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DIFFUSIVE REPRESENTATION OF A FRACTIONAL CONTROL USING ADAPTIVE PARTITIONING ALGORITHM

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Abstract. This article presents optimal fractional control. This control is based on the property of the invariance of a fractional order differential equation. The problem formulation of the used control is expressed by diffusive representation. The fractional control problem is described in a minimization form, where the global optimum represents the diffusive realization of the controller. To determine the optimal fractional diffusive control, an adaptive partitioning algorithm is used. As an application, we have chosen the control of a DC motor with uncertain parameters.

1. Introduction

The fractional operators become an interesting tool in the systems mathematical modeling and design, their use appeared strongly in different disciplines. Moreover, their convenient interest has been proved in the last decade. These operators are widely used in automatic control systems to construct robust fractional controllers like fractional PI^{α} or PD^{β}, ...etc. [1, 2, 3, 4, 5, 6].

In controlling uncertain dynamic systems, the use of robust control laws that are able to ensure the best compromise between performances and Robustness is highly required. In order to satisfy this requirement researchers used the fractional control as an alternative choice, where the concept of the robustness is based on the property of the invariance of the fractional differential equation.

The fundamental property of this control is to preserve as much as possible, and on all the domain of uncertainty, the dynamic features imposed by the control of the nominal system, up to time scaling (or to frequency scaling). However, the use of fractional operators leads to some difficulties and problems, which come mainly from the fact that these operators are hereditary with singular kernels, and hence the numerical approximation becomes very difficult and requires large memory storage capacities. To remedy these problems, the fractional control will be achieved by

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using the diffusive representation [9]. This representation allows the realization of the fractional operators in non-hereditary way using linear dynamical systems of diffusive nature.[9]

We apply this concept to the control of a DC motor of which the transfer function is uncertain. The uncertainty is carried at the mechanical load and the current loop constant time. The non strict invariance consists in the minimization of an adequate cost functional. The optimal controller is achieved by using new algorithm so called 'Adaptive Partitioning Algorithms.

This article is organized as follows. Section 2 gives an overview on fractional order control systems and its realization by a irrational function . Section 3 describes the principle of the control design. where the mathematical formulation of the optimal control are presented by diffusive model. In section 4, we discuss an experimental design that is used to construct the new sub-regions and to generate the new populations. This design produces a set of individuals; each one occupies a subregion of the feasible region. Finally, an application are given in section 5

2. Fractional Calculus

In this part, We present some concepts of the fractional calculus, its scope and the difficulties associated with it. We also present the fractional operator realization using the diffusive representation.

2.1. **Fractional Operator.** The fractional derivation and integration of order $\alpha \in [0+\infty]$ of a causal function f, formulated by Riemann-Liouville, is given by [1, 5, 6].

$$I^{\alpha}(t) = \int_0^{\infty} \frac{(t-\tau)^{(\alpha-1)}}{\Gamma(\alpha)} f(t) d\tau \tag{2.1}$$

$$D_t^{\alpha} f(t) = \frac{d}{dt} \int_0^{\infty} \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(t) d\tau \tag{2.2}$$

where Γ is the gamma function defined by the expression: $\Gamma(\alpha) = \int_0^{\infty} t^{(\alpha-1)} e^{-t} dt$

2.2. **Fractional systems.** A linear fractional system with constant coefficient can be represented by a fractional order differential equation given by

$$\sum_{k=0}^{K} a_k \frac{d}{dt}^{\alpha_k} y(t) = \sum_{k=0}^{M} b_k \frac{d}{dt}^{\beta_k} f(t) \tag{2.3}$$

where K and M are integers, a_k, b_m are real numbers and α_k, β_k are arbitrary constants.

Using Laplace transform of 2.3, we obtain the following transfer function

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^{\beta_k}}{\sum_{k=0}^{K} a_k s^{\alpha_k}} \tag{2.4}$$

If there are k, l and m as $\alpha_k a = m.a_1$ and $\beta_m = k\beta_1$, then the fractional system (2.4) is called to commensurable order system, if this is not the case it is not a commensurable order.

2.3. **Problems related to the fractional calculus.** The difficulties encountered in the study of systems containing fractional operators, come mainly from the fact that these operators are hereditary with singular kernels, making the numerical approximation very difficult and requires large memory storage.

