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NUMERICAL SOLUTIONS OF THE MRLW EQUATION USING MOVING LEAST SQUARE COLLOCATION METHOD

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Abstract. In this paper, the Modified Regularized Long Wave (MRLW) equation is solved by using moving least square collocation (MLSC) method. To show the accuracy of the used method several numerical test examples are given. The motion of single solitary waves, the interaction of two solitary waves and the Maxwellian initial condition problems are chosen as test problems. For the single solitary wave motion whose analytical solution is known L_2, L_∞ error norms are calculated. Also mass, energy and momentum invariants are calculated for every test problem. Obtained numerical results are compared with some earlier works. According to the obtained results, the method is very efficient and reliable.

1. Introduction

The regularized long wave (RLW) equation was defined for the first time by Peregrine [1] to introduce the behavior of the bore development. Later Benjamin et al. used as model for a larger class of physical phenomena. The modified regularized long wave equation (MRLW) is a special form of RLW equation. This equation plays a very important role at the modelling of the nonlinear, dispersive media being modeled feature small-amplitude, long-wave length disturbances. The MRLW equation has the following form

$$U_t + U_x + \varepsilon U^2 U_x - \mu U_{xxt} = 0 \tag{1}$$

where ε and μ are positive integers and subscripts x and t denote space and time derivatives, respectively.

The exact solution of the MRLW equation was found by as follows [3]

$$U(x,t) = \sqrt{\frac{6c}{\varepsilon}}\,\operatorname{sech}\left(k\left(x-(c+1)t-x_0\right]\right) \tag{2}$$

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where $k=\sqrt{\frac{c}{\mu(c+1)}}$, x_0 and c are arbitrary constants. For the single solitary wave $6c/\varepsilon$ is the amplitude of solitary wave and its width is represented by k, also x_0 is the peak position of the wave. This solitary wave propagates to the right side by keeping its original shape at a steady velocity $c+1$ while time is increasing, [14]. In this study numerical experiments will be performed for the parameter $\varepsilon=6$, $\mu=1$, $c=1$ and $x_0=40$. In the present study, in order to find a numerical solution of the MRLW equation on the finite domain $[a,b]$, boundary conditions and initial condition are chosen as follow, respectively:

$$\begin{aligned} &U(a,t)=0,\;\; U(b,t)=0,\;\; a\le x\le b,\, t>0\\ &U(x,0)=f(x). \end{aligned} \tag{3}$$

The MRLW equation has been solved numerically by using numerical techniques. These include finite element, finite difference, Fourier method and meshless methods. The MRLW equation was solved by various types of B-spline functions by using finite element method such as the collocation method with quintic B-splines finite element method in [3], the collocation method using cubic B-splines finite element in [7], a numerical scheme based on quartic B-spline method in [17], based on collocation of quintic B-splines finite elements in [18] was presented. Also the collocation method with quadratic, cubic, quartic and quintic B-splines [10], cubic B-spline lumped Galerkin finite element method [11], a Petrov-Galerkin method [13] were used.

A finite difference scheme and Fourier stability analysis in [5], two finite difference approximations for the space dicretization and a multi-time step method for the time discretization for the MRLW equation in [15] and a fully implicit finite difference method in [19] were presented for the numerical solution of the MRLW equation. Also, the Adomian decomposition method was applied to solve numerically the MRLW equation in [6].

The meshless methods were applied to the MRLW equation in the literature. The meshless kernel based method of lines was used in [8] where gaussian, multiquadric and Wendland's compactly supported radial basis functions were used as kernel functions in computations. The radial basis functions collocation method by using different shape functions which were multiquadric, gaussian, inverse multiquadric and inversequadric radial basis functions was applied in [9]. A meshless method based on the moving least-squares approximation for the nonlinear generalized regularized long wave equation was used in [16]. In where the cubic spline function was used as weight function also convergence of the iterative process was presented.

In this study, the moving least square collocation method by using gaussian weight function will be applied to the MRLW equation and obtained results will be compared with other results in the literature.

