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SHAPE CURVATURES OF THE LORENTZIAN PLANE CURVES

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Abstract. In this paper, we examine the Lorentzian similar plane curves using the hyperbolic structure and spherical arc length parameter. We classify all self-similar Lorentzian plane curves and give formulas for pseudo shape curvatures of evolute, involute and parallel curves of a nonnull plane curve.

1. **Introduction**

A similarity transformation (or similitude), which consists of a rotation, a translation and an isotropic scaling, is an automorphism preserving the angles and ratios between lengths. These mappings are the smallest extension of one parameter motion. The similarity transformations are used in many areas of the pure and applied mathematics. KS. Chou and C. Qu [16, 17] showed that the motions of curves in two-, three- and n-dimensional ($n > 3$) similarity geometries correspond to the Burgers hierarchy, Burgers-mKdV hierarchy and a multi-component generalization of these hierarchies by using the similarity invariants of curves. Alcazar et. al. [14] presented a novel and deterministic algorithm to detect whether two given rational plane curves are related by means of a similarity transformation, which is a central question in Pattern Recognition. On the other hand, the self-similar objects, whose images under the similarity map are themselves, have had a wide range of applications in areas such as fractal geometry, dynamical systems, computer networks and statistical physics. Mandelbrot called these objects fractals, which are the systems that present such self-similar behavior and the examples in nature are many. The Cantor set, the von Koch snowflake curve and the Sierpinski gasket are some of most famous examples of such sets (see [4, 13, 15]).

Berger [18] represented the broad content of similarity transformations in the Euclidean spaces. Some geometric properties of a Euclidean plane curve as frames, curvature and so on were examined by [1] using the complex structure which is

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defined by a linear map $J:\mathbb{R}^2\rightarrow\mathbb{R}^2$

$$J\left(x_1,x_2\right)=\left(-x_2,x_1\right).$$

Encheva and Georgiev [20] studied the differential geometric invariants of Frenet curves under a similarity map in 2-dimensional Euclidean space. Schwenk-Schellscmidt et. al. [3] characterized conic sections by using their spherical image in terms of appropriate eigenvalue equations of second order in the Euclidean plane. Then, they investigated the evolutes and involutes and their geometric properties in relation to the eigenvalue equations considered in [21]. On the other hand, in the Lorentzian plane, Öztekin and Ergüt studied the Lorentzian version of the paper [21] for nonnull Lorentzian plane curves. Saloom and Tari [2] handled the caustic, evolute, Minkowski symmetry set and parallels of a smooth and regular Lorentzian plane curve. Simsek and Özdemir [10] introduced the hyperbolic structure in the Clifford algebra $Cl_{1,1}$ and gave a formula for the curvature function of Lorentzian plane curves by means of the hyperbolic structure. Also, they [11] investigated the Lorentzian similarity geometry of nonnull Frenet curves in any dimensional space.

The content of paper is as follows. We give basic informations about the Lorentzian plane geometry and pseudo similarity map by means of the geometric product. We examine the differential geometry of a nonnull plane curve under the similarity map in view of the hyperbolic structure. We find formulas related to the pseudo shape curvatures of the evolutes, involutes and parallel curves of nonnull plane curves. We determine all nonnull self-similar plane curves and show that the evolutes and parallel curves of hyperbolic logarithmic spirals are self-similar and similar curves, respectively.

2. Preliminaries

The Clifford algebra $Cl_{p,q}$ is an associative and distributive geometric algebra generated by a pseudo-Euclidean vector space $\mathcal{M}^{p,q}$ equipped with a quadratic form Q. We can think of it as a structure generalizing the hypercomplex number systems such as the complex numbers, quaternions, split quaternions, double numbers. The algebra operation $\mathbf{xy}$, called the *geometric product*, for any $\mathbf{x}$, $\mathbf{y}\in\mathcal{M}^{p,q}$ is defined by

$$\mathbf{xx}=\mathbf{x}^2=Q\left(\mathbf{x}\right),$$
$$\mathbf{xy}=\mathbf{x}\cdot\mathbf{y}+\mathbf{x}\wedge\mathbf{y}$$

where $\mathbf{x}\cdot\mathbf{y}$ and $\mathbf{x}\wedge\mathbf{y}$ are inner product and outer product of $\mathcal{M}^{p,q}$ and $Q\left(\mathbf{x}\right)=-\sum\limits_{t=1}^{q}x_t^2+\sum\limits_{t=q+1}^{p+q}x_t^2$ for $\mathbf{x}=\left(x_1,...,x_{p+q}\right)$. We can express the inner product and outer product in terms of the geometric product:

$$\begin{aligned}\mathbf{x}\cdot\mathbf{y} &= \frac{1}{2}(\mathbf{xy}+\mathbf{yx})\\ \mathbf{x}\wedge\mathbf{y} &= \frac{1}{2}(\mathbf{xy}-\mathbf{yx}).\end{aligned}$$

In this paper, we shall deal with the Clifford algebra $Cl_{1,1} = gen\{\mathbf{i},\mathbf{j}\}$ defined by the geometric product rules

$$\mathbf{i}^2 = -1, \mathbf{j}^2 = \mathbf{1} \text{ and } \mathbf{ij} = \mathbf{i}\wedge\mathbf{j} = -\mathbf{ji}$$

where $\{\mathbf{i},\mathbf{j}\}$ is the standard basis of Minkowski plane $\mathcal{M}^{1,1}$. Any element of $Cl_{1,1}$, called a *multivector* or *geometric number*, has the form

$$s + \mathbf{x} + t\mathbf{ij},$$

where $s, t \in \mathbb{R}$ and $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}$ for $x_1 x_2 \in \mathbb{R}$. In other words, the multivectors in $Cl_{1,1}$ are linear combinations of scalars (0-vector) s, vectors (1-vector) $\mathbf{i},\mathbf{j}$, bivector (2-vector) $\mathbf{ij}$. One can find more information about the Clifford algebras in [6, 7, 12].

