\left(\left(\left(f' f\right) \eta\right)\left(n\right)\right) c' \\
&= \eta_{\left(f' f\right)_*}\left(\overline{c n c'}\right) \\
&= \left(id_{\mathrm{C}} \eta_{\left(f' f\right)_*}\right)\left(\overline{c n c'}\right)
\end{aligned}$$

on generators, which means $\left(\beta_{\mathrm{N}}, id_{\mathrm{C}}\right) : \left(f' f\right)_* \mathcal{N} \longrightarrow \left(f_*' f_*\right) \mathcal{N}$ is a crossed C-module morphism.

Now, for all generators $\overline{c\left(\overline{b n b'}\right) c'}$ of $\left(f_*' f_*\right) \mathrm{N}$

$$\begin{aligned}
\left(\beta_{\mathrm{N}} \alpha_{\mathrm{N}}\right)\left(\overline{c\left(\overline{b n b'}\right) c'}\right) &= \beta_{\mathrm{N}}\left(\alpha_{\mathrm{N}}\left(\overline{c\left(\overline{b n b'}\right) c'}\right)\right) \\
&= \beta_{\mathrm{N}}\left(\overline{\left(c\left(f' b\right)\right) n \left(\left(f' b'\right) c'\right)}\right) \\
&= \overline{\left(c\left(f' b\right)\right) \overline{1 n 1} \left(\left(f' b'\right) c'\right)} \\
&= \overline{c\left({}^{b}\overline{1 n 1}^{b'}\right) c'} \\
&= \overline{c\left(\overline{\left(b 1\right) n \left(1 b'\right)}\right) c'} \\
&= \overline{c\left(\overline{b n b'}\right) c'},
\end{aligned}$$

i.e. $\beta_{\mathrm{N}}\alpha_{\mathrm{N}} = id_{(f'_* f_*)\mathrm{N}}$ and for all generators $\overline{cnc'}$ of $(f'f)_* \mathrm{N}$

$$\begin{aligned}
(\alpha_{\mathrm{N}}\beta_{\mathrm{N}})\left(\overline{cnc'}\right) &= \alpha_{\mathrm{N}}\left(\beta_{\mathrm{N}}\left(\overline{cnc'}\right)\right) \\
&= \alpha_{\mathrm{N}}\left(\overline{c\overline{1n1}c'}\right) \\
&= \overline{\left(c\left(f'1_{s((f\eta)(n))}\right)\right)n\left(\left(f'1_{t((f\eta)(n))}\right)c'\right)} \\
&= \overline{\left(c\left(1_{s((f'f\eta)(n))}\right)\right)n\left(\left(1_{t((f'f\eta)(n))}\right)c'\right)} \\
&= \overline{\left(c\left(1_{tc}\right)\right)n\left(\left(1_{sc'}\right)c'\right)} \\
&= \overline{cnc'},
\end{aligned}$$

i.e. $\alpha_{\mathrm{N}}\beta_{\mathrm{N}} = id_{(f'f)_*\mathrm{N}}$. That is, α_{N} is an isomorphism from $(f'_* f_*)\,\mathrm{N}$ to $(f'f)_*\,\mathrm{N}$.

Moreover, for all $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A})\,, \mathcal{N}' = (\eta' : \mathrm{N}' \longrightarrow \mathrm{A}) \in \mathrm{XAlg}(R)/\mathrm{A}$, for all crossed module morphisms $(g, id_{\mathrm{A}}) : \mathcal{N} \longrightarrow \mathcal{N}'$ and for all generators $\overline{c\left(\overline{bnb'}\right)c'}$ of $(f'_* f_*)\,\mathrm{N}$

$$\begin{aligned}
\left(\left((f'f)_*\, g\right)\alpha_{\mathrm{N}}\right)\left(\overline{c\left(\overline{bnb'}\right)c'}\right) &= \left((f'f)_*\, g\right)\left(\alpha_{\mathrm{N}}\left(\overline{c\left(\overline{bnb'}\right)c'}\right)\right) \\
&= \left((f'f)_*\, g\right)\left(\overline{\left(c\left(f'b\right)\right)n\left(\left(f'b'\right)c'\right)}\right) \\
&= \overline{\left(c\left(f'b\right)\right)\left(gn\right)\left(\left(f'b'\right)c'\right)} \\
&= \alpha_{\mathrm{N}'}\left(\overline{c\left(\overline{b\left(gn\right)b'}\right)c'}\right) \\
&= \alpha_{\mathrm{N}'}\left(\overline{c\left(\left(f_* g\right)\left(\overline{bnb'}\right)\right)c'}\right) \\
&= \alpha_{\mathrm{N}'}\left(\left(f'_*\left(f_* g\right)\right)\left(\overline{c\left(\overline{bnb'}\right)c'}\right)\right) \\
&= \left(\alpha_{\mathrm{N}'}\left(\left(f'_* f_*\right)g\right)\right)\left(\overline{c\left(\overline{bnb'}\right)c'}\right)
\end{aligned}$$

which means the diagram in Figure 4 is commutative:

