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SURFACES WITH CONSTANT MEAN CURVATURE IN EUCLIDEAN SPACE

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Abstract. This is an expository article. It discusses some topics on the theory of constant mean curvature (CMC) surfaces with non-empty boundary. The paper starts with a simple introduction to the mean curvature of a surface giving different physical and mathematical motivations. Next we analyze the mean curvature equation giving the Tangency Principle and the Alexandrov reflection method. The main part of the work focuses in surfaces with non-empty boundary showing how the geometry of the boundary imposes geometrical restrictions to the surface. Finally we discuss the Dirichlet problem associated with the mean curvature equation and some of the techniques employed in this context.

Contents

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1. The mean curvature of a surface

What is the curvature of a curve? What does it mean that a surface is curved? The curvature of a curve is its *acceleration*. For this, we need to move throughout the curve at constant speed. Consider a (regular) curve α in Euclidean space $\mathbb{R}^3$, that is, a differentiable map $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$, $\alpha = \alpha(s)$, such that $\alpha'(s) \neq 0$ for any $s \in I$. We say that α is parametrized by the arc length if $|\alpha'(s)| = 1$, $s \in I$. We point out that any regular curve α can reparametrize by the arc length. So, it suffices to define $s(t) = \int_{t_0}^t |\alpha'(u)|du$. Because $s'(t) \neq 0$, $s : I \to J \subset \mathbb{R}$ is a diffeomorphism, where $J \subset \mathbb{R}$ is an interval. Then taking $\beta(s) = \alpha(\phi(s))$, we have

$$\beta'(s) = \frac{1}{|s'(t)|}\alpha'(\phi(s)) = \frac{1}{|\alpha'(t)|}\alpha'(\phi(s)) \Rightarrow |\beta'(s)| = 1.$$

Definition 1.1. Given a curve $\alpha : I \to \mathbb{R}^3$ parametrized by the arc lenght, the curvature of α at s is defined by

$$\kappa(s) = |\alpha''(s)|.$$

When the curve is defined in Euclidean plane $\mathbb{R}^2$, $\alpha : I \to \mathbb{R}^2$, we can assign a sign on the curvature κ. We say that κ is positive if the curve goes to left and negative if it goes to right. Exactly, because $\langle \alpha'(s), \alpha'(s)\rangle = 1$, then $\langle \alpha''(s), \alpha'(s)\rangle = 0$ and this means that the acceleration $\alpha''(s)$ is orthogonal to the velocity $\alpha'(s)$. Let $\vec{n}(s)$ be the $\pi/2$-rotation of the vector $\alpha'(s)$ in the counterclockwise sense. Then $\alpha''(s)$ is proportional to $\vec{n}(s)$: $\alpha''(s) = \kappa(s)\vec{n}(s)$. We say that $\kappa(s)$ is the signed curvature of α. We point out that in this case, $|\kappa(s)| = |\alpha''(s)|$. Some elementary properties and facts about κ are the following:

(1) If $\beta(s) = \alpha(-s)$, then $\kappa_\beta(s) = -\kappa_\alpha(-s)$.
(2) If α parametrizes a straight-line, $\alpha(s) = sv + p$, $|v| = 1$, then $\kappa \equiv 0$.
(3) If α is a circle of radius $r > 0$, $\alpha(s) = (r\cos(s/r), r\sin(s/r))$, then $\alpha'(s) = (\cos(s/r), \sin(s/r))$ and $\vec{n}(s) = (-\sin(s/r), \cos(s/r))$. Thus $\kappa(s) = 1/r$.
(4) If α is the graph of a function $y = f(x)$ and we write $\alpha(t) = (t, f(t))$, then

$$\kappa(t) = \frac{f''(t)}{(1 + f'(t)^2)^{3/2}}. \tag{1.1}$$

Now, we consider a surface $M \subset \mathbb{R}^3$, or $X = X(u,v) : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ a differentiable map such that $X(D)$ is a surface of $\mathbb{R}^3$. For each $p \in M$ there exists a tangent plane T_pM formed by all velocity vectors at p: $T_pM = \{\alpha'(0); \alpha : I \to M, \alpha(0) = p\}$. Fix $N(p)$ a unit vector orthogonal to $T_p(s)$. Consider all planes P containing $N(p)$, which are transverse to M at p. Then $P \cap M$ is a planar curve containing p, called a normal section. See Figure 1. We take the orientation on the curve such that the normal vector to this curve is $N(p)$. Each plane P is given by a tangent direction $v \in T_pM$. We parametrize $P = P_v$, where $v \in \mathbb{S}^1(p) = \{v \in T_pM; |v| = 1\}$. Denote $\alpha_v = P_v \cap M$ and let

$$\kappa_v(p) = \kappa_{\alpha_v}(0).$$

By the compactness of $\mathbb{S}^1(p)$, there exists $v_1, v_2 \in T_pM$ such that

$$\lambda_1(p) := \kappa_{v_1}(0) = \max\{\kappa_v(0); v \in \mathbb{S}^1(p)\}.$$

$$\lambda_2(p) := \kappa_{v_2}(0) = \min\{\kappa_v(0); v \in \mathbb{S}^1(p)\}.$$

Definition 1.2. The numbers $\lambda_i(p)$ are the principal curvatures of M at p, and v_i are the principal directions.

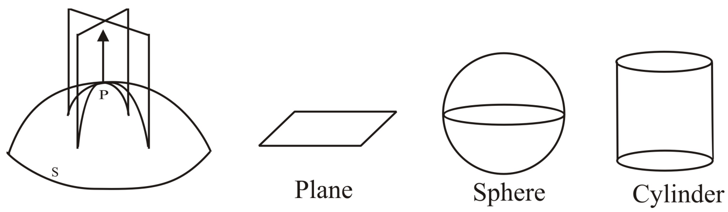


FIGURE 1. Normal sections; a plane, a sphere, a cylinder.