We can study the Minkowski plane $\mathcal{M}^{1,1}$ by means of the Clifford algebra $Cl_{1,1}$ by defining as $\mathcal{M}^{1,1} = \{x_1\mathbf{i} + x_2\mathbf{j} : x_1, x_2 \in \mathbb{R}\}$. The vector $\mathbf{x}$ is called a spacelike vector, lightlike (or null) vector and timelike vector if $\mathbf{x}^2 > 0$ or $\mathbf{x} = 0$, $\mathbf{x}^2 = 0$ or $\mathbf{x}^2 < 0$, respectively. The norm of the vector $\mathbf{x}$ is described by $\|\mathbf{x}\| = \sqrt{|\mathbf{x}^2|}$. Also, the inverse of any nonnull vector $\mathbf{x}$ can be defined in the Clifford algebra as the following

$$\mathbf{x}^{-1} = \frac{\mathbf{x}}{\mathbf{x}^2}.$$

The Lorentzian rotation in $\mathcal{M}^{1,1}$ can be expressed with a *spinor*, is a linear combination of a scalar and a bivector. If we take any vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{B} = \mu_1 + \mu_2 J$, where $J = \mathbf{ji}$, then the geometric product of $\mathbf{v}$ and $\mathbf{B}$ is equal to

$$\mathbf{vB} = (v_1\mu_1 + v_2\mu_2)\,\mathbf{i} + (v_1\mu_2 + v_2\mu_1)\,\mathbf{j} = \begin{bmatrix}\mu_1 & \mu_2\\ \mu_2 & \mu_1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix},$$

which is a vector in $\mathcal{M}^{1,1}$. When $\mu_1 = \cosh\theta$ and $\mu_2 = \sinh\theta$, the spinor has the form $\mathbf{B} = \cosh\theta + \sinh\theta J = e^{\theta J}$ and $\mathbf{vB}$ is a vector obtained by rotation of $\mathbf{v}$ through θ. The geometric product of two spinor gives a new spinor. Thus, the spinors form a subgroup of $Cl_{1,1}$.

The similarity transformation in any finite dimensional Minkowski space is a composition of an homothety and Lorentzian motion and called *p-similarity (pseudo-similarity) transformation*. This map preserves the angles between any two vectors and the causal character of a vector. Also, the p-similarity of Minkowski plane is orientation-preserving map. The set of p-similarity transformations form a group under the composition of maps and is denoted by $\mathbf{Sim}(\mathcal{M}^{1,1})$ (see [11]).

The *hyperbolic structure* of $\mathcal{M}^{1,1}$ is the linear map $\mathcal{J} : \mathcal{M}^{1,1} \to \mathcal{M}^{1,1}$ given by

$$\mathcal{J}\mathbf{x} = \mathbf{x}\mathbf{i}\mathbf{j} = (x_1\mathbf{i} + x_2\mathbf{j})\,\mathbf{i}\mathbf{j} = -x_2\mathbf{i} - x_1\mathbf{j}, \quad \text{for any } \mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}. \tag{1}$$

It is easy to prove that the hyperbolic structure has the following properties

$$\begin{aligned} \mathcal{J}^2 &= I, \\ (\mathcal{J}\mathbf{x}) \cdot (\mathcal{J}\mathbf{y}) &= -\mathbf{x} \cdot \mathbf{y}, \\ \mathcal{J}\mathbf{x} \cdot \mathbf{x} &= 0, \\ \mathbf{x}\mathbf{y} &= \mathbf{x} \cdot \mathbf{y} + (\mathbf{x} \cdot \mathcal{J}\mathbf{y})\,\mathbf{i}\mathbf{j} \end{aligned} \tag{2}$$

for $\mathbf{x}$, $\mathbf{y} \in \mathcal{M}^{1,1}$ where $I : \mathcal{M}^{1,1} \to \mathcal{M}^{1,1}$ is the identity linear map. In the rest of the paper, we will show the hyperbolic structure with $\mathcal{J}$ (see [10]).

Let's consider a smooth and regular nonnull curve $\gamma : U \to \mathcal{M}^{1,1}$

$$\boldsymbol{\gamma}(s) = \gamma_1(s)\,\mathbf{i} + \gamma_2(s)\,\mathbf{j}$$

parameterized by arc length s, where U is an open interval in $\mathbb{R}$. Let us denote by $\varphi(s)$ the hyperbolic angle between the tangent vector and the positive direction at s. The (oriented) *curvature* at a point measures the rate of bending as the point moves along the curve with unit speed and can be defined as

$$\kappa(s) = \frac{d\varphi}{ds}. \tag{3}$$

If we denote $\mathbf{T}$ as a tangent vector of γ, we can say the following equations

$$\frac{d\mathbf{T}}{ds} = \varepsilon_{\mathcal{J}\mathbf{T}}\kappa\mathcal{J}\mathbf{T}, \quad \frac{d\mathcal{J}\mathbf{T}}{ds} = \varepsilon_{\mathcal{J}\mathbf{T}}\kappa\mathbf{T}$$

where $\varepsilon_{\mathbf{x}} = 1$ if $\mathbf{x}$ is spacelike or $\varepsilon_{\mathbf{x}} = -1$ if $\mathbf{x}$ is timelike.