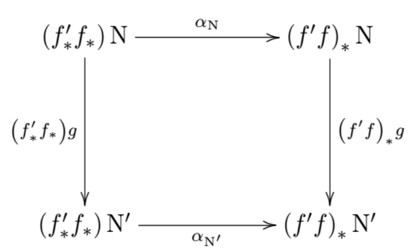

Figure 4

So, we can conclude that

$$\{(\alpha_{\mathrm{N}}, id_{\mathrm{C}}) : (f'_* f_*)\,\mathcal{N} \longrightarrow (f'f)_*\,\mathcal{N} \mid \mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{A}) \in \mathrm{XAlg}(R)/\mathrm{A}\}$$

is a natural isomorphism between $f'_* f_*$ and $(f'f)_*$. □

Theorem 1. *For any R-algebroids* A *and* B*, and any R-algebroid morphism f :* A $\longrightarrow$ B *the induced crossed module functor f_* is the left adjoint of the pullback crossed module functor f^*.*

Proof. We must find a natural equivalence

$$\Phi : \quad (\mathrm{XAlg}(R)/\mathrm{B})\,(f_*\,(-)\,,-) \quad \cong \quad (\mathrm{XAlg}(R)/\mathrm{A})\,(-,f^*\,(-))$$

which is required to give a map

$$\begin{array}{rrcl} \Phi : & \mathrm{Ob}\,(\mathrm{XAlg}(R)/\mathrm{A}) \times \mathrm{Ob}\,(\mathrm{XAlg}(R)/\mathrm{B}) & \longrightarrow & \mathrm{Sets} \\ & (\mathcal{M},\mathcal{N}) & \longmapsto & \Phi\,(\mathcal{M},\mathcal{N}) \end{array}$$

where $\Phi\,(\mathcal{M},\mathcal{N})$, from $(\mathrm{XAlg}(R)/\mathrm{B})\,(f_*\mathcal{M},\mathcal{N})$ to $(\mathrm{XAlg}(R)/\mathrm{A})\,(\mathcal{M},f^*\mathcal{N})$, is a bijection and natural in both $\mathcal{M}$ and $\mathcal{N}$.

For all crossed modules $\mathcal{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A})$ and $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{B})$ define $\Phi\,(\mathcal{M},\mathcal{N})$ as $\Phi\,(\mathcal{M},\mathcal{N})\,(h,id_{\mathrm{B}}) = (\Phi\,(\mathcal{M},\mathcal{N})\,(h)\,,id_{\mathrm{A}})$ such that $(\Phi\,(\mathcal{M},\mathcal{N})\,(h))\,(m)$ $= \left(h\left(\overline{1m1}\right),\mu m\right)$ for all $(h,id_{\mathrm{B}}) \in (\mathrm{XAlg}(R)/\mathrm{B})\,(f_*\mathcal{M},\mathcal{N})$ and $m \in \mathrm{M}$. $\Phi\,(\mathcal{M},\mathcal{N})\,(h)$ is well defined since $\eta\left(h\left(\overline{1m1}\right)\right) = \mu_{f_*}\left(\overline{1m1}\right) = 1\,((f\mu)\,m)\,1 = f\,(\mu m)$ for all $m \in \mathrm{M}$. Moreover, it can easily be seen that $(\Phi\,(\mathcal{M},\mathcal{N})\,(h)\,,id_{\mathrm{A}})$ is a crossed A-module morphism and $\Phi\,(\mathcal{M},\mathcal{N})$ is 1-1.