The principal directions at each point are orthogonal. Moreover, if we change N by $-N$, the signs of the principal curvatures change.

We are in position to define the curvature of M at p as a type of "average" of the principal curvatures, for example, geometrical or arithmetic average:

Definition 1.3. The Gauss curvature $K(p)$ and the mean curvature $H(p)$ are defined respectively by

$$K(p) = \lambda_1(p)\lambda_2(p), \qquad H(p) = \frac{\lambda_1(p) + \lambda_2(p)}{2}.$$

All concepts are invariant by rigid motions of space, except perhaps, by a sign. In fact, the change of N by $-N$ implies that H changes of sign but do not K.

Definition 1.4. An orientation (or a Gauss map) on a surface M is a differentiable map $N : M \to \mathbb{R}^3$ such that $|N(p)| = 1$ and $N(p) \perp T_pM$ for each $p \in M$.

Any surface is locally orientable, that is, given a point $p \in M$, there exists a neighborhood V of p at M such that V is an orientable surface: if $X : U \to \mathbb{R}^3$ is a local parametrization of the surface around p, we define

$$N(X(u,v)) = N(u,v) = \frac{X_u \times X_v}{|X_u \times X_v|}(u,v).$$

Here $\times$ is the vectorial product and the subscripts u and v denote the corresponding derivatives. Therefore $X(U) \subset M$ is an open set of M oriented by $N \circ X$. We also point out that closed surfaces (compact with not boundary) are orientable thanks to the existence of a interior domain of the surface: see Figure 2

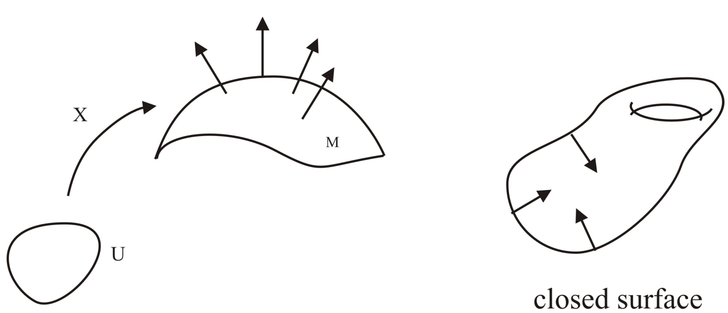


FIGURE 2. Local orientability in a surface; a closed surface is orientable.

We show some calculations of principal curvatures:

(1) M is a plane. Because N is constant, the normal sections are straight-lines. Then $\lambda_i = 0$ and $K = H = 0$ on M.
(2) M is a sphere of radius $r > 0$. Take N pointing inside. The normal sections are maximal circles on the sphere, that is, circles with radius r. Then $\lambda_i = 1/r$ and $K = 1/r^2$ and $H = 1/r$.
(3) We say that p is a umbilical point if $\lambda_1(p) = \lambda_2(p)$. The surface is called umbilical if all its points are umbilical. The only umbilical surfaces are (open of) planes and spheres.
(4) M is a cylinder of radius $r > 0$. Consider N pointing inside. Then $\lambda_1 = 1/r$ and $\lambda_2 = 0$ corresponding to each circle of radius r and any straight-line on the surface, respectively. Thus $K = 0$ and $H = 1/(2r)$.

In all above cases, the principal curvatures are constant, and so, K and H.

We focus our attention on the <u>mean curvature</u>.

Facts:

(1) The inequality $(\lambda_1 - \lambda_2)^2 \geq 0$ writes as $H^2 \geq K$. Moreover p is a umbilical point iff $(H^2 - K)(p) = 0$.
(2) Let $p \in M$ and $v, w \in \mathbb{S}_1(p)$ two tangent vectors such that $\langle v, w\rangle = 0$. Then
$$H(p) = \frac{1}{2}\,(\kappa_v(0) + \kappa_w(0)).$$
As consequence, one can compute the mean curvature H along two any orthogonal directions.
(3) As an application, consider a surface given as a graph $z = f(x, y)$ and we calculate the mean curvature at a point p. After a rigid motion of the space, we assume that $p = (0, 0, 0)$ and that the tangent plane is horizontal. Consider the parametrization $X(x, y) = (x, y, f(x, y))$, with $f(0, 0) = 0$. The tangent plane at p is generated by the vectors $\partial_x X$ and $\partial_y X$, that is, $(1, 0, f_x)$ and $(0, 1, f_y)$. Because the tangent plane is horizontal, $f_x(0, 0) = f_y(0, 0) = 0$. Thus the intersection of M wih each one of the coordinate planes $\{y = 0\}$ and $\{x = 0\}$ determine orthogonal curves at p. These curves are $(x, 0, f(x, 0))$ and $(0, y, f(0, y))$ and the curvatures as curves are given by (1.1):
$$\frac{f_{xx}}{(1+f_x^2)^{3/2}}(0,0) = f_{xx}(0,0), \qquad \frac{f_{yy}}{(1+f_y^2)^{3/2}}(0,0) = f_{yy}(0,0).$$
Finally
$$H(p) = \frac{1}{2}\Big(f_{xx}(0,0) + f_{yy}(0,0)\Big) = \frac{1}{2}\Delta f(0,0).$$
Here Δ is the Laplacian operator on $\mathbb{R}^2$. This equation shows that the theory of surfaces with constant mean curvature are stronger related with the study of elliptic PDE's.