2. The moving least square approximation

To solve the MRLW equation (1) numerically, the moving least square collocation (MLSC) approximation will be applied. Lancaster and Salkauskas [12] defined a local approximation for $u(\mathbf{x})$ as follow

$$u^h(\mathbf{x})=\sum_{i=1}^{m} p_i(\mathbf{x})a_i(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{4}$$

where $p_i(\mathbf{x})$ is the given monomial basis function of order m, $a_i(\mathbf{x})$ are the unknown coefficients of basis functions at spatial coordinates $\mathbf{x}$. $u^h(\mathbf{x})$ represents the MLS approximation of $u(\mathbf{x})$. The unknown coefficients $a_i(\mathbf{x})$ will be determined by minimizing the weighted discrete error norm given by

$$\begin{aligned} J(\mathbf{x}) &= \sum_{i=1}^{N}\omega(\mathbf{x}-\mathbf{x}_i)[u^h(\mathbf{x})-u(\mathbf{x}_i)]^2 \\ &= \sum_{i=1}^{N}\omega(\mathbf{x}-\mathbf{x}_i)\left[\sum_{j=0}^{m} p_j(\mathbf{x})a_j(\mathbf{x})-u(\mathbf{x}_i)\right]^2 \end{aligned}$$

where $\omega(\mathbf{x}-\mathbf{x}_i)$ are weight functions, $\mathbf{x}_i$ are nodes of spatial coordinates $\mathbf{x}$. To find the value of vector $\mathbf{a}(\mathbf{x})$ by minimizing the J we obtain

$$\frac{\partial J}{\partial \mathbf{a}} = \mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{u} = 0$$

so this can be written in the equation system

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\mathbf{u},$$

from this resulting equation $\mathbf{a}(\mathbf{x})$ is found as

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{u}$$

where the matrices $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $\mathbf{u}$, $\mathbf{W}(\mathbf{x})$ and $\mathbf{p}(\mathbf{x})$ are defined as follow:

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \mathbf{p}^T(\mathbf{x})\mathbf{W}(\mathbf{x})\mathbf{p}(\mathbf{x}), \\ \mathbf{B}(\mathbf{x}) &= \mathbf{p}^T(\mathbf{x})\mathbf{W}(\mathbf{x}), \\ \mathbf{u}^T &= (u_1, u_2, ..., u_N), \end{aligned}$$

$$\mathbf{W}(\mathbf{x}) = \begin{bmatrix} \omega(\mathbf{x}-\mathbf{x}_1) & 0 & \cdots & 0 \\ 0 & \omega(\mathbf{x}-\mathbf{x}_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega(\mathbf{x}-\mathbf{x}_N) \end{bmatrix},$$

$$\mathbf{p}(\mathbf{x}) = \begin{bmatrix} p_0(\mathbf{x}_1) & p_1(\mathbf{x}_1) & \cdots & p_m(\mathbf{x}_1) \\ p_0(\mathbf{x}_2) & p_1(\mathbf{x}_2) & \cdots & p_m(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_0(\mathbf{x}_N) & p_1(\mathbf{x}_N) & \cdots & p_m(\mathbf{x}_N) \end{bmatrix}.$$

Therefore, substituting $\mathbf{a}(\mathbf{x})$ into Eq. (4) approximation function of $u(\mathbf{x})$ is obtained as

$$u(\mathbf{x}) \approx u^h(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{u}$$

or

$$u(\mathbf{x}) \approx u^h(\mathbf{x}) = \sum_{i=1}^{N} \phi_i(\mathbf{x})u_i = \mathbf{\Phi}^T(\mathbf{x})\mathbf{u}. \tag{5}$$

$\mathbf{\Phi}^T(\mathbf{x})$ is a matrix of shape function and has the following form

$$\begin{aligned} \mathbf{\Phi}(\mathbf{x}) &= (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), ..., \phi_N(\mathbf{x})) \\ &= \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) \end{aligned} \tag{6}$$

where $\phi_i(\mathbf{x})$ has the following form

$$\phi_i(\mathbf{x}) = \sum_{j=0}^{m} p_j(\mathbf{x}) \left[(\mathbf{A}^{-\mathbf{1}}(\mathbf{x})\mathbf{B}(\mathbf{x}))\right]_{ji}. \tag{7}$$