Lemma 1. *Let $\boldsymbol{\gamma} = \boldsymbol{\gamma}(t)$ parameterized by t be a nonnull curve and κ be the curvature of $\boldsymbol{\gamma}$. Then, we have*

$$\kappa = \frac{\varepsilon\,(\ddot{\boldsymbol{\gamma}} \cdot \mathcal{J}\dot{\boldsymbol{\gamma}})}{\|\dot{\boldsymbol{\gamma}}\|^3} \tag{4}$$

where $\dot{\boldsymbol{\gamma}} = \dfrac{d\gamma}{dt}$ and $\varepsilon = 1$ or -1 if $\boldsymbol{\gamma}$ is timelike or spacelike, respectively ([10]).

3. Similar Curves in the Lorentzian Plane

Now, we define p-similarity map using the geometric product in $\mathcal{M}^{1,1}$. A *p-similarity* of Minkowski plane, $f : \mathcal{M}^{1,1} \to \mathcal{M}^{1,1}$, can be given by

$$f(\mathbf{x}) = \lambda\mathbf{B}\mathbf{x} + \mathbf{b}, \tag{5}$$

where $\lambda \neq 0$, $\mathbf{B}$ and $\mathbf{b}$ are a real constant, a spinor and a fixed translation vector, respectively. The constant λ is called p-similarity ratio of f.

Let $\gamma : t \in I \to \boldsymbol{\gamma}(t) = \gamma_1(t)\,\mathbf{i} + \gamma_2(t)\,\mathbf{j} = \mathbf{m}(t) \in \mathcal{M}^{1,1}$ be a nonnull curve of class C^2. We let $\gamma^* = f \circ \gamma$ for $f \in \mathbf{Sim}(\mathcal{M}^{1,1})$ such that

$$\gamma^*(t) = \lambda\mathbf{B}\mathbf{m}(t) + \mathbf{b}. \tag{6}$$

The arc length functions of γ and γ^* starting at $t_0 \in I$ are

$$s(t) = \int_{t_0}^{t} \left\| \frac{d\gamma(u)}{du} \right\| du, \qquad s^*(t) = \int_{t_0}^{t} \left\| \frac{d\gamma^*(u)}{du} \right\| du = |\lambda| \, s(t).$$

Lemma 2. *Let σ be the spherical arc-length parameter of γ. The arc-length element $d\sigma$ and the function $\tilde{\kappa} = -\dfrac{d\kappa}{\kappa d\sigma}$ are invariants under the p-similarity map.*

Proof. Let σ^* and κ^* be the spherical arc-length parameter and curvature of γ^* defined by (6), respectively. From (4) we can calculate the curvature κ^* as follows

$$\kappa^* = \varepsilon \frac{d^2\gamma^*}{ds^{*2}} \cdot \mathcal{J} \frac{d\gamma^*}{ds^*} = \frac{1}{|\lambda|} \varepsilon \frac{d^2\gamma}{ds^2} \cdot \mathcal{J} \frac{d\gamma}{ds} = \frac{1}{|\lambda|} \kappa. \tag{7}$$

Then, we can find

$$d\sigma = \kappa ds = \kappa^* ds^* = d\sigma^*. \tag{8}$$

Thus, the spherical arc-length element $d\sigma$ is invariant under the p-similarities of $\mathcal{M}^{1,1}$. Using (7) and (8), we can write

$$-\frac{d\kappa^*}{\kappa^* d\sigma^*} = -\frac{d\kappa}{\kappa d\sigma}$$

so that it is obtained $\tilde{\kappa}$ is an invariant. □

From the Lemma 1, we can use the spherical arc-length parameter in order to study the geometry of nonnull plane curves under the p-similarity motion. The derivative formulas of γ with respect to σ are

$$\frac{d\gamma}{d\sigma} = \frac{1}{\kappa} \mathbf{T}, \qquad \frac{d\mathbf{T}}{d\sigma} = \varepsilon_{\mathcal{J}\mathbf{T}} \kappa \mathcal{J} \frac{d\gamma}{d\sigma}, \qquad \frac{d(\mathcal{J}\mathbf{T})}{d\sigma} = \varepsilon_{\mathcal{J}\mathbf{T}} \kappa \frac{d\gamma}{d\sigma} \tag{9}$$

and

$$\begin{aligned} \frac{d^2\gamma}{d\sigma^2} &= \tilde{\kappa} \frac{d\gamma}{d\sigma} + \varepsilon_{\mathcal{J}\mathbf{T}} \mathcal{J} \frac{d\gamma}{d\sigma} \\ \frac{d}{d\sigma}\left(\mathcal{J} \frac{d\gamma}{d\sigma} \right) &= \tilde{\kappa} \mathcal{J} \frac{d\gamma}{d\sigma} \varepsilon_{\mathcal{J}\mathbf{T}} \frac{d\gamma}{d\sigma}. \end{aligned} \tag{10}$$

Similarly, the equations (9) and (10) are also valid for the nonnull curve γ^*. The function $\tilde{\kappa}$ takes the form

$$\tilde{\kappa}(\sigma) = \frac{\frac{d^2\gamma}{d\sigma^2} \cdot \frac{d\gamma}{d\sigma}}{\frac{d\gamma}{d\sigma} \cdot \frac{d\gamma}{d\sigma}} \tag{11}$$

by the equation (10).