Consider a Gauss map $N : M \to \mathbb{S}^2$ and the differentiable map $dN_p : T_pM \to T_{N(p)}\mathbb{S}^2 \equiv T_pM$. This map is defined by
$$dN_p(v) = \frac{d}{dt}\bigg|_{t=0} N(\alpha(t)),$$

where $\alpha : I \to M$ is curve on S such that $\alpha(0) = p$ and $\alpha'(0) = v$. Then dN_p is an endomorphism, which is self-adjoint, that is,

$$\langle dN_p(u), v\rangle = \langle u, dN_p(v)\rangle, \quad u, v \in T_pM,$$

or equivalently, the bilinear form $\sigma_p : T_pM \times T_pM \to \mathbb{R}$ given by

$$\sigma_p(u, v) = -\langle dN_p(u), v\rangle$$

is *symmetric*. In particular, both dN_p and σ_p are diagonalizable. The proof of this fact is as follows. Take a local parametrization $X = X(s, t)$. A basis of the tangent plane is X_s, X_t. Then we have to show

$$\langle dN_p(X_s), X_t\rangle = \langle X_s, dN_p(X_t)\rangle.$$

On the other hand, N is orthogonal to X_s and X_t. Then

$$\langle N(X(s, t)), X_s\rangle = 0 \Rightarrow \langle N_t, X_s\rangle + \langle N, X_{st}\rangle = 0$$
$$\langle N(X(s, t)), X_t\rangle = 0 \Rightarrow \langle N_s, X_t\rangle + \langle N, X_{ts}\rangle = 0$$

and this concludes the proof.

Definition 1.5. The map $A_p : -dN_p : T_pM \to T_pM$ is the Weingarten map and $\sigma_p : T_pS \times T_pM \to \mathbb{R}$ given by $\sigma_p(u, v) = -\langle dN_p(u), v\rangle$ is the second fundamental form. Moreover, the Weingarten map is diagonalizable.

Theorem 1.1. *The eigenvalues of A_p are the principal curvatures.*

Thus

$$H(p) = -\frac{1}{2} \text{ trace } (dN_p).$$

We rediscover the mean curvature

(1) If M is a plane, then N is constant, $dN_p = 0$ and $H = 0$.
(2) If M is a sphere of radius $r > 0$, then $N(p) = -p/r$. Then $dN_p(u) = -u/r$, $u \in T_pM$ and $H = 1/r$.
(3) If p is an umbilical point, then $-dN_p = \lambda(p)I_p$, where I_p is the identity map on T_pM.

Theorem 1.2. *In local coordinates $X = X(u, v)$, the mean curvature H is given by the formula*

$$H = \frac{1}{2}\ \frac{eG - 2fG + gE}{EG - F^2}, \tag{1.2}$$

where

$$E = \langle X_u, X_u\rangle,\ F = \langle X_u, X_v\rangle,\ G = \langle X_v, X_v\rangle.$$
$$e = \frac{det(X_{uu}, X_u, X_v)}{\sqrt{EG - F^2}},\ f = \frac{det(X_{uv}, X_u, X_v)}{\sqrt{EG - F^2}},\ g = \frac{det(X_{vv}, X_u, X_v)}{\sqrt{EG - F^2}}.$$

Proof. Consider $X = X(u, v)$ a local parametrization of M with the Gauss map $N = (X_u \times X_v)/|X_u \times X_v|$. If $B = \{X_u, X_v\}$ is a basis of the tangent plane of M at $X(u, v)$, we denote the matrix of $-dN_p$ with respect to B as

$$-dN_p \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} := A.$$

On the other hand,

$$\sigma_p(w_1, w_2) = \langle -dN_p(w_1), w_2\rangle = \langle Aw_1, w_2\rangle.$$

Denote the matricial expressions of $\langle,\rangle$ and σ as

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}, \qquad \begin{pmatrix} e & f \\ f & g \end{pmatrix},$$

respectively. For $\langle,\rangle$ is evident; for σ, we only do the proof for e:

$$-\langle dN_p X_u, X_u\rangle = \langle N, x_{uu}\rangle = \frac{\langle X_u \times X_v, X_{uu}\rangle}{|X_u \times X_v|} = \frac{\det(X_u, x_v, X_{uu})}{\sqrt{EG-F^2}} = e.$$

Then

$$w_1^T \begin{pmatrix} e & f \\ f & g \end{pmatrix} w_2 = (Aw_1)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} w_2 \Rightarrow A^T = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}.$$

Because trace$(A) =$ trace$(A^T$, we deduce (1.2). □

We compute the mean curvature H for two spacial surfaces of Euclidean space.

(1) If the surface is the graph of a function $z = f(x,y)$, we take $X(x,y) = (x,y,f(x,y))$ as a parametrization. Then (1.2) gives

$$\begin{aligned} H(x,y,z) &= \frac{1}{2}\,\frac{(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy}}{(1+f_x^2+f_y^2)^{3/2}} \\ &= \frac{1}{2}\left(\frac{f_x}{\sqrt{1+|\nabla f|^2}}\right)_x + \frac{1}{2}\left(\frac{f_y}{\sqrt{1+|\nabla f|^2}}\right)_y \\ &= \frac{1}{2}\,\mathrm{div}\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}. \end{aligned} \tag{1.3}$$

(2) Consider a surface of revolution obtained by rotation with respect to the z-axis the curve $(r(s),0,s)$, $s \in I$. With the parametrization $X(s,\theta) = (r(s)\cos\theta, r(s)\sin\theta, s)$, the expression of H is

$$H = \frac{1 + r'^2 - rr''}{2r(1+r'^2)^{3/2}}. \tag{1.4}$$

2. CMC surfaces

The simplest case of mean curvature function is that H is *constant*. Does have any physical meaning the fact that the mean curvature is constant? Surfaces with constant mean curvature are solutions of a variational problem. We consider a compact surface M with possible non-empty boundary ∂M. Let $x : M \to \mathbb{R}^3$ be a isometric immersion. A *variation* of x is a differentiable map $X : M \times (-\epsilon,\epsilon) \to \mathbb{R}^3$ such that

(1) For each t, $X_t : M \to \mathbb{R}^3$ given by $X_t(p) = X(p,t)$, is an immersion.
(2) $X(p,0) = x(p)$, that is, $X_0 = x$.
(3) $X(p,t) = x(p)$ for any $t \in (-\epsilon,\epsilon)$ and $p \in \partial M$. This means that the variation preserves the boundary.