2.4. **Diffusive Representation.** The theory of diffusive representation allows the realization of fractional operators in non-hereditary way using linear dynamical systems of diffusive nature. It is very suited to the analysis and study of systems containing these operators.

Let $H(s)$ be a transfer function (non-rational) associated with the causal convolution operator $H(d/dt)$, the diffusive canonic realization of this operator is expressed, if any, by the realization of the state $f \to y = h(d/dt)f = h * f$ [10, 11].

$$\begin{cases} \partial_t \varphi(t,\xi) & = & \xi(t,\xi) + f(t) \\ y & = & \int_0^\infty \mu(\xi)\varphi(t,\xi)d\xi \end{cases} \tag{2.5}$$

where $\mu(\xi)$ is called diffusive representation of $H(d/dt)$.

The diffusive representation $\mu(\xi)$ of an invariant symbol pseudo-differential time operator $H(s)$ is defined, if any, as the integral equation solution [7, 8].

$$H(s) = \int_0^\infty \frac{\mu(\xi)}{s+\xi} d\xi \tag{2.6}$$

Equation 2.6 is the transfer function associated with the diffusive representation, with $\mu = L^{-1}\{h\}$, and h is the impulse response.

$$h(t) = \int_0^{+\infty} \mu(\xi) e^{-\xi t} d\xi \tag{2.7}$$

The diffusive representation is therefore the use of the Laplace transform in the opposite direction , wherein t plays the role of the Laplace variable.

3. Fractional Control

3.1. **Principle.** Let's consider the fractional differential equation of order β with uncertain parameter λ:

$$\lambda \frac{d^\beta}{dt^\beta} y(t) + y(t) = e(t) \quad 1 \le \beta < 2 \tag{3.1}$$

The corresponding transfer function is given by:

$$H_\lambda(s) = \frac{1}{\lambda s^\beta + 1} \tag{3.2}$$

The system (3.2) is the closed loop transfer function of the open loop transfer function H_{ol} given by

$$H_{ol}(s) = \frac{1}{\lambda s^{\beta}}. \tag{3.3}$$

The transfer function 3.3 is the Bode's ideal transfer function where the gain crossover frequency ω_c is $\frac{1}{\lambda^{\frac{1}{\beta}}}$.

The transfer function given by (3.2) is closed to a damping second order transfer function, and the damping ration is related directly to β and insensitive to gain variations [12] which give a time responses with iso-damping.

In mathematical point of view, the gain variations may be presented are equivalent to a change of time scale. Using the change of frequency $\tilde{s} = \lambda^{\frac{1}{\beta}} s$, the transfer function (3.2) will be written in the following form:

$$H_{\lambda}(s) = \frac{1}{\tilde{s}^{\beta} + 1}. \tag{3.4}$$

All changes of the value of λ are equivalent to a change of frequency scale. In other way, the closed loop transfer function presents the property of the invariance under transformation noted T_{σ} and defined by $\sigma_{\lambda} = \lambda^{\frac{1}{\beta}}$ [13].

The set of changes of frequency defined by a function $\sigma_{\lambda} > 0$ are a group of transformations under change of frequency [13]. We can write then:

$$\left(T_{\sigma} H_{\lambda}\right)(s) = H_{\lambda}\left(\sigma_{\lambda} s\right) \cong H_{\lambda_0}(s) \tag{3.5}$$

where λ_0 is the nominal parameter

3.2. **Optimal fractional controller design.** In control system, The system 3.3 may be used as a reference model to design a controller . In the case it is impossible to determine analytically a controller that permits to give a system closed to (3.3), the problem are solved through an optimization problem. So, the mathematical formulation of this problem are given by

$$\min_{C \in \kappa} \Im\left(F(C), F(C_0)\right) \tag{3.6}$$

where $F(C)$ is the closed loop transfer function of the uncertain system controlled by a fractional compensator C, and $F(C_0)$ is the desired closed loop transfer function for the nominal system controlled by the classical controller C_0.