In the literature there are some weight functions and in our algorithms Gaussian weight function with compact supported is chosen. Gaussian weight function is defined as follows

$$\omega(\mathbf{x}-\mathbf{x}_i) = \begin{cases} \dfrac{e^{-(d_i/c_i)^2} - e^{-(r_i/c_i)^2}}{1 - e^{-(r_i/c_i)^2}}, & 0 \le d_i \le r_i \\ 0, & d_i > r_i \end{cases} \tag{8}$$

where c_i is the shape parameter and $d_i = |\mathbf{x}-\mathbf{x}_i|$ is the distance between collocation points $\mathbf{x}$ and $\mathbf{x}_i$. The support size r_i for weight functions $\omega(\mathbf{x}-\mathbf{x}_i)$ determines the support of node $\mathbf{x}_i$. Derivatives of the approximate solution $u^h(\mathbf{x})$ can be obtained as

$$\frac{\partial^k}{\partial x^k} u^h(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial^k \phi_i(\mathbf{x})}{\partial x^k} u_i. \tag{9}$$

3. Discretization of the MRLW equation

In this section, the MRLW equation is discretized by using a Crank-Nicolson scheme for U and a forward difference rule for U_t in time as following:

$$\frac{U^{n+1}-U^n}{\Delta t}+\frac{U_x^{n+1}+U_x^n}{2}+6\frac{\left(U^2U_x\right)^{n+1}+\left(U^2U_x\right)^n}{2}-\mu\frac{U_{xx}^{n+1}-U_{xx}^n}{\Delta t}=0. \quad (10)$$

Eq. (10) can be rewritten as

$$U^{n+1}-U^n+\frac{\Delta t}{2}\left(U_x^{n+1}+U_x^n\right)+3\Delta t\left(\left(U^2U_x\right)^{n+1}+\left(U^2U_x\right)^n\right)-\mu\left(U_{xx}^{n+1}-U_{xx}^n\right)=0. \quad (11)$$

The nonlinear term $(U^2U_x)^{n+1}$ in Eq. (11) may be linearized by using Taylor series expansion as follows

$$\left(U^2U_x\right)^{n+1}=(U^2)^nU_x^{n+1}+2U^nU_x^nU^{n+1}-2\left(U^2\right)^nU_x^n. \quad (12)$$

Substituting Eq. (12) in to the Eq. (11) time-discretized the MRLW equation can be written as

$$\begin{aligned}U^{n+1}&+\frac{\Delta t}{2}U_x^{n+1}-\mu U_{xx}^{n+1}+3\Delta t\left((U^2)^nU_x^{n+1}+2U^nU_x^nU^{n+1}\right)\\&=U^n-\frac{\Delta t}{2}U_x^n-\mu U_{xx}^n+3\Delta t\left(U^2)^nU_x^n\right).\end{aligned} \quad (13)$$

4. Implementation of the MLSC method

In this section, the MLSC method will be applied to the discretized form of the MRLW equation (13). To find the numerical value of U used approximation is given as follows

$$U^n=\sum_{i=1}^{N}\phi_i(\mathbf{x}_k)\lambda_i^n, \quad k=1,2,\cdots,N \quad (14)$$

where $\phi_i(\mathbf{x}_k)$ are shape functions at each collocation points $\mathbf{x}_k$ and λ_i are the unknown values. Eq. (14) can be written as the following matrix form

$$U^n=\tilde{A}\lambda^n \quad (15)$$

where

$$\tilde{A}=[\phi_i(x_k):i,k=\overline{1,N}], \quad \lambda^n=[\lambda_1^n,\lambda_2^n,\cdots,\lambda_N^n]^T.$$

First and second derivatives of Eq. (14) given by

$$U_x^n=\sum_{i=1}^{N}\phi_i^{'}(\mathbf{x}_k)\lambda_i^n, \quad U_{xx}^n=\sum_{i=1}^{N}\phi_i^{''}(\mathbf{x}_k)\lambda_i^n, \quad k=1,2,\cdots,N \quad (16)$$

respectively. We put our trial functions Eqs. (14) and (16) into the Eq. (13) and Eq. (3) at the collocation points $\mathbf{x}_k$, gives the following systems of algebraic equations:

$$\begin{aligned}
&\sum_{i=1}^{N}\phi_i(\mathbf{x}_k)\lambda_i^{n+1}+\frac{\Delta t}{2}\sum_{i=1}^{N}\phi_i^{'}(\mathbf{x}_k)\lambda_i^{n+1}+3\Delta t\sum_{i=1}^{N}\phi_i^{2}(\mathbf{x}_k)\lambda_i^{n}\cdot\sum_{i=1}^{N}\phi_i^{'}(\mathbf{x}_k)\lambda_i^{n+1}\\
&+6\Delta t\sum_{i=1}^{N}\phi_i(\mathbf{x}_k)\lambda_i^{n}\cdot\sum_{i=1}^{N}\phi_i^{'}(\mathbf{x}_k)\lambda_i^{n}\cdot\sum_{i=1}^{N}\phi_i(\mathbf{x}_k)\lambda_i^{n+1}-\mu\sum_{i=1}^{N}\phi_i^{''}(\mathbf{x}_k)\lambda_i^{n+1}\\
&=\sum_{i=1}^{N}\phi_i(\mathbf{x}_k)\lambda_i^{n}-\frac{\Delta t}{2}\sum_{i=1}^{N}\phi_i^{'}(\mathbf{x}_k)\lambda_i^{n}+3\Delta t\sum_{i=1}^{N}\phi_i^{2}(\mathbf{x}_k)\lambda_i^{n}\cdot\sum_{i=1}^{N}\phi_i^{'}(\mathbf{x}_k)\lambda_i^{n}\\
&-\mu\sum_{i=1}^{N}\phi_i^{''}(\mathbf{x}_k)\lambda_i^{n},\quad k=2,\dots,N-1\\
&\sum_{i=1}^{N}\phi_i(\mathbf{x}_k)\lambda_i^{n+1}=\alpha,\quad k=1\\
&\sum_{i=1}^{N}\phi_i(\mathbf{x}_k)\lambda_i^{n+1}=\beta,\quad k=N
\end{aligned}\tag{17}$$

Substituting Eq. (7) into Eq. (17) and simplifying we have obtain following linear equation system

$$\begin{aligned}
&M_1\boldsymbol{\lambda}^{n+1}=\alpha,\quad k=1\\
&\sum_{j=1}^{5}M_j\boldsymbol{\lambda}^{n+1}=\sum_{j=1}^{3}M_j\boldsymbol{\lambda}^{n}-M_4\boldsymbol{\lambda}^{n}\ ,k=2,\dots,N-1\\
&M_1\boldsymbol{\lambda}^{n+1}=\beta,\quad k=N
\end{aligned}\tag{18}$$

where coefficients M_j are defined as follows:

$$\begin{aligned}
M_1&=\sum_{i=1}^{N}\sum_{j=0}^{m}p_j(\mathbf{x}_k)[\mathbf{A}^{-1}(\mathbf{x}_k)\mathbf{B}(\mathbf{x}_k)]_{ji}\\
M_2&=-\mu\sum_{i=1}^{N}\sum_{j=0}^{m}p_j(\mathbf{x}_k)[\mathbf{A}^{-1}(\mathbf{x}_k)\mathbf{B}(\mathbf{x}_k)]_{ji}^{''}\\
M_3&=3\Delta t\sum_{i=1}^{N}\sum_{j=0}^{m}p_j(\mathbf{x}_k)[\mathbf{A}^{-1}(\mathbf{x}_k)\mathbf{B}(\mathbf{x}_k)]_{ji}^{2}\lambda_i^{n}\cdot\sum_{i=1}^{N}\sum_{j=0}^{m}p_j(\mathbf{x}_k)[\mathbf{A}^{-1}(\mathbf{x}_k)\mathbf{B}(\mathbf{x}_k)]_{ji}^{'}\\
M_4&=6\Delta t\sum_{i=1}^{N}\sum_{j=0}^{m}p_j(\mathbf{x}_k)[\mathbf{A}^{-1}(\mathbf{x}_k)\mathbf{B}(\mathbf{x}_k)]_{ji}\lambda_i^{n}\cdot\sum_{i=1}^{N}\sum_{j=0}^{m}p_j(\mathbf{x}_k)[\mathbf{A}^{-1}(\mathbf{x}_k)\mathbf{B}(\mathbf{x}_k)]_{ji}^{'}\lambda_i^{n}\\
&\cdot\sum_{i=1}^{N}\sum_{j=0}^{m}p_j(\mathbf{x}_k)[\mathbf{A}^{-1}(\mathbf{x}_k)\mathbf{B}(\mathbf{x}_k)]_{ji}\\
M_5&=\sum_{i=1}^{N}\sum_{j=0}^{m}p_j(\mathbf{x}_k)[\mathbf{A}^{-1}(\mathbf{x}_k)\mathbf{B}(\mathbf{x}_k)]_{ji}^{'}
\end{aligned}$$