For any $(g,id_{\mathrm{A}}) \in (\mathrm{XAlg}(R)/\mathrm{A})\,(\mathcal{M},f^*\mathcal{N})$ the morphism $g : \mathrm{M} \longrightarrow f^*\mathrm{N}$ must be defined as $gm = (g_1m,g_2m)$, for all $m \in \mathrm{M}$, such that $g_1m \in \mathrm{N}$, $g_2m \in \mathrm{A}$ and $\eta g_1m = fg_2m$. But $g_2m = \eta_{f^*}\,(g_1m,g_2m) = \eta_{f^*}\,(gm) = \left(\eta_{f^*}g\right)m = \mu m$ since (g,id_{A}) is a crossed A-module morphism. So we can write $gm = (g_1m,\mu m)$ where $\eta g_1m = f\mu m$. Define $h : f_*\mathrm{M} \longrightarrow \mathrm{N}$ as $h\left(\overline{bmb'}\right) = {}^{b}g_1m^{b'}$ on generators. Clearly h is an R-algebroid morphism preserving B-action and (h,id_{B}) is a crossed B-module morphism since

$$\begin{aligned} (\eta h)\left(\overline{bmb'}\right) &= \eta\left({}^{b}g_1m^{b'}\right) = b\,(\eta g_1m)\,b' = b\,(f\mu m)\,b' \\ &= b\,((f\mu)\,(m))\,b' = \mu_{f_*}\left(\overline{bmb'}\right) = \left(id_{\mathrm{B}}\mu_{f_*}\right)\left(\overline{bmb'}\right) \end{aligned}$$

on generators. That is $(h,id_{\mathrm{B}}) \in (\mathrm{XAlg}(R)/\mathrm{B})\,(f_*\mathcal{M},\mathcal{N})$. Moreover

$$(\Phi\,(\mathcal{M},\mathcal{N})\,(h))\,(m) = \left(h\left(\overline{1m1}\right),\mu m\right) = \left({}^{1}g_1m^{1},\mu m\right) = (g_1m,\mu m) = gm$$

for all $m \in \mathrm{M}$ which means $\Phi\,(\mathcal{M},\mathcal{N})$ is onto and so is a bijection.

Moreover, provided that $(-)^{\bullet}$ is composition with $(-)$ from right, for all crossed module $\mathcal{M}' = (\mu' : \mathrm{M}' \longrightarrow \mathrm{A})$, for all $(g,id_{\mathrm{A}}) \in (\mathrm{XAlg}(R)/\mathrm{A})\,(\mathcal{M},\mathcal{M}')$, $(h,id_{\mathrm{B}}) \in (\mathrm{XAlg}(R)/\mathrm{B})\,(f_*\mathcal{M}',\mathcal{N})$ and $m \in \mathrm{M}$

$$\begin{aligned} \left(\left(\Phi\,(\mathcal{M},\mathcal{N})\,(f_*g)^{\bullet}\right)(h)\right)(m) &= \left(\Phi\,(\mathcal{M},\mathcal{N})\left((f_*g)^{\bullet}\,h\right)\right)(m) \\ &= (\Phi\,(\mathcal{M},\mathcal{N})\,(h\,(f_*g)))\,(m) \\ &= \left((h\,(f_*g))\left(\overline{1m1}\right),\mu m\right) \\ &= \left(h\left(\overline{1\,(gm)\,1}\right),(\mu'g)\,(m)\right) \\ &= \left(h\left(\overline{1\,(gm)\,1}\right),\mu'\,(gm)\right) \\ &= (\Phi\,(\mathcal{M}',\mathcal{N})\,(h))\,(gm) \\ &= ((\Phi\,(\mathcal{M}',\mathcal{N})\,(h))\,g)\,(m) \\ &= (g^{\bullet}\,(\Phi\,(\mathcal{M}',\mathcal{N})\,(h)))\,(m) \\ &= ((g^{\bullet}\Phi\,(\mathcal{M}',\mathcal{N}))\,(h))\,(m) \end{aligned}$$

which means the diagram in Figure 5 is commutative and so $\Phi(\mathcal{M},\mathcal{N})$ is natural in $\mathcal{M}$:

$$\begin{array}{ccc}
(\mathrm{Alg}(R)/\mathrm{B})(f_*\mathcal{M},\mathcal{N}) & \xrightarrow{\Phi(\mathcal{M},\mathcal{N})} & (\mathrm{Alg}(R)/\mathrm{A})(\mathcal{M},f^*\mathcal{N}) \\
\Big\uparrow{\scriptstyle ((f_*g)^\bullet, id_{\mathrm{B}})} & & \Big\uparrow{\scriptstyle (g^\bullet, id_{\mathrm{A}})} \\
(\mathrm{Alg}(R)/\mathrm{B})(f_*\mathcal{M}',\mathcal{N}) & \xrightarrow[\Phi(\mathcal{M}',\mathcal{N})]{} & (\mathrm{Alg}(R)/\mathrm{A})(\mathcal{M}',f^*\mathcal{N})
\end{array}$$