In the problem described by (3.6), the functional cost under group of transformation can be expressed in a Hilbert space as follows [14]:

$$\min_{T \in \varsigma, C \in \kappa} \left\{ \left\| T_{\sigma} F(C) - F_{\lambda_0}(C_0) \right\|_{\lambda, s}^2 \right\} \tag{3.7}$$

where ς is the group of continuous functions defined on Λ, κ is the space of controllers and $\| \, . \, \|_{\lambda,s}^2$ is the H_2 Hilbertien norm. This formulation permits to get a compensated transfer function with proximity to the reference response defined by a standard controllers C_0 , where $T_{\sigma} F(C)$ is closed as possible to $F_{\lambda_0}(C_0)$, $\forall \lambda \in \Lambda$.

The transfer function of the optimal controller C (solution of the problem 3.7) are a rational function. So, the controller C is a fractional controller that can be achieved via diffusive representation given by (2.6). Therefore, The problem of minimization (3.6) can be formulated under diffusive formulation by (3.8) [14].

$$\min_{T\in\varsigma}\min_{\mu,K_c\in\kappa}\left\{\ \|T_\sigma F(C_\mu)-F_{\lambda_0}(C_0)\|^2_{\lambda,\omega}\ \right\} \tag{3.8}$$

where the compensator $C_\mu(s)$ is defined by:

$$C_\mu(s)=K_c\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{s+\xi}d\xi \tag{3.9}$$

4. The optimization algorithm

In the optimization part we used an adaptive partitioning algorithm. The principle of this algorithm is based on successive division of the search space until a narrow area is achieved about the global optimum. The initial search space denoted by α_0 is partitioned into C_2^m sub-regions denoted by α_i $(i=1..k)$, where t indicates the iteration number and m is the number of the best points selected after sampling the initial search space. m is defined at the beginning of the optimization operation. The partitioning and sampling operations are established using new experimental technique that will be described in what follows.

4.1. **The proposed experimental design.** The proposed technique that we call "circular design" is an experimental technique that permits to generate a set of points (individuals) around a central point expected to be the global optimum located between two points (that we called "parents") X^t_{P1} and X^t_{P2} [15, 16]. These individuals are selected form a population that occupies a limited region in the search space. The new population has a property that the point's distribution density decreases when we go far from its center (the expected global optimum). This is due to the fact that the points' distribution should have higher density about the central point.

To illustrate the proposed idea, let's suppose an n dimensional problem, thus at an iteration time t, a population of q individuals can be generated from two individuals X^t_{P1} and X^t_{P2}. This population is located inside a hypersphere centered at

$$X^t_c=[x^t_{c1},x^t_{c2},...x^t_{cn}] \tag{4.1}$$

where:

$$X^t_c=\frac{1}{2}(X^t_{p1}+X^t_{p2}) \tag{4.2}$$

The coordinates of each individual $X_k^t = [x_{k1}^t, x_{k2}^t, ..., x_{kn}^t]$ for (k=1,..., q), are calculated using the system equation 4.3.

$$\begin{cases} x_{k1}^t = x_{c1}^t + \frac{R}{2\pi} \prod_{l=2}^{n} \cos\theta_l \\ x_{kn}^t = x_{cn}^t + \frac{R}{2\pi} \theta_l \sin\theta_2 and \\ x_{kr}^t = x_{cr}^t + \frac{R}{2\pi} \left(\prod_{l=2}^{n-r+1} \cos\theta_l \right) \sin\theta_{n-r+2} \end{cases} \tag{4.3}$$

for : $r = 2, ..., (n-1)$ and $k = 1, ..., q$. and

$$\theta_l = \frac{2\pi}{q} ud(j,l) l = 1, .., n \tag{4.4}$$

Eq. 4.3 represents the circular transformation of $ud = [ud_{ij}]_{q \times n}$, a matrix of points uniformly scattered and is obtained by applying the linear uniform design technique discussed in [15, 16] over the considered research space. R is the radius of the hypersphere and is given by (4.5).

$$R = \sqrt{(X_{P1}^t - X_{P2}^t)^T (X_{P1}^t - X_{P2}^t)} \tag{4.5}$$

Equation 4.5 shows that the error offset equals to 1/2 of the population radius i.e. the real selected sub-region $\alpha_i'(t)$ to be repartitioned at iteration $t+1$ is greater than $\alpha_i(t)$. Hence, we reduce the probability to miss the global optimum, supposing that $\alpha_i(t)$ is the most promising subregion and contains the global optimum.