By solving the equations system (18) numerical values of λ^{n+1} are obtained at each nodal points. Substituting these values into the Eq. (14) numerical values of the MRLW equation at the each nodal points on solution interval is obtained.

5. Numerical examples and comparisons

In this section, MRLW equation are solved for three test problems to demonstrate accuracy and efficiency of the method. Also, conserved quantities and error norms will be calculated for test problems.

The MRLW equation has three conservation laws which are [4]

$$C_1 = \int_a^b U dx, \quad C_2 = \int_a^b \left(U^2 + \mu \left(U_x \right)^2 \right) dx, \quad C_3 = \int_a^b (U^4 - \mu \left(U_x \right)^2) dx$$

and corresponding to conservation of mass, momentum and energy respectively. In our computations numerical values of invariants are computed by using rectangular rule.

The root mean square error L_2 and maximum error L_∞ will be used to measure the error between the analytical and numerical solutions:

$$L_2 = \sqrt{h \sum_{j=1}^{N} \left| U_j^{exact} - U_j^{num.} \right|^2}, \quad L_\infty = \max_{1 \le j \le N} \left| U_j^{exact} - U_j^{num.} \right|.$$

5.1. **Test 1: Single solitary wave motion.** The single solitary wave solution of the MRLW equation is written as follow:

$$U(x,t) = \sqrt{c}\,\mathrm{sech}\left(k\left[x - (c+1)t - x_0\right]\right), \ k = \sqrt{\frac{c}{\mu(c+1)}}$$

For numerical calculations, boundary conditions $U(0,t) = U(100,t) = 0$ and initial condition $U(x,0) = \sqrt{c}\,\mathrm{sech}\left(k\left[x - x_0\right]\right)$ are used. Simulations are done over the solution domain $0 \le x \le 100$ in the time period $0 \le t \le 10$ with parameters $h = 0.2$, $\Delta t = 0.025$, $\mu = 1$, $c = 1$ and $x_0 = 40$. The analytical values of invariants are given as [3]

$$\begin{aligned} C_1 &= \frac{\pi\sqrt{c}}{k} = 4.44288 \\ C_2 &= \frac{2c}{k} + \frac{2\mu k c}{3} = 3.29983 \\ C_3 &= \frac{4c^2}{3k} - \frac{2\mu k c}{3} = 1.41421 \end{aligned} \tag{19}$$

At the time $t = 10$, calculated values of invariants C_1, C_2, C_3 and error norms L_2, L_∞ are listed in Table 1. It is clear that, obtained results are all in very good agreement with the analytical and other numerical results [3, 7, 9, 11]. The solitary wave profile is depicted in Figure 1 at different times. From Figure 1, it is seen that solitary wave propagates to the right along the x-axis without changing its shape.

Table 1 : Comparison of invariants and error norms.

Method	$L_2 \times 10^3$	$L_\infty \times 10^3$	C_1	C_2	C_3
MLSC	1.69905	0.79871	4.44288	3.29979	1.41416
[3]	16.39	9.24	4.442	3.299	1.413
[7]	9.30196	5.43718	4.44288	3.29983	1.41420
G[9]	4.17678	2.17187	4.44280	3.29957	1.41395
MQ[9]	3.98518	2.05606	4.44209	3.29974	1.41413
IQ[9]	5.87856	3.20703	4.44176	3.29795	1.41233
IMQ[9]	2.91797	1.54209	4.44323	3.30032	1.41470
[11]	2.41750	1.08099	4.44319	3.30030	1.41469