We define the area and volume functionals $A, V : (-\epsilon,\epsilon) \to \mathbb{R}$, as

$$A(t) = \mathrm{area}(X_t), \qquad V(t) = \mathrm{volume}(X_t),$$

or

$$A(t) = \int_M 1\, dM_t, \qquad V(t) = -\frac{1}{3}\int_M \langle X_t, N_t\rangle\, dM_t.$$

We focus in the variations that preserve the volume, that is, $V(t) = V(0)$ for any t. We ask for those immersions x, such that $A'(0) = 0$.

Theorem 2.1. *An immersion of a compact surface in Euclidean space has constant mean curvature (CMC) iff it is a critical point of the area functional for any preserving volume variation.*

General texts on CMC surfaces, with physical interpretation of these surfaces, are [12, 14, 28]

We show Theorem 2.1 in the case that the surface is a graph. Consider M given as a graph of a function f defined on $D \subset \mathbb{R}^2$. Consider a variation M_t as *graphs* on D where

(1) $g : D \times (-\epsilon, \epsilon) \to \mathbb{R}$, with $M_t = g(M \times \{t\})$ and $g(x, y, 0) = f(x, y)$. This means that at $t = 0$, we have the original graph M.
(2) $g(x, y, t) = f(x, y)$ for any $(x, y) \in \partial D$. With this condition, the variation preserves the boundary of M.

The area of M_t is

$$A(t) = \int_M 1 \, \mathrm{d}M_t = \int_D \sqrt{1 + g_x^2 + g_y^2} \, \mathrm{d}x\mathrm{d}y.$$

We differentiate with respect to t, and let $t = 0$. We obtain

$$A'(0) = \int_D \frac{f_x g_{xt} + f_y g_{yt}}{\sqrt{1 + |\nabla f|^2}}(x, y, 0) \, \mathrm{d}x\mathrm{d}y.$$

Denote $T(f) = \nabla f / \sqrt{1 + |\nabla f|^2}$. We integrate by parts:

$$\begin{aligned} A'(0) &= \int_D \left(\left(\frac{g_t f_x}{\sqrt{1 + |\nabla f|^2}} \right)_x + \left(\frac{g_t f_y}{\sqrt{1 + |\nabla f|^2}} \right)_y \right)(x, y, 0) \, \mathrm{d}x\mathrm{d}y \\ &- \int_D g_t \, \mathrm{div}T(f)(x, y, 0) \, \mathrm{d}x\mathrm{d}y \\ &= \int_{\partial D} g_t \langle T(f), \vec{n} \rangle \mathrm{d}s - \int_D g_t \, \mathrm{div}T(f)(x, y, 0) \, \mathrm{d}x\mathrm{d}y. \end{aligned}$$

In the last identity we have used the divergence theorem, where $\vec{n}$ is the outer unit normal vector to ∂D. The first integral vanishes because $g(x, y, t) = f(x, y)$ for any $(x, y) \in \partial D$, and so, $g_t(x, y, 0) = 0$ on ∂D. Then

$$A'(0) = -\int_D g_t \mathrm{div}T(f)(x, y, 0) \, \mathrm{d}x\mathrm{d}y = -\int_D (2H) g_r(x, y, 0) \mathrm{d}x\mathrm{d}y.$$

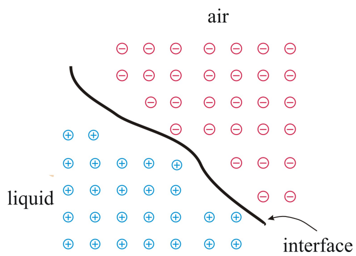
If $H = 0$, then $A'(0) = 0$ for any g. Assume now that $A'(0) = 0$ for any preserving volume variation of M. We consider an appropriate variation g given by:

$$g(x, y, t) = f(x, y) + tHp(x, y)(x, y),$$

where $p(x, y) > 0$ on D and $p = 0$ on ∂D. Then $g_t = Hp$ and so, $\int_M pH^2 dM = 0$ and thus, $H = 0$. As conclusion we have proved that $H = 0$ on the surface if and only if the surface is a critical point of the functional area.

If we assume that the variation preserves the volume, we have the constraint $V(t) = \int_D g \, \mathrm{d}x\mathrm{d}y = $ constant. If $t = 0$ is a critical point of $A(t)$, the theory of Lagrange multipliers implies the existence of a constant $\lambda \in \mathbb{R}$ such that

$$A'(0) + \lambda V'(0) = 0.$$



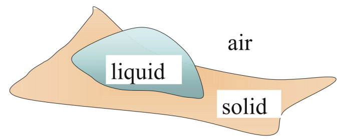


FIGURE 3. A drop resting in a support surface.

Now

$$V'(0) = \int_D g_t(x, y, 0) \; \mathrm{d}x\mathrm{d}y.$$

Then

$$A'(0) = -\int_D g_t(2H + \lambda)(x, y, 0) \; \mathrm{d}x\mathrm{d}y.$$

If this happens for any g, for appropriate variations g, we have $2H + \lambda = 0$, that is, H is constant. On the other hand, if H is constant, and because $V'(0) = 0$, we have

$$A'(0) = -\int_D (2H) g_t(x, y, 0) \; \mathrm{d}x\mathrm{d}y = -2H \int_D g_t(x, y, 0) \; \mathrm{d}x\mathrm{d}y = 0$$

Definition 2.1. A minimal surface is a surface whose mean curvature vanishes on the surface.

Physically, CMC surfaces correspond with the following physical setting. We deposit an amount of liquid on a planar substrate, and assume that there are no chemical and physical reactions between liquid, air and solid. Also, we delete the gravitational forces. We denote by L, A and S the liquid, air and solid phases, and by S_{IJ} the interface between the I and J phases. In mechanical equilibrium, the liquid drop attains its shape when the following equation holds:

$$P_L(p) - P_A(p) = \gamma(2H)(p) \qquad \text{(Laplace)}$$

for each $p \in S_{LA}$. Here P_L and P_A are the pressures in the liquid and air. The constant γ is the surface tension coefficient of the liquid and H is the mean curvature of the interface S_{LA}. The coefficient γ is determined by chemical and physical properties of the liquid and it measures the intermolecular forces that exist in the liquid which are necessary to move the molecules from inside to the S_{LA} interface. If the pressures in both sides of the interface are constant, then the interface is a surface with constant mean curvature.

In our system, the only force acting on the interface is the surface tension. This force is proportional to the area of this interface. Then the energy is proportional to the area of S_{LA}. We remark that the volume of the drop remains constant. If we perturb the drop, the liquid tries to reduce its energy (proportional to the area of S_{LA}) and when this occurs, this interface has constant mean curvature. Thus we can say that the shapes of (small) liquid drops are modeled by CMC surfaces. We suggest the amazing book of de Gennes [7], Nobel Prize in Physics, where shows a number of physical settings where the mean curvature appears in equilibrium shapes.

From the mathematical viewpoint, CMC surfaces can be introduced by the isoperimetric problem: among all compact surfaces in Euclidean space with the same volume, which is the one with smaller area? For minimal surfaces, the analogous problem is the so-called minimizing area: Characterize those surfaces which have least area among all surfaces with the same boundary. In both cases, because the area is a minimum and then, $A'(0)=0$.

We end this section showing the relation between the theory of minimal surfaces and the Complex Analysis: see [29]. Consider $X:D\rightarrow\mathbb{R}^3$ an immersion, $X=X(u,v)$, where (u,v) are isothermal parameters, that is, $E=G, F=0$. We compute the Laplacian of X, $\Delta X=X_{uu}+X_{vv}$. We know that $X_{uu}=aX_u+bX_v+cN$. Now

$$a=\frac{1}{E}\langle X_{uu},X_u\rangle=\frac{E_u}{2E},\qquad b=\frac{1}{G}\langle X_{uu},X_v\rangle=-\frac{E_v}{2G}X_v,\qquad c=e.$$

Thus

$$X_{uu}=\frac{E_u}{2E}X_u-\frac{E_v}{2G}X_v+eN.$$

Similarly,

$$X_{vv}=-\frac{G_u}{2G}X_u+\frac{G_v}{2G}X_v+gN.$$

Using that $E=G$, we conclude

$$\Delta X=X_{uu}+X_{vv}=\Big(\frac{E_u}{2E}-\frac{G_u}{2G}\Big)X_u+\Big(-\frac{E_v}{2G}+\frac{G_v}{2G}\Big)X_v+2HE^2N=2HE^2N.$$

Thus we have proved

Theorem 2.2. *Let $X:D\rightarrow\mathbb{R}^3$ an immersion in isothermal coordinates. Then $H=0$ if and only if X is a harmonic map.*

This result says us the strong relation between minimal surfaces and harmonic maps. We can continue as follows. Let write $X(u,v)=(x_1(u,v),x_2(u,v),x_3(u,v))$ the coordinates functions of a minimal surface and let $z=u+iv$. We define $\phi:D\subset\mathbb{C}\rightarrow\mathbb{C}$, $1\leq i\leq 3$ as

$$\phi_1=(x_1)_u-i(x_1)_v,\qquad \phi_2=(x_2)_u-i(x_2)_v,\qquad \phi_3=(x_3)_u-i(x_3)_v.$$

The fact that the coordinates are *isothermal* writes as

$$\sum_{i=1}^3\phi_i^2=|X_u|^2-|X_v|^2-2i\langle X_u,X_v\rangle=0.$$

By the armonicity of X, the functions ϕ_i satisfy the Cauchy-Riemann relations, that is,

$$\frac{\partial(x_i)_u}{\partial u}=\frac{\partial(-x_i)_v}{\partial v}\Leftrightarrow(x_i)_{uu}+(x_i)_{vv}=0.$$
$$\frac{\partial(x_i)_u}{\partial v}=-\frac{\partial(-x_i)_v}{\partial u}\Leftrightarrow(x_i)_{uv}=(x_i)_{uv}.$$

Thus ϕ_i are holomorphic functions. Finally, because X is an immersion, we obtain

$$\sum_{i=1}^3|\phi_1|^2=\sum_{i=1}^3(x_i)_u^2+\sum_{i=1}^3(x_i)_v^2=|X_u|^2+|X_v|^2=2E.$$

It is possible the converse in the following sense:

Theorem 2.3. *Let $D \subset \mathbb{C}$ be a simply-connected domain. Let ϕ_i be holomorphic functions on D, $1 \leq i \leq 3$. Assume that $\sum_i |\phi_i|^2 \neq 0$ and $\sum_i \phi_i^2 = 0$. Then the map $X : D \to \mathbb{R}^3$, $X = (x_1, x_2, x_3)$ given by*

$$x_i(z) = \Re \int_{z_0}^{z} \phi_i(z) dz$$

defines a minimal immersion in Euclidean space $\mathbb{R}^3$. The integral is calculated along any path on D joining a fix point z_0 with z.

Because D is a simply-connected domain, the above integrals do not depend on the chosen path joining z_0 with z.

3. Some special CMC surfaces

In this section we present examples of surfaces with constant mean curvature whose geometry is particular in some sense.

(1) *Surfaces of revolution.* Rotational surfaces in the Euclidean space with constant mean curvature are known as Delaunay surfaces. From the expression of the mean curvature in (1.4), we have

$$\frac{1}{\sqrt{1+r'^2}} - \frac{rr''}{(1+r'^2)^{3/2}} = 2Hr \Rightarrow \frac{r'}{\sqrt{1+r'^2}} - \frac{rr'r''}{(1+r'^2)^{3/2}} = 2Hrr'.$$

This writes as

$$\frac{d}{ds}\left(Hr^2 - \frac{r}{\sqrt{1+r'^2}}\right) = 0,$$

and a first integral is

$$Hr^2 - \frac{r}{\sqrt{1+r'^2}} = c \tag{3.1}$$

for an integration constant c. The function r cannot be completely integrated because the ordinary differential equation involves elliptic integrals. Delaunay discovered that the geometry of the generating curve $(r(s), 0, s)$ is focus of a conic that rolls on a straight-line, being this line the axis of revolution: see Figure 4. See also [6].

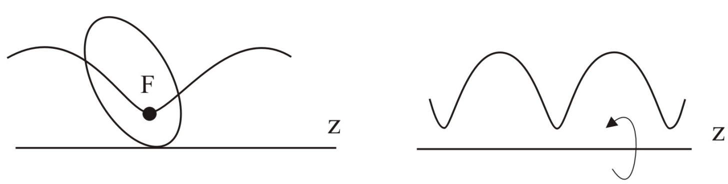


Figure 4. The trace of a focus of a conic.

Besides planes and catenoids ($H = 0$), Delaunay surfaces are unduloids and nodoids, and the limit cases, spheres and cylinders. See Figure 5. Unduloids are embedded (Figure 6) and nodoids are non-embedded.

If $H = 0$, the equation (3.1) can completely integrated: we have

$$\frac{r}{\sqrt{1+r'^2}} = c \Rightarrow \frac{r'}{\sqrt{r^2-c^2}} = \frac{1}{c} \Rightarrow \text{arc}\cosh(\frac{r}{c}) = \frac{s}{c} + \lambda,\ \lambda \in \mathbb{R}.$$

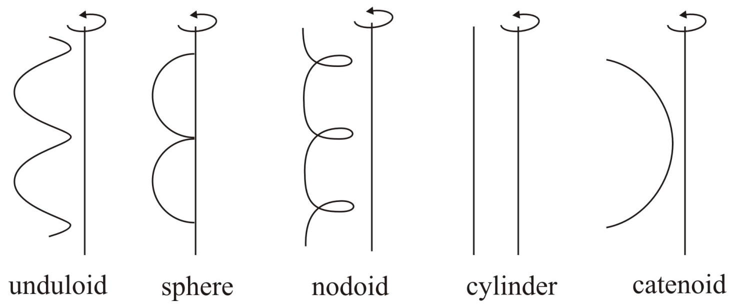


FIGURE 5. Profiles curves of Delaunay surfaces.

Then

$$r(s) = c\cosh(\frac{s}{c} + \lambda).$$

The profile curve is a catenary and the corresponding surface is called a catenoid. See Figure 6.

Theorem 3.1. *Planes and catenoids are the only minimal rotational surfaces.*

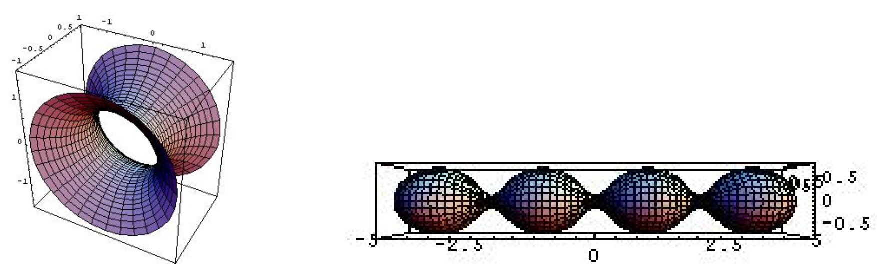


FIGURE 6. A catenoid and an unduloid.

(2) *Ruled surfaces.*

Theorem 3.2. *The only ruled surfaces with constant mean curvature are helicoids ($H = 0$) and right circular cylinders ($H \neq 0$).*

A helicoid is the ruled surface generated by a helix and the rulings are horizontal straight-line at each point of the helix. If we take the helix $\alpha(s) = (r\cos(s), r\sin(s), as)$, $a > 0$, a parametrization of the helicoid is

$$X(s,t) = \alpha(s) + t(\alpha_1(s), \alpha_2(s), 0) = (r(1+t)\cos s, r(1+t)\sin s, as).$$

See Figure 7.

(3) *Translation surfaces.* A translation surface is a surface that is a graph of a function $z = f(x) + g(y)$.

Theorem 3.3. *The only CMC surfaces that are translations surfaces are planes, Scherk's surfaces ($H = 0$) and right circular cylinders ($H \neq 0$).*

Proof. For $H = 0$, the result is classic and it is due to Scherk. When H is a non-zero constant, the theorem is proved in [18]. Here we do the proof for the minimal case. The mean curvature satisfies

$$(1 + g'^2)f'' + (1 + f'^2)g'' = 2H(1 + f'^2 + g'^2)^{3/2}.$$

Then

$$\frac{f''}{1 + f'^2} + \frac{g''}{1 + g'^2} = 2H\frac{(1 + f'^2 + g'^2)^{3/2}}{(1 + f'^2)(1 + g'^2)}.$$

If $H = 0$, then

$$\frac{f''}{1 + f'^2} = -\frac{g''}{1 + g'^2}.$$

Since one side is a function depends on x and the right one depends on y, there exists a constant $c \in \mathbb{R}$ such that

$$\frac{f''}{1 + f'^2} = -\frac{g''}{1 + g'^2} = c.$$

A simple integration gives, up constants,

$$f(x) = -\frac{1}{c}\log\Big(\cos(cx)\Big), \quad g(y) = \frac{1}{c}\log\Big(\cos(cx)\Big),$$

or

$$z(x, y) = \frac{1}{c}\log\left(\frac{\cos(cy)}{\cos(cx)}\right).$$

This surface is called a Scherk's surface and it appears in Figure 7. □

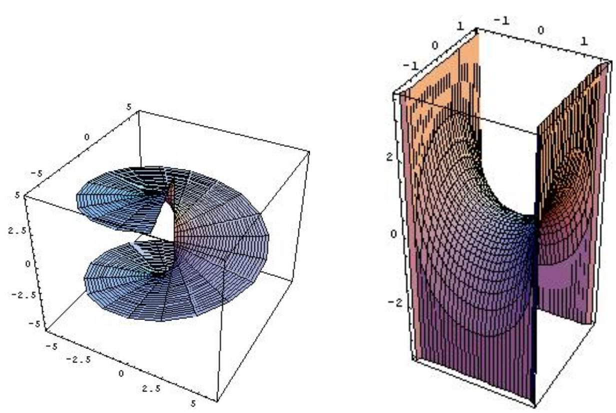


FIGURE 7. A helicoid and a Scherk surface.

4. The constant mean curvature equation

Let M be a surface in $\mathbb{R}^3$ and we locally write as the graph of a smooth function $z = f(x, y)$. Consider N the orientation on M given by

$$N(x, y, f(x, y)) = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}, \tag{4.1}$$

where the subscripts indicate the correspondent derivatives. We know from (1.3) that the mean curvature H of S satisfies the following partial differential equation:

$$(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy} = 2H(1+f_x^2+f_y^2)^{\frac{3}{2}}. \tag{4.2}$$

Theorem 4.1 (Comparison)**.** *Consider two tangent surfaces M_1 and M_2 at some point $p \in M_1 \cap M_2$. We orient both surfaces so $N_1(p) = N_2(p)$. If $M_1 \leq M_2$, then $H_1(p) \leq H_2(p)$.*

Proof. We consider that both surfaces are graphs of functions $z = f_i(x, y)$, whose tangent planes are horizontal and that p is the origin. Then the normal vectors at p agree if we take the usual parametrization of the surface as a graph of a function. We know that $H_i(p) = (f_{i_{xx}} + f_{i_{yy}})(0, 0)$. If $M_1 \leq M_2$ then $f_1(x, y) \leq f_2(x, y)$ around the origin. In particular, the function $f_2 - f_1$ attains a local minimum at $(0, 0)$ and thus, $(f_2 - f_1)_{xx}(0, 0) \geq 0$ and $(f_2 - f_1)_{yy}(0, 0) \geq 0$. This implies $H_2(p) \geq H_1(p)$. □

Corollary 4.1. *Let M be a compact CMC graph on a domain D included in a plane P. Assume that the boundary of M is the boundary curve ∂D. If $H \neq 0$, then M lies in one side of P. If $H = 0$, then $M = D$. Moreover, if the orientation points downward, $H > 0$ (resp. $H < 0$), then M lies over P (resp. below P).*

Proof. Compare the surface with the tangent planes at the lowest and highest points. □

Corollary 4.2. *Any minimal compact surface with boundary lies in the convex hull of its boundary*

Proof. Compare the surface with spheres with sufficiently big radius: see Figure 8. □

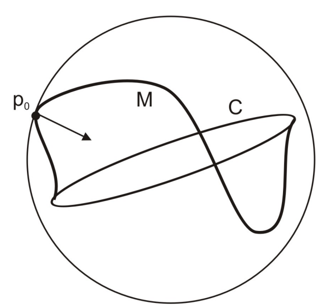


FIGURE 8. Proof of Corollary 4.2.

Corollary 4.3. *Let M_1 and M_2 be two graphs with constant mean curvature H_1 and H_2 with respect to the downward orientations. Assume that $\partial M_1 = \partial M_2$. If $H_1 < H_2$, then $M_1 \leq M_2$.*

Proof. On the contrary, that is, if M_1 has points over M_2, we move down M_1 until the last position. This occurs in some interior point, where both surfaces are tangent. However the orientations agree at that point and M_1 lies over M_2: contradiction. □

The next two results say that CMC surfaces minimize area, at least locally.

Theorem 4.2. *Let M be a minimal compact surface with boundary ∂M. If M is a graph, then M has least area among all graphs with the same boundary than M.*

Proof. Let $z = f(x, y)$ a minimal graph, where f is defined in some domain $D \subset \mathbb{R}^2$, and the boundary of the surface is the curve $C = \{(x, y, f(x, y)); (x, y) \in \partial D\}$. Let g other differentiable function on D such that $f = g$ along ∂D and denote $M' = \text{graph}(g)$. Consider the Gauss map N on M:

$$N(x, y, f(x, y))) = \frac{1}{\sqrt{1 + |\nabla f|^2}}(-\nabla f, 1).$$

From (1.3), we deduce $\text{div}T(f) = 0$. Consider W the enclosed domain by $M \cup M'$ and define on $W \subset \mathbb{R}^3$ the vector field

$$X(x, y, z) = N(x, y, f(x, y)).$$

See Figure 9. We compute the divergence (in $\mathbb{R}^3$) of X, that is,

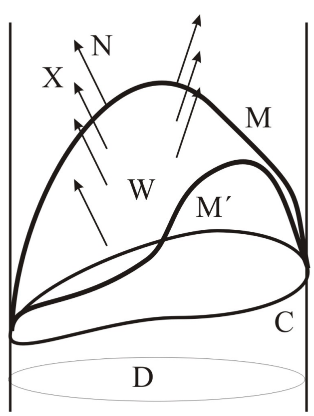


FIGURE 9. The vector field X in the proof of Theorem 4.2.

$$\text{DIV}(X) = \left(\frac{-f_x}{\sqrt{1 + f_x^2 + f_y^2}}\right)_x + \left(\frac{-f_y}{\sqrt{1 + f_x^2 + f_y^2}}\right)_y + \left(\frac{1}{\sqrt{1 + f_x^2 + f_y^2}}\right)_z = 0.$$

The divergence theorem assures that

$$\begin{aligned} 0 &= \int_W \text{DIV}(X) = \int_{\partial W} \langle X, N_{\partial W}\rangle = \int_M \langle X, N_M\rangle + \int_{M'} \langle X, N_{M'}\rangle \\ &= \int_M 1 + \int_{M'} \langle X, N_{M'}\rangle = \text{area}(M) + \int_{M'} \langle X, N_{M'}\rangle. \end{aligned}$$

Then

$$\text{area}(M) = -\int_{M'} \langle N_M, N_{M'}\rangle \leq \int_{M'} 1 = \text{area}(M').$$

□

Theorem 4.3. *Let M be a graph with constant mean curvature H. If M' is other graph on the same domain, with the same boundary and volume, then area$(M) \leq$ area(M').*

Proof. Because the boundaries and volumes agree, then the volume of W is zero. Then

$$\int_W \mathrm{DIV}(X) = \int_W (2H) = 2H \int_W 1 = 0,$$

and the proof is as in the above theorem. ☐

5. The Alexandrov theorem

Round spheres are closed CMC in Euclidean 3-space $\mathbb{R}^3$. A pioneering work in the theory of CMC closed surfaces is the following result due to Hopf.

Theorem 5.1 (Hopf). *Spheres are the only CMC closed surfaces with genus* 0 [11].

The proof uses Complex Analysis and the idea is as follows. Given $X : M \to \mathbb{R}^3$ an immersion, and using isothermal coordinates, we define $\phi : D \subset \mathbb{C} \to \mathbb{C}$ as

$$\phi(z) = \frac{e-g}{2} - if,$$

where $\{e, f, g\}$ are the coefficients of the second fundamental form. Then

$$|\phi| = \frac{E^2}{2}|\lambda_1 - \lambda_2|.$$

This means that the zeroes of ϕ agree with the umbilical points. Moreover,

$$\frac{(e-g)_v}{2} - f_u = -E^2 H_v, \qquad \frac{(e-g)_u}{2} + f_v = E^2 H_u.$$

Then

Theorem 5.2. *An immersion X has constant mean curvature iff ϕ is holomorphic. In particular, umbilical points are isolated, or the immersion is umbilical.*

The proof of Hopf theorem consists to consider a complex structure on M. Since M is a topological sphere, M is conformally equivalente to $\overline{\mathbb{C}}$ with its usual analytic structure. The function ϕ defines a holomorphic function on $\overline{\mathbb{C}}$ which is bounded. Thus $\phi = 0$, that is, X is umbilical, that is, $X(M)$ is a round sphere.

In order to study the closed surfaces with constant mean curvature in Euclidean space, Alexandrov proved in 1956 that any embedded closed CMC surface in $\mathbb{R}^3$ must be a round sphere. An embedded surface in $\mathbb{R}^3$ is a surface without self-intersections. For longtime, it was an open question if spheres were the only closed CMC surfaces in $\mathbb{R}^3$. If a such surface would exist, it would be a surface with self-intersections and higher genus. In 1986, Wente succeeded by constructing an explicit immersed torus with constant mean curvature [34]. This discovery activated a great work in the search of new examples of closed CMC surfaces.

We show the Alexandrov result.

Theorem 5.3 (Alexandrov). *The sphere is the only CMC closed surface that is embedded* [1].

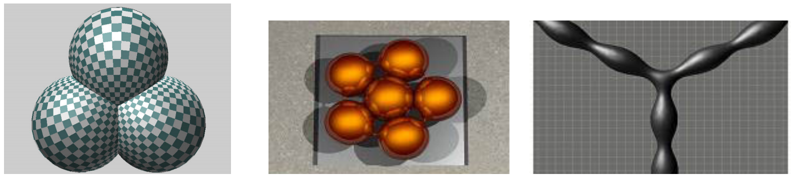


FIGURE 10. (left) Wente torus; (right) CMC surfaces with higher genus and topology.

The proof of this theorem is "geometric" and it is based on the Maximum Principle of linear elliptic equations. Equation (4.2) may written as

$$Q(f) := \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 2H, \tag{