The proposed experimental technique mechanism is summarized as follows:

At each iteration number t, we generate C_2^m populations; each population occupies a sub-region of the search space; and each population is defined by its own vector $X_{k,i=1..ni}^t$, where $ni = 1..C_2^m$. We should notice that the space includes X_i^t represents exactly α_i.

The following definition describes the partitioning scheme of the APA.

Definition. The collection of the sub-sets generated by the circular design at the iteration time t is defined as follows

$$S = \{\alpha_i, i = 1...k\} \tag{4.6}$$

where $k = C_2^m$, and

$$\{\alpha_1(t+1) \cup \alpha_2(t+1) \cup \alpha_3(t+1) ... \alpha_k(t+1) \cap \alpha_i(t)\} \neq \phi \tag{4.7}$$

The last equation shows that the union of the new produced sub-regions is greater than their parent region, which means that the set S does not represent an exact partition of $\alpha_i(t)$. The left side of the last relation describes the set that has a weak probability of the global optimum existence. The real new sub-regions are given by a modified sub regions denoted by $\alpha_i'(t+1)$; these sub-regions are a modified form of the selected sub-region $\alpha_i(t+1)$. Where

$$\alpha'_i(t+1) = \beta_i(t+1).\alpha_i(t+1) \tag{4.8}$$

where $\beta_i(t+1)$ represents an error margin coefficient which depends strictly on the parameters of a given sub region. This coefficient allows the algorithm to reallocate the new sub-regions when the expected localization of the global optimum at iteration t do not confirm those given by the stochastic tests at the iteration $t+1$.

4.2. **The potential of a sub region and the decision factor.** The way deciding which $\alpha_i(t)$ containing the global optimum is based on the sub-regions' potential. The potential of a space can be evaluated using interval, and statistical estimation techniques [15] or fuzzy approaches [16]. For the fuzzy approaches, the degree of a membership of any point is measured by the membership function $\mu_{i,k}$ of the function to minimize $f(x_{ij})$, $x_{ij} \in D_i(t)$, and $D_i(t)$ represents a sample set of the sub-region $\alpha'_i(t)$. For the evaluation of $\mu_{i,k}$, several formulations are possible, we can mention the S-membership function, the Gaussian membership function or the linear membership function. In this paper, we have chosen to compute $\mu_{i,k}$, using the modified linear membership function, described in [15]. This member ship function maps sample points functional values $f(x_{ij})$ in $\alpha'_i(t)$ to a unit interval $[0,1]$.

$$\mu_{t,ij} = (f(x_{ij}) - f(x^*))/R(t) \tag{4.9}$$

where $f(x^*)$ is the best (minimum) functional value obtained, and $R(t)$ is the range of all functional values gathered up to and including the iteration t. the member ship function measures the location of $f(x_{ij})$ on the rang scale. Now, we can define the potential of a sub region using (4.10).

$$r_{ij} = \frac{1}{q}\sum_{j=1}^{q} \mu_{t,ij}.\exp(1-\mu_{t,ij}) \tag{4.10}$$

q is the size of the sample set $D_i(t)$.

It is very important to mention here that the circular design forces the sub region, whose central point is located at the nearest position to the global optimum to have the smallest potential value by ensuring a good point distribution around this expected optimum. Thus, we ensure that the probability of missing the global optimum when t the iteration time increases goes to zero.

5. Application to a DC motor control

The open loop uncertain transfer function of the motor can be written as

$$H_\lambda(s) = \frac{1}{Js\,(1+T_{bc}s)} \tag{5.1}$$

The uncertainty is carried at the moment of inertia J "motor + load", the time constant T_{bc} of the current loop or on the two at the same time.

Let's consider the nominal parameter λ_0 which contains the nominal values of J and T_{bc} noted by J_0 and T_{bc0} respectively. Where J is the moment of inertia. We can distinguish three possible cases:

(1) $\lambda_0 = [J]$, the moment of inertia J is uncertain and the constant of time T_{bc} is fixed where: $\Lambda = [J_0] \times [T_{bc0}]$.
(2) $\lambda_0 = [J_0, T_{bc}]$, T_{bc} is uncertain, J is fixed where: $\Lambda = [J_0] \times [T_{bc_{\min}}, T_{bc_{\max}}]$
(3) $\lambda_0 = [J, T_{bc}]$, J and T_{bc} are uncertain where: $\Lambda = [J_{\min},\ J_{\max}] \times [T_{bc_{\min}},\ T_{bc_{\max}}]$

We consider that the moment of inertia, the load and the time constant in the current loop are uncertain.