Definition 3. *The function $\tilde{\kappa} = -\dfrac{d\kappa}{\kappa d\sigma}$ is called a* p-shape curvature *of a curve in $\mathcal{M}^{1,1}$.*

Remark 4. *The p-shape curvature of a Minkowski plane curve parameterized by an arbitrary parameter t can be expressed by*

$$\begin{aligned}\tilde{\kappa} &= \frac{-\dot{\kappa}}{\kappa^2 \left\|\dot{\gamma}\right\|} \\ &= \varepsilon \frac{3\left(\ddot{\gamma}\cdot\dot{\gamma}\right)\left(\ddot{\gamma}\cdot\mathcal{J}\dot{\gamma}\right)-\left(\dddot{\gamma}\cdot\mathcal{J}\dot{\gamma}\right)\left|\dot{\gamma}\cdot\dot{\gamma}\right|}{\left(\ddot{\gamma}\cdot\mathcal{J}\dot{\gamma}\right)^2}. \end{aligned} \tag{12}$$

If the spacelike or timelike curve γ is given as the graph of a function $y = f(x)$, the p-shape curvature is

$$\tilde{\kappa}(x) = 3f'(x) - \frac{f'''(x)\left((f'(x))^2 - 1\right)}{\left(f''(x)\right)^2}$$

or

$$\tilde{\kappa}(x) = 3f'(x) + \frac{f'''(x)\left(1 - (f'(x))^2\right)}{\left(f''(x)\right)^2},$$

respectively, using the formula (12)*.*

Now, let's show that two curves having same p-shape curvature are equivalent to each other under the p-similarity map by means of the hyperbolic structure. The following proposition is also showed in [11] without the hyperbolic structure.

Proposition 5. *Let $\gamma, \gamma^* : I \to \mathcal{M}^{1,1}$ be two nonnull curves of class C^2 parameterized by the same spherical arc-length parameter σ and have the same causal character, where $I \subset \mathbb{R}$ is an open interval. Suppose that γ and γ^* have the (oriented) non-zero curvature and $\tilde{\kappa} = \tilde{\kappa}^*$ for any $\sigma \in I$. Then, there exists a p-similarity f such that $\gamma^* = f \circ \gamma$.*

Proof. Let κ and κ^* be the curvature of γ and γ^*. Using the equality $\tilde{\kappa} = \tilde{\kappa}^*$, we get $\kappa = \lambda\kappa^*$ for some real constant $\lambda > 0$. We can choose any point $\sigma_0 \in I$. There exists a Lorentzian motion ϱ of $\mathcal{M}^{1,1}$ such that

$$\varrho\left(\gamma\left(\sigma_0\right)\right) = \gamma^*\left(\sigma_0\right), \quad \varrho\left(\mathbf{T}\left(\sigma_0\right)\right) = -\varepsilon_{\mathcal{J}\mathbf{T}}\mathbf{T}^*\left(\sigma_0\right), \quad \varrho\left(\mathcal{J}\mathbf{T}\left(\sigma_0\right)\right) = \varepsilon_{\mathcal{J}\mathbf{T}}\mathcal{J}\mathbf{T}^*\left(\sigma_0\right)$$

Let's consider the function $\Psi : I \to \mathbb{R}$ defined by

$$\Psi(\sigma) = \left(\varrho\left(\mathbf{T}(\sigma)\right) + \varepsilon_{\mathcal{J}\mathbf{T}}\mathbf{T}^*(\sigma)\right)^2 + \left(\varrho\left(\mathcal{J}\mathbf{T}(\sigma)\right) - \varepsilon_{\mathcal{J}\mathbf{T}}\mathcal{J}\mathbf{T}^*(\sigma)\right)^2.$$

Taking the derivative of Ψ with respect to σ, we get

$$\frac{d\Psi}{d\sigma} = 0$$

Since we know $\Psi(\sigma_0) = 0$, we can write $\Psi(\sigma) = 0$ for any $\sigma \in I$. As a result, we can say that

$$\varrho\left(\mathbf{T}(\sigma)\right) = -\varepsilon_{\mathcal{J}\mathbf{T}}\mathbf{T}^*(\sigma) \quad \text{and} \quad \varrho\left(\mathcal{J}\mathbf{T}(\sigma)\right) = \varepsilon_{\mathcal{J}\mathbf{T}}\mathcal{J}\mathbf{T}^*(\sigma) \quad \forall\sigma \in I.$$

The map $g = \lambda\varrho : \mathcal{M}^{1,1} \to \mathcal{M}^{1,1}$ is a p-similarity of $\mathcal{M}^{1,1}$. We examine the other function $\Phi : I \to \mathbb{R}$ such that

$$\Phi(\sigma) = \left(\frac{d}{d\sigma} g(\gamma(\sigma)) + \varepsilon_{\mathcal{J}\mathbf{T}} \frac{d}{d\sigma} \gamma^*(\sigma)\right)^2 \qquad \text{for } \forall\sigma \in I.$$

Taking the derivative of this function, we can easily find

$$\frac{d\Phi}{d\sigma} = 0$$

by using (9) and (10). Since we have $\Phi(\sigma_0) = 0$, the function $\Phi(\sigma)$ is equal to zero for $\forall\sigma \in I$. This means that

$$\frac{d}{d\sigma} g(\alpha(\sigma)) = -\varepsilon_{\mathcal{J}\mathbf{T}} \frac{d}{d\sigma} \gamma^*(\sigma)$$

or equivalently $\gamma^*(\sigma) = -\varepsilon_{\mathcal{J}\mathbf{T}} g(\gamma(\sigma)) + \mathbf{v}_0$ where $\mathbf{v}_0$ is a constant vector. Then, the theorem is proved. □

The next proposition states the existence of a unique nonnull plane curve whose p-shape curvature is given.