Figure 5

Finally, provided that $(-)_\bullet$ is composition with $(-)$ from left, for all crossed module $\mathcal{N}' = (\eta' : \mathrm{N}' \longrightarrow \mathrm{B})$, for all $(g, id_{\mathrm{B}}) \in (\mathrm{XAlg}(R)/\mathrm{B})(\mathcal{N},\mathcal{N}')$, $(h, id_{\mathrm{B}}) \in (\mathrm{XAlg}(R)/\mathrm{B})(f_*\mathcal{M},\mathcal{N})$ and $m \in \mathrm{M}$

$$\begin{aligned}
((\Phi(\mathcal{M},\mathcal{N}')\, g_\bullet)(h))(m) &= (\Phi(\mathcal{M},\mathcal{N}')(g_\bullet h))(m) \\
&= (\Phi(\mathcal{M},\mathcal{N}')(gh))(m) \\
&= \left((gh)\left(\overline{1m1}\right), \mu m\right) \\
&= \left(g\left(h\left(\overline{1m1}\right)\right), \mu m\right) \\
&= (f^*g)\left(h\left(\overline{1m1}\right), \mu m\right) \\
&= (f^*g)((\Phi(\mathcal{M},\mathcal{N})(h))(m)) \\
&= ((f^*g)(\Phi(\mathcal{M},\mathcal{N})(h)))(m) \\
&= ((f^*g)_\bullet(\Phi(\mathcal{M},\mathcal{N})(h)))(m) \\
&= (((f^*g)_\bullet\, \Phi(\mathcal{M},\mathcal{N}))(h))(m)
\end{aligned}$$

which means the diagram in Figure 6 is commutative and so $\Phi(\mathcal{M},\mathcal{N})$ is natural in $\mathcal{N}$:

$$\begin{array}{ccc}
(\mathrm{Alg}(R)/\mathrm{B})(f_*\mathcal{M},\mathcal{N}) & \xrightarrow{\Phi(\mathcal{M},\mathcal{N})} & (\mathrm{Alg}(R)/\mathrm{A})(\mathcal{M},f^*\mathcal{N}) \\
\Big\downarrow{\scriptstyle (g_\bullet, id_{\mathrm{B}})} & & \Big\downarrow{\scriptstyle ((f^*g)_\bullet, id_{\mathrm{A}})} \\
(\mathrm{Alg}(R)/\mathrm{B})(f_*\mathcal{M},\mathcal{N}') & \xrightarrow[\Phi(\mathcal{M},\mathcal{N}')]{} & (\mathrm{Alg}(R)/\mathrm{A})(\mathcal{M},f^*\mathcal{N}')
\end{array}$$

Figure 6

□

5. Consequences of the Adjunction

Theorem 1 has some consequences:

1. In the proof of the Theorem 1, since $\Phi(\mathcal{M},\mathcal{N})$ is a bijection, its inverse $\Phi^{-1}(\mathcal{M},\mathcal{N})$ is also a bijection from $(\mathrm{XAlg}(R)/\mathrm{A})(\mathcal{M}, f^*\mathcal{N})$ to $(\mathrm{XAlg}(R)/\mathrm{B})(f_*\mathcal{M},\mathcal{N})$ and defined for all $(g, id_{\mathrm{A}}) \in (\mathrm{XAlg}(R)/\mathrm{A})(\mathcal{M}, f^*\mathcal{N})$ as $\Phi^{-1}(\mathcal{M},\mathcal{N})(g, id_{\mathrm{A}}) = \left(\Phi^{-1}(\mathcal{M},\mathcal{N})(g), id_{\mathrm{B}}\right)$ such that $\left(\Phi^{-1}(\mathcal{M},\mathcal{N})(g)\right)\left(\overline{bmb'}\right) = {}^{b}\widehat{f}gm^{b'}$ on generators.