In this case it is impossible to find a compensator that confers a transfer function to the uncertain system of the form given by 3.9. Therefore, it is necessary to choose the best group of transformation that is well adapted to the chosen application that permits to have some invariant responses.

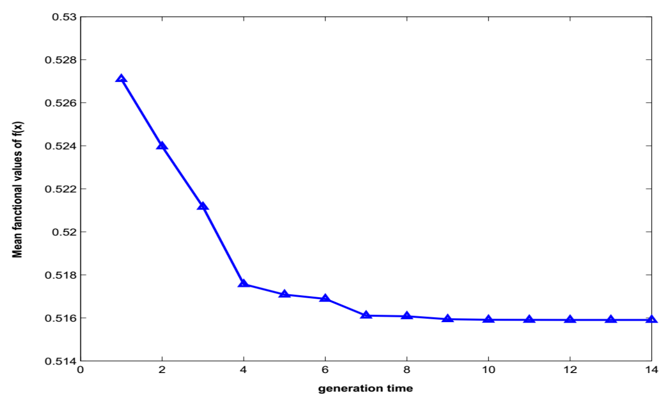

FIGURE 1. Mean functional values of the minimization problem of equation (3.9)

We used the proposed optimization algorithm to find out the optimal fractional corrector. The variations of the mean functional values are shown in figure 1. Table 1 shows the center and the radius of the selected subregions during the optimization operation. Figures 2 and 3 present the step responses of the velocity for different values of T_{bc}, where, *J=0.1* in the case of a classical PI controller and a fractional one. It is very clear, that the fractional controller use permits to obtain pseudo invariant responses around nominal system response.

6. Conclusion

In this paper, optimal fractional control realization via diffusive representation is proposed. The use of the diffusive representation of pseudo differential operators

TABLE 1. The center and the radius of the selected subregions

The Center x_m	$R(t)$	t
-1.3115 -7.3770	2.8013	1
-0.8476 -6.3604	1.8702	2
-0.7711 -5.9313	1.1871	3
-0.6957 -5.5377	0.5019	4
-0.6846 -5.4650	0.4043	5
-0.7045 -5.0592	0.1495	8
-0.6979 -5.0091	0.0632	9
-0.6958 -4.9579	0.0013	13
-0.6956 -4.9571	0.0006	14

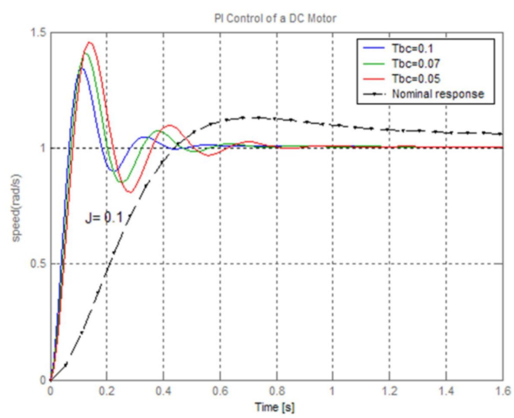

FIGURE 2. Step response with classical PI controller

allow to solve a number of problems involving fractional operators or more generally long memory non oscillating ones by transforming them into input-output well posed differential equations. The optimal solution obtained using the Adaptive partitioning algorithm was able to produce a stable controlled system with iso-damping step responses over the uncertainty domain of system parameters.

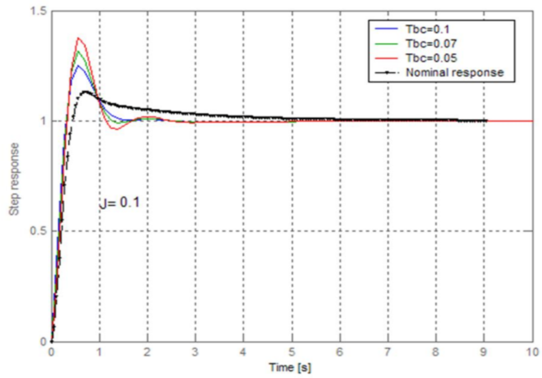

FIGURE 3. Step response with fractional controller

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