Proposition 6. *([11]) Let $z : I \to \mathbb{R}$ be the function of class C^1 and $\mathbf{e}_1^0$, $\mathbf{e}_2^0$ be an orthonormal 2-frame at $\mathbf{x}_0$ in $\mathcal{M}^{1,1}$ where $\mathbf{x}_0 \in \mathcal{M}^{1,1}$. There exists an unique nonnull curve $\gamma : I \to \mathcal{M}^{1,1}$ parameterized by the spherical arc-length parameter σ under the p-similarity mapping such that $\gamma(\sigma_0) = \mathbf{x}_0$ for any $\sigma_0 \in I$, the moving frame of γ at $\mathbf{x}_0$ is $\{\mathbf{e}_1^0, \mathbf{e}_2^0\}$ and the invariant $\tilde{\kappa}$ is equal to z.*

From the Proposition 5 and Proposition 6, we get the following fundamental theorem for plane curves under the p-similarity motion in $\mathcal{M}^{1,1}$.

Theorem 7. *Let $z : I \to \mathbb{R}$ be the function of class C^1. There exists a unique nonnull curve with the p-shape curvature z.*

The Theorem 7 implies that we can define a unique nonnull curve $\gamma : I \to \mathcal{M}^{1,1}$ parameterized by

$$\gamma(\sigma) = \mathbf{x}_0 + \int_{\sigma_0}^{\sigma} e^{\int \tilde{\kappa}(\sigma) d\sigma} \mathbf{e}_1(\sigma)\, d\sigma, \qquad \sigma \in I \tag{13}$$

such that its moving frame $\{\mathbf{e}_1,\ \mathbf{e}_2\}$ satisfy the system

$$\frac{d\mathbf{e}_1}{d\sigma} = \varepsilon_{\mathbf{e}_2} \mathbf{e}_2, \quad \frac{d\mathbf{e}_2}{d\sigma} = \varepsilon_{\mathbf{e}_2} \mathbf{e}_1 \tag{14}$$

with the initial conditions $\{\mathbf{e}_1^0,\ \mathbf{e}_2^0\}$.

Example 8. *Let $\gamma : I \to \mathcal{M}^{1,1}$ be a timelike curve with the p-shape curvature $\tilde{\kappa} = \dfrac{1}{\sigma}$. Choose initial conditions*

$$\mathbf{e}_1^0 = \mathbf{i}, \quad \mathbf{e}_2^0 = \mathbf{j}.$$

The system (14) *defines a vector* $\mathbf{e}_1(\sigma) = (\cosh\sigma)\,\mathbf{i} + (\sinh\sigma)\,\mathbf{j}$ *with* $\mathbf{e}_1(0) = \mathbf{e}_1^0$. *Solving the equation* (13) *we get the parametrization of timelike curve* γ *as the following*

$$\gamma(\sigma) = (2 + \sigma\sinh\sigma - \cosh\sigma)\,\mathbf{i} + (\sigma\cosh\sigma - \sinh\sigma)\,\mathbf{j} \quad \text{(see Figure 1).} \tag{15}$$

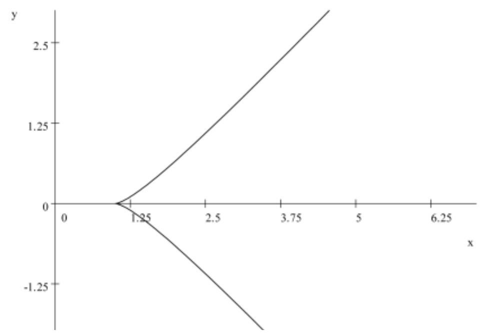

FIGURE 1. .

The locus of centres of osculating pseudo-circles of a nonnull curve, with inflection points removed, is called *evolute* of it. For a unit speed curve γ, we can find the evolute $\mathcal{E}_\gamma$ as

$$\mathcal{E}_\gamma(s) = \gamma(s) - \frac{\varepsilon_{\mathcal{J}\mathbf{T}}}{\kappa}\mathcal{J}\mathbf{T} \tag{16}$$

$\mathcal{E}_\gamma$ is singular at the vertices of γ and a spacelike (timelike) curve when γ is a timelike (spacelike) curve. We can easily compute $\kappa_{\mathcal{E}} = \varepsilon_{\mathcal{J}\mathbf{T}}\dfrac{\kappa^3}{|\kappa'|}$ from the equation (4) where $\kappa' = \dfrac{d\kappa}{ds}$. Also, we know that the arc-length parameters of spherical image of $\mathcal{E}_\gamma$ and γ are equal to each other; namely $\sigma_{\mathcal{E}} = \sigma$ (see also [9]).