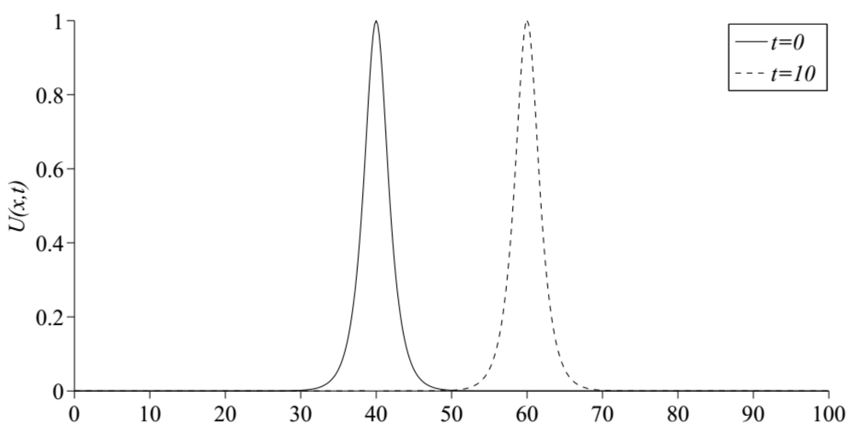

FIGURE 1. Motion of the single solitary wave.

5.2. **Test 2: Interaction of Two Solitary Waves.** Interaction of two positive solitary waves for MRLW equation is modelled by using the following initial conditions

$$U(x,0) = \sum_{j=1}^{2} \sqrt{c_j}\ \mathrm{sech}\left(k_i(x - x_j)\right), \quad k_j = \sqrt{\frac{c_j}{\mu(c_j+1)}} \tag{19}$$

where x_j, c_j are arbitrary constants, $j = 1, 2$. In this test problems, parameters are chosen $\mu = 1$, $c_1 = 4$, $c_2 = 1$, $x_1 = 25$, $x_2 = 55$ over the solution domain $[0, 250]$ in the time period $0 \leq t \leq 20$ with space step $h = 0.2$, time step $\Delta t = 0.025$. The analytical values of invariants are given as [3]

$$C_1=\sum_{j=1}^{2}\frac{\pi\sqrt{c_j}}{k_j}=11.4677,$$

$$C_2=\sum_{j=1}^{2}\left(\frac{2c_j}{k_j}+\frac{2\mu k_j c_j}{3}\right)=14.6292, \tag{20}$$

$$C_3=\sum_{j=1}^{2}\left(\frac{4c_j^2}{3k_j}-\frac{2\mu k_j c_j}{3}\right)=22.8805.$$

Calculated values of invariants are tabulated in Table 2. Obtained results are compared with analytical and other numerical results [7, 9, 11]. Two solitary waves profiles are depicted in Figure 2 and 3 at different times. As seen from these figures, the larger wave moves faster than the smaller one and it catches up and passes the smaller wave as time progresses. It is seen from figures a small tail occurs after completed of interaction of waves because these waves are solitary waves not solitons. It is known that solitary waves don't obey the principle of superposition. When the faster wave overtakes a slower wave, again solitary waves don't combine and add together [15].

Table 2: Comparison of invariants.

Method	C_1	C_2	C_3
MLSC	11.4677	14.5830	22.6965
[7]	11.4677	14.6292	22.8805
G[9]	11.4677	14.5834	22.6978
MQ[9]	11.4644	14.5819	22.6927
IQ[9]	11.4698	14.5811	22.6859
IMQ[9]	11.4602	14.5708	22.6524
[11]	11.4662	14.6253	22.8650

5.3. **Test 3: The Maxwellian initial condition.** Maxwellian initial condition for MRLW equation is defined as

$$U(x,0)=\exp(-(x-x_0)^2).$$

Boundary conditions are taken as $U(0,t)=U(80,t)=0$ in the time period $0\leq t\leq 10$. The computed values of the invariants for $\mu=0.1$, $\mu=0.05$, $\mu=0.025$, $x_0=20$, $h=0.2$ and $\triangle t=0.01$ are recorded in Table 3. Obtained results are compared with numerical results in [13]. It is seen that from Figures 4, 5 and 6 the number of solitary wave is increased when μ is reduced. For instance, a single solitary wave is occurred in Figure 4 while a few solitary wave is occurred in Figure 5 and 6.