2. The family, called the *unit* of the adjunction,

$$\left\{(\alpha_{\mathrm{M}}, id_{\mathrm{A}}) = \Phi(\mathcal{M},\mathcal{N})(id_{f_*\mathrm{M}}, id_{\mathrm{B}}) : \mathcal{M} \longrightarrow f^* f_* \mathcal{M} \mid \mathcal{M} \in \mathrm{Ob}(\mathrm{XAlg}(R)/\mathrm{A})\right\}$$

is a natural transformation from $\mathbf{1}_{\mathrm{XAlg}(R)/\mathrm{A}}$ to $f^* f_*$ where $\mathbf{1}_{\mathrm{XAlg}(R)/\mathrm{A}}$ is the identity functor on $\mathrm{XAlg}(R)/\mathrm{A}$. Moreover $\alpha_{\mathcal{M}} = (\alpha_{\mathrm{M}}, id_{\mathrm{A}})$ is universal for each $\mathcal{M} = (\mu : \mathrm{M} \longrightarrow \mathrm{A}) \in \mathrm{Ob}(\mathrm{XAlg}(R)/\mathrm{A})$, i.e. for each $\mathcal{N} \in \mathrm{Ob}(\mathrm{XAlg}(R)/\mathrm{B})$ and for each morphism $(g, id_{\mathrm{A}}) : \mathcal{M} \longrightarrow f^*\mathcal{N}$ there exists a unique morphism $(g', id_{\mathrm{B}}) : f_*\mathcal{M} \longrightarrow \mathcal{N}$ making the universal diagram in Figure 7 commutative:

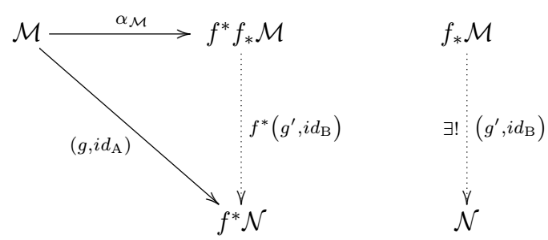

Figure 7

It can be shown that $\alpha_{\mathrm{M}}(m) = \left(\overline{1m1}, \mu m\right)$ for all $m \in \mathrm{M}$ and $(g', id_{\mathrm{B}}) = \Phi^{-1}(\mathcal{M},\mathcal{N})(g, id_{\mathrm{A}})$ which requires g' to be defined on generators as $g'\left(\overline{bmb'}\right) = {}^{b}\widehat{f}gm^{b'}$.

3. The family, called the *counit* of the adjunction,

$$\left\{(\beta_{\mathrm{N}}, id_{\mathrm{B}}) = \Phi^{-1}(\mathcal{M},\mathcal{N})(id_{f^*\mathrm{N}}, id_{\mathrm{A}}) : f_* f^* \mathcal{N} \longrightarrow \mathcal{N} \mid \mathcal{N} \in \mathrm{Ob}(\mathrm{XAlg}(R)/\mathrm{B})\right\}$$

is a natural transformation from $f_* f^*$ to $\mathbf{1}_{\mathrm{XAlg}(R)/\mathrm{B}}$ where $\mathbf{1}_{\mathrm{XAlg}(R)/\mathrm{B}}$ is the identity functor on $\mathrm{XAlg}(R)/\mathrm{B}$. Moreover $\beta_{\mathcal{N}} = (\beta_{\mathrm{N}}, id_{\mathrm{B}})$ is universal for each $\mathcal{N} = (\eta : \mathrm{N} \longrightarrow \mathrm{B}) \in \mathrm{Ob}(\mathrm{XAlg}(R)/\mathrm{B})$, i.e. for each $\mathcal{M} \in \mathrm{Ob}(\mathrm{XAlg}(R)/\mathrm{A})$ and for each morphism $(h, id_{\mathrm{B}}) : f_*\mathcal{M} \longrightarrow \mathcal{N}$ there exists a unique morphism $(h', id_{\mathrm{A}}) : \mathcal{M} \longrightarrow f^*\mathcal{N}$ making the universal diagram in Figure 8 commutative:

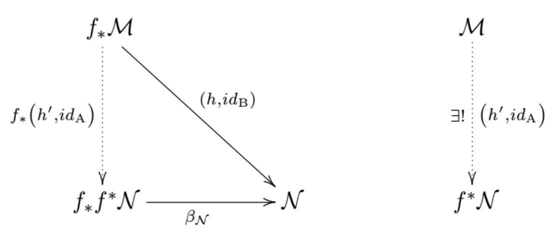

Figure 8

It can be shown that $\beta_{\mathrm{N}}\left(\overline{b\,(n,a)\,b'}\right) = {}^{b}n^{b'}$ on generators and $(h', id_{\mathrm{A}}) = \Phi\,(\mathcal{M},\mathcal{N})\,(h, id_{\mathrm{B}})$ which requires h' to be defined as $h'\,(m) = \left(h\left(\overline{1m1}\right), \mu m\right)$ for all $m \in \mathrm{M}$.