The curve $\mathbf{In}_{\gamma,a}$, whose normal directions are tangent directions of nonnull curve γ, is called *involute of* γ. The equation of $\mathbf{In}_{\gamma,a}$ is given by

$$\mathbf{In}_{\gamma,a}(s) = \gamma(s) - (s-a)\,\mathbf{T}, \qquad a \in \mathbb{R}.$$

Also, we have $\dfrac{ds_\imath}{ds} = \kappa(c-a)$ where $s_\imath$ is arc-length parameter of $\mathbf{In}_{\gamma,a}$. The involute $\mathbf{In}_{\gamma,a}$ is singular at the points satisfying the equation $s - a = 0$. From (4), we can find the curvature $\kappa_\imath$ of $\mathbf{In}_{\gamma,a}$ as $\kappa_\imath = \dfrac{sign(\kappa)}{|s-a|}$ and know that the spherical arc-length parameters coincide $\sigma_{\mathbf{In}} = \sigma$.

Lemma 9. *Let* $\gamma : I \to \mathcal{M}^{1,1}$ *be a unit-speed nonnull curve and* β *be involute of* γ. *Then, the evolute of* β *is the curve* γ.

The next proposition show how we can find the p-shape curvatures of the evolute and involute of general nonnull plane curves.

Proposition 10. *Let $\gamma : I \to \mathcal{M}^{1,1}$ be a nonnull curve parameterized by spherical arc length parameter σ and $\tilde{\kappa} \neq 0$ be p-shape curvature of γ.*

i) The p-shape curvature of evolute $\mathcal{E}_\gamma$ of γ is

$$\tilde{\kappa}_{\mathcal{E}_\gamma}(\sigma) = \tilde{\kappa}(\sigma) + \frac{d(|-\tilde{\kappa}(\sigma)|)}{|-\tilde{\kappa}(\sigma)|\, d\sigma}. \tag{17}$$

ii) The p-shape curvature of involute $\mathbf{In}_{\gamma,a}$ of γ is

$$\tilde{\kappa}_{\mathbf{In}_{\gamma,a}}(\sigma) = \frac{1}{\kappa\left(\int \frac{1}{\kappa(\sigma)} d\sigma - a\right)}. \tag{18}$$

Proof. *i)* From the definition of p-shape curvature and $\kappa' = -\tilde{\kappa}\kappa^2$, we can write

$$\tilde{\kappa}_{\mathcal{E}_\gamma}(\sigma) = -\frac{d\left(\kappa^3/|\kappa'|\right)}{\left(\kappa^3/|\kappa'|\right) d\sigma} = \tilde{\kappa}(\sigma) + \frac{d(|-\tilde{\kappa}(\sigma)|)}{|-\tilde{\kappa}(\sigma)|\, d\sigma}.$$

ii) The involute of γ parameterized by $\sigma_{\mathbf{In}} = \sigma$ can be stated as

$$\mathbf{In}_{\gamma,a}(\sigma) = \gamma(\sigma) - \left(\int \frac{1}{\kappa(\sigma)} d\sigma - a\right) \mathbf{T}(\sigma).$$

Using the formulas (9), we can write

$$\begin{aligned} \frac{d\mathbf{In}_{\gamma,a}}{d\sigma} &= -\varepsilon_{\mathcal{J}\mathbf{T}} \left(\int \frac{1}{\kappa(\sigma)} d\sigma - a\right) \mathcal{J}\mathbf{T}, \\ \frac{d^2\mathbf{In}_{\gamma,a}}{d\sigma^2} &= \frac{-\varepsilon_{\mathcal{J}\mathbf{T}}}{\kappa} \mathcal{J}\mathbf{T} - \left(\int \frac{1}{\kappa(\sigma)} d\sigma - a\right) \mathbf{T}. \end{aligned}$$

Then, the formula (11) implies the equation (18). □

Corollary 11. *The vertices of a nonnull curve γ in $\mathcal{M}^{1,1}$ are the points where the p-shape curvature $\tilde{\kappa}$ vanish.*

A *parallel curve* α to a nonnull plane curve $\gamma : I \to \mathcal{M}^{1,1}$ with unit speed at a Lorentzian distance $r \neq 0$ is defined by

$$\alpha(s) = \gamma(s) + r\mathcal{J}\mathbf{T}(s), \quad t \in I \tag{19}$$

(see also [19]). Using the formula (4), we can obtain the curvature of parallel curve as the following

$$\kappa_\alpha = \frac{\varepsilon_{\mathcal{J}\mathbf{T}}\kappa}{|1 + r\varepsilon_{\mathcal{J}\mathbf{T}}\kappa|}.$$

For p-shape curvature of the parallel curve α, we have the following formula.