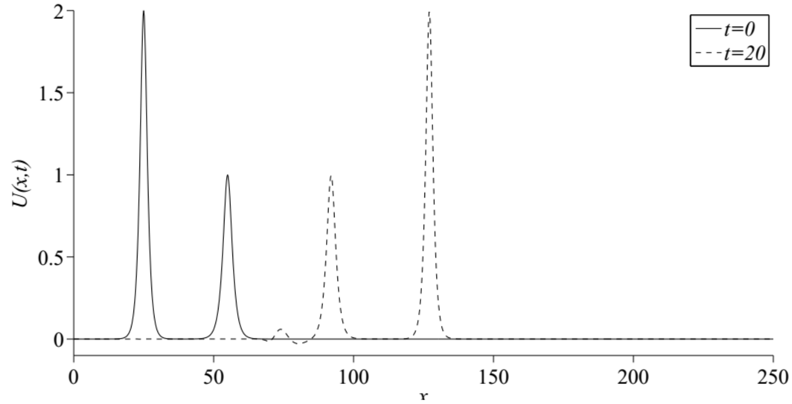

FIGURE 2. Interaction of two solitary waves at different times.

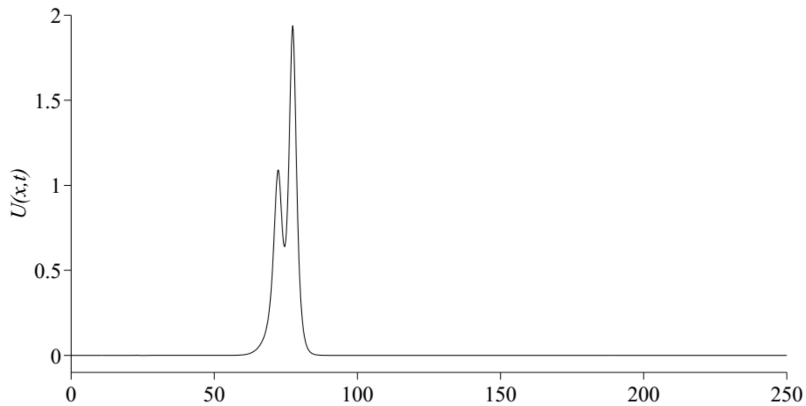

FIGURE 3. Interaction of two solitary waves at $t = 10$.

Table 3: Computed values of invariants.

Method	μ	C_1	C_2	C_3
MLSC	0.1	1.77245	1.38255	0.75714
[13]	0.1	1.77245	1.3808	0.7618
MLSC	0.05	1.77245	1.32353	0.82520
[13]	0.05	1.77246	1.31898	0.825787
MLSC	0.025	1.77245	1.27829	0.894994
[13]	0.025	1.77245	1.28935	0.863683

Figure 4. Maxwellian initial condition at $t = 10$ for $\mu = 0.1$.

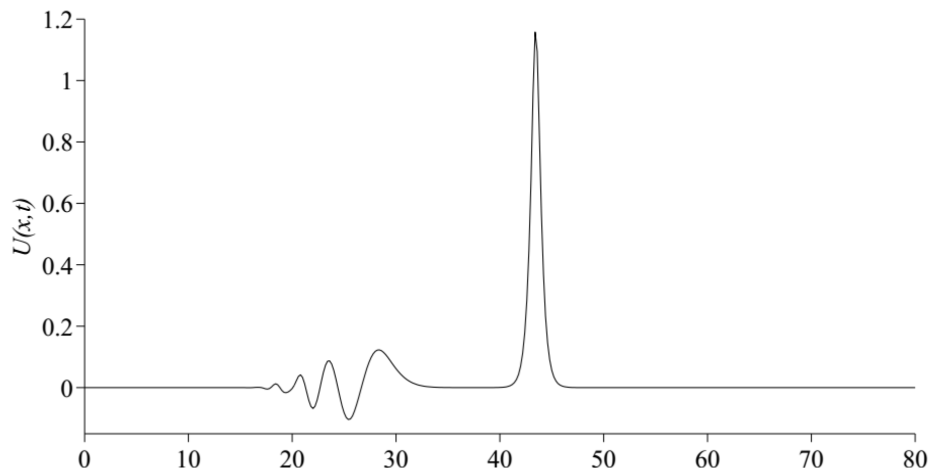

FIGURE 4. Maxwellian initial condition at $t = 10$ for $\mu = 0.1$.

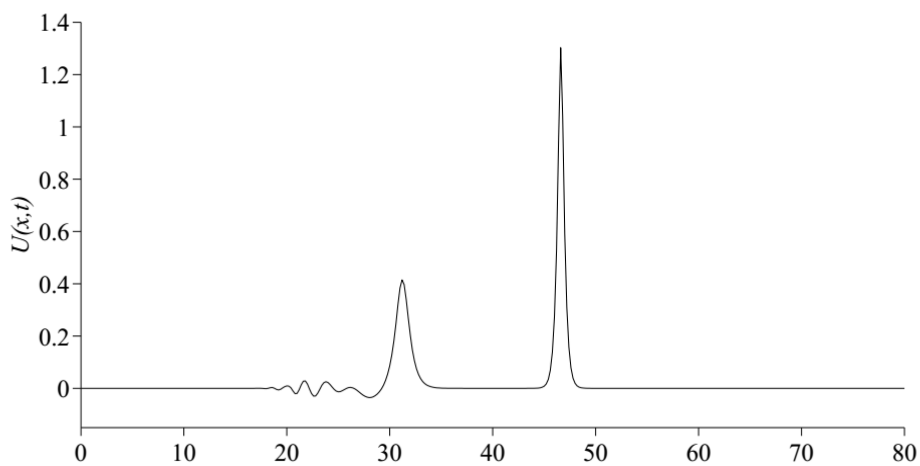

FIGURE 5. Maxwellian initial condition at $t = 10$ for $\mu = 0.05$.

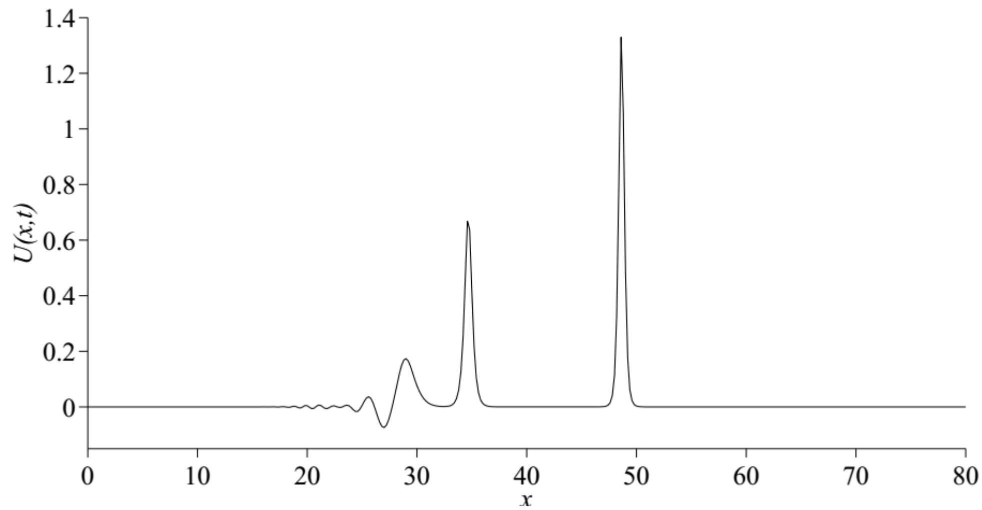

FIGURE 6. Maxwellian initial condition at $t = 10$ for $\mu = 0.025$.

6. CONCLUSION

The MRLW equation is solved numerically by using moving least square collocation method. The performance of the used numerical method with the different three test problems which are the single solitary wave motion, interaction of two solitary waves and the Maxwellian initial condition have been examined. The error norms L_2 and L_∞ for the single solitary wave motion have been calculated. Also, for the examined test problems numerical values of three conserved quantities have been evaluated. It has been seen that evaluated numerical results are in good agreement with the results obtained by previous studies. According to obtained results, the method satisfied very highly acceptable results for the MRLW equation.
This method is a meshless method hence has the simplicity of implementation and very high accuracy. The obtained results indicate that the present method is very effective and reliable numerical technique for solving this type nonlinear equations. It should point that the used numerical technique can be applied to nonlinear problems.

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