4. For each $\mathcal{M} \in \mathrm{Ob}\,(\mathrm{XAlg}(R)/\mathrm{A})$ and for each $\mathcal{N} \in \mathrm{Ob}\,(\mathrm{XAlg}(R)/\mathrm{B})$

$$\beta_{f_*\mathcal{M}} f_*\,(\alpha_{\mathcal{M}}) = id_{f_*\mathcal{M}} \text{ and } f^*\,(\beta_{\mathcal{N}})\,\alpha_{f^*\mathcal{N}} = id_{f^*\mathcal{N}}$$

References

[1] Alp, M., Pullback crossed modules of algebroids, *Iran. J. Sci. Technol.* (2008), 32(A1), 1-5.
[2] Alp, M., Pushout crossed modules of algebroids, *Iran. J. Sci. Technol.* (2008), 32(A3), 175-181.
[3] Amgott, S.M., Separable Categories, *J. Pure. Appl. Algebra* (1986), 40, 1-14.
[4] Arvasi, Z. and Porter, T., Simplicial and crossed resolutions of commutative algebras, *J. Algebra* (1996), 181, 426-448.
[5] Arvasi, Z., Crossed Modules of Algebras, *Math. Comput. Appl.* (2004), 9(2), 173-182.
[6] Avcıoğlu, O. and Akça, İ.İ., Free modules and crossed modules of R-algebroids, Submitted to *Turk. J. Math.*
[7] Brown, R. and Higgins, P.J., On the Connection Between the Second Relative Homotopy Groups of Some Related Spaces, *Proc. London Math. Soc.* (1978), 36, 193-212.
[8] Brown, R., Higgins, P.J. and Sivera, R., Nonabelian Algebraic Topology. *EMS Tracts in Mathematics Vol.15* 2011.
[9] Brown, R. and İçen İ., Homotopies and Automorphisms of Crossed Modules of Groupoids, *Applied Categorical Structures* (2003), 11, 185-206.
[10] Brown, R. and Wensley C.D., On Finite Induced Crossed Modules, and The Homotopy 2-Type of Mapping Cones, *Theory and Applications of Categories* (1995), 1(3), 54-71.
[11] Gerstenhaber, M., On the deformation of rings and algebras: II, *Ann. of Math.* (1966), 84, 1-19.
[12] Lichtenbaum, S. and Schlessinger, M., The Cotangent Complex of a Morphism, *Trans. Amer. Math. Soc.* (1967), 128, 41-70.
[13] Lue, A.S.T., Non-abelian cohomology of associative algebras, *Q. J. Math.* (1968), 19(1), 159-180.
[14] Mitchell, B., Rings with several objects, *Adv. in Math.* (1972), 8(1), 1-161.
[15] Mitchell, B., Some applications of module theory to functor categories, *Bull. Amer. Math. Soc.* (1978), 84, 867-885.
[16] Mitchell, B., Separable Algebroids, *Mem. Amer. Math. Soc.* 57 (1985), no. 333, 96 pp.
[17] Mosa, G.H., Higher Dimensional Algebroids and Crossed Complexes, PhD Thesis, University of Wales, Bangor, 1986. groupoids.org.uk/mosa-thesis.html
[18] Shammu, N.M., Algebraic and categorical structure of categories of crossed modules of algebras, PhD Thesis, University of Wales, Bangor, 1992. http://ethos.bl.uk/OrderDetails.do?uin=uk.bl.ethos.304282
[19] Porter, T., The Crossed Manegarie: An introduction to crossed gadgetry and cohomology in algebra and topology, 9 February 2010. https://ncatlab.org/timporter/files/menagerie10.pdf
[20] Porter, T., Homology of commutative algebras and an invariant of Simis and Vasconcelos. *J. Algebra* (1986), 99, 458-465.
[21] Porter, T., Some categorical results in the theory of crossed modules in commutative algebras, *J. Algebra* (1987), 109, 415-429.
[22] Whitehead, J.H.C., On adding relations to homotopy groups, *Ann. of Math.* 1941, 42, 409-428.

[23] Whitehead, J.H.C., Note on a previous paper entitled "On adding relations to homotopy groups.", *Ann. of Math.* (1946), 47, 806-810.

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