Lemma 12. *Let $\gamma : I \to \mathcal{M}^{1,1}$ be a nonnull curve with p-shape curvature $\tilde{\kappa} \neq 0$. For parallel curve, the p-shape curvature is given by*

$$\tilde{\kappa}_\alpha(\sigma) = \tilde{\kappa}(\sigma) + \left| \frac{r\kappa(\sigma)\,\tilde{\kappa}(\sigma)}{\varepsilon_{\mathcal{J}\mathbf{T}} + r\kappa(\sigma)} \right| \tag{20}$$

Proof. Let σ_α and s_α be a spherical arc length parameter and arc length parameter of α, respectively. Since we have $ds_\alpha = \|\alpha'\|\, ds = |1 + \varepsilon_{\mathcal{J}\mathbf{T}} r\kappa|\, ds$ from the definition (19), we can write $d\sigma_\alpha = d\sigma$, which means that the spherical arc length elements coincide. Then, the p-shape curvature of parallel curve is

$$\tilde{\kappa}_\alpha(\sigma) = \frac{-d\kappa_\alpha}{\kappa_\alpha d\sigma_\alpha} = \tilde{\kappa}(\sigma) + \left| \frac{\varepsilon_{\mathcal{J}\mathbf{T}} r\, d\kappa/d\sigma}{1 + \varepsilon_{\mathcal{J}\mathbf{T}} r\kappa} \right| = \tilde{\kappa}(\sigma) + \left| \frac{r\kappa(\sigma)\,\tilde{\kappa}(\sigma)}{\varepsilon_{\mathcal{J}\mathbf{T}} + r\kappa(\sigma)} \right|.$$

□

4. Self-Similar Curves in the Minkowski Plane

A curve γ is called *self-similar* curve in $\mathcal{M}^{1,1}$ if every p-similarity $f \in G$ conserve globally γ and G acts transitively on the nonnull curve γ where G is a one-parameter subgroup of $\mathbf{Sim}\left(\mathcal{M}^{1,1}\right)$. We can say that the invariant $\tilde{\kappa}$ is a constant for every self-similar curves with $\kappa \neq 0$ in $\mathcal{M}^{1,1}$.

Let's find the parametrizations of nonnull self similar curves with a constant p-shape curvature $\tilde{\kappa}$ in Lorentzian plane. For a curve $\gamma : I \to \mathcal{M}^{1,1}$ of class C^3 parameterized by the spherical arc-length parameter σ, we know

$$\frac{d\gamma}{d\sigma} = \frac{1}{\kappa}\mathbf{T} \quad \text{and} \quad \frac{d^2\gamma}{d\sigma^2} = \tilde{\kappa}\frac{d\gamma}{d\sigma} + \varepsilon_{\mathcal{J}\mathbf{T}}\mathcal{J}\frac{d\gamma}{d\sigma}. \tag{21}$$

The differential equation (21) can be rewritten as the system

$$\begin{aligned} \frac{d^2\gamma_1}{d\sigma^2} &= \tilde{\kappa}\frac{d\gamma_1}{d\sigma} - \varepsilon_{\mathcal{J}\mathbf{T}}\frac{d\gamma_2}{d\sigma} \\ \frac{d^2\gamma_2}{d\sigma^2} &= \tilde{\kappa}\frac{d\gamma_2}{d\sigma} - \varepsilon_{\mathcal{J}\mathbf{T}}\frac{d\gamma_1}{d\sigma} \end{aligned} \tag{22}$$

where $\gamma(\sigma) = \gamma_1(\sigma)\,\mathbf{i} + \gamma_2(\sigma)\,\mathbf{j}$.

If γ is a timelike curve, the solution of the above system with initial conditions

$$\gamma_1(0) = 0,\ \frac{d\gamma_1(0)}{d\sigma} = 1,\ \gamma_2(0) = 1,\ \frac{d\gamma_2(0)}{d\sigma} = 0$$

is given by

$$\gamma_1(\sigma) = \frac{-\tilde{\kappa} + e^{\tilde{\kappa}\sigma}[\tilde{\kappa}\cosh\sigma - \sinh\sigma]}{\tilde{\kappa}^2 - 1},\ \gamma_2(\sigma) = \frac{\tilde{\kappa}^2 - 2 - e^{\tilde{\kappa}\sigma}[\tilde{\kappa}\sinh\sigma - \cosh\sigma]}{\tilde{\kappa}^2 - 1}. \tag{23}$$

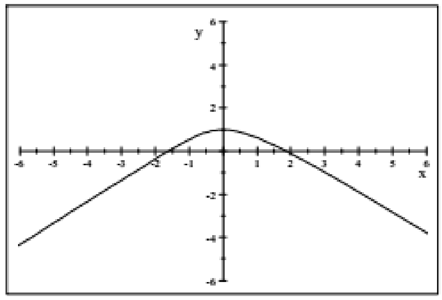

FIGURE 2. The graph of timelike self-similar curve for $\tilde{\kappa} = 0.5$.

Thus, the parametrization of a timelike self-similar curve is expressed by (23) (see Figure 2).

If γ is a spacelike curve, the solution of the system (22) with initial conditions

$$\gamma_1(0) = 1, \ \frac{d\gamma_1(0)}{d\sigma} = 0, \ \gamma_2(0) = 0, \ \frac{d\gamma_2(0)}{d\sigma} = 1$$

is given by

$$\gamma_1(\sigma) = \frac{\tilde{\kappa}^2 + e^{\tilde{\kappa}\sigma}[\tilde{\kappa}\sinh\sigma - \cosh\sigma]}{\tilde{\kappa}^2 - 1}, \quad \gamma_2(\sigma) = \frac{-\tilde{\kappa} + e^{\tilde{\kappa}\sigma}[\tilde{\kappa}\cosh\sigma - \sinh\sigma]}{\tilde{\kappa}^2 - 1}, \tag{24}$$

which is the equation of a spacelike self-similar curve (see figure 3).

Suppose that $\tilde{\kappa} = 0$ for $\kappa \neq 0$. Then, we get the timelike curve $\gamma(\sigma) = (\sinh\sigma, \ \cosh\sigma)$ and spacelike curve $\gamma(\sigma) = (\cosh\sigma, \ \sinh\sigma)$ by using (23) and (24). In addition, there is a one-parameter group of pseudo-rotations preserving a pseudo-circle which acts transitively on pseudo-circle. Thus, pseudo-circles are the unique Minkowski plane curves that satisfy $\tilde{\kappa} = 0$ with $\kappa \neq 0$. On the other hand, there is no a self-similar nonnull curve which has the property $\tilde{\kappa} = \pm 1$.

Now, we examine Lorentzian self-similar curves whose curvature can be equal to zero. Let $\alpha : I \to \mathcal{M}^{1,1}$ be a nonnull self-similar curve and $\kappa(s_0) = 0$ for some $s_0 \in I$. There exists a orientation-preserving p-similarity f of $\mathcal{M}^{1,1}$ such that $f(\alpha) = \alpha$ and $f(\alpha(s)) = \alpha(s_0)$ for any $s \in I$. Then, $\tilde{\kappa}(s)$ does not exist, or equivalently $\kappa(s) = 0$. Thus, we find that $\kappa(s) = 0$ for all $s \in I$, i.e., the nonnull self-similar curve α is a straight line. Conversely, every straight line is clearly a self-similar curve.

The logarithmic spirals are the unique self-similar curves except the straight lines and circles in the Euclidean space (see [20]). Then, we may think that the curves parameterized by (23) and (24) are the hyperbolic logarithmic spirals of Lorentzian plane since they are the unique self-similar curves except the lines and pseudo-circles in $\mathcal{M}^{1,1}$. As a result, from the above considerations, we get the following Lemma.

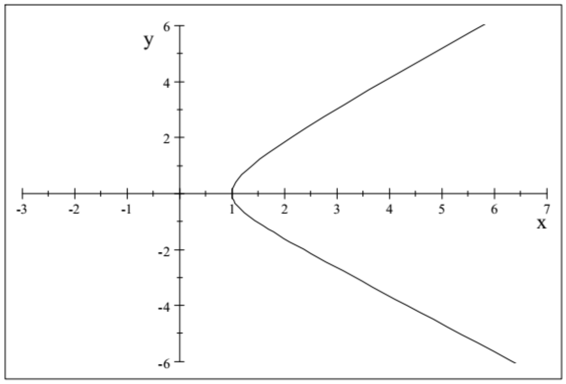

FIGURE 3. The graph of spacelike self-similar curve for $\tilde{\kappa} = 0.5$.

Lemma 13. *The unique self-similar curves are straight lines, pseudo-circles and hyperbolic logaritmic spirals in Lorentzian plane. Moreover, there is no the self-similar curve whose p-shape curvature is equal to* ± 1.

Lastly, we consider the evolutes, involutes and parallel curves of hyperbolic logarithmic spirals. We can say the following results.

Lemma 14. *Let* $\tilde{\kappa}$ *be the constant p-shape curvature of hyperbolic logarithmic spirals.*

i) The evolutes of the hyperbolic logarithmic spirals are also self-similar curves.

ii) The involutes of γ *the hyperbolic logarithmic spirals are not self-similar curves for* $\tilde{\kappa} \neq 0$.

iii) The parallel curves of timelike and spacelike hyperbolic logarithmic spirals are similar curves but not self-similar curves for $\tilde{\kappa} \neq 0$.

Proof. *i)* Let β_1 and β_2 be the timelike and spacelike hyperbolic logarithmic spirals, respectively. Using the formula (10), we obtain the p-shape curvatures of evolutes as $\tilde{\kappa}_{\mathcal{E}_{\beta_1}} = \tilde{\kappa}_{\mathcal{E}_{\beta_2}} = \tilde{\kappa}$.

ii) From the Lemma 1, the curvatures of β_1 and β_2 can be found as $\kappa_{\beta_1} = e^{-\tilde{\kappa}\sigma}$ and $\kappa_{\beta_2} = -e^{-\tilde{\kappa}\sigma}$. Then, the p-shape curvatures of involutes of β_1 and β_2 are

$$\tilde{\kappa}_{\mathbf{In}_{\beta_1,a}} = \frac{\tilde{\kappa} e^{\tilde{\kappa}\sigma}}{e^{\tilde{\kappa}\sigma} - \tilde{\kappa} a}, \quad \tilde{\kappa}_{\mathbf{In}_{\beta_2,a}} = \frac{\tilde{\kappa} e^{\tilde{\kappa}\sigma}}{e^{\tilde{\kappa}\sigma} + \tilde{\kappa} a}$$

by the formula (18), which says that the involutes are not self-similar curves.

iii) Using the formula (20) and the curvatures $\kappa_{\beta_1} = e^{-\tilde{\kappa}\sigma}$ and $\kappa_{\beta_2} = -e^{-\tilde{\kappa}\sigma}$, the p-shape curvatures of parallel curves α_1 and α_2 are

$$\tilde{\kappa}_{\alpha_1} = \tilde{\kappa}_{\alpha_2} = \tilde{\kappa} + \left| \frac{r \tilde{\kappa} e^{-\tilde{\kappa}\sigma}}{1 + r e^{-\tilde{\kappa}\sigma}} \right|,$$

which proves the hypothesis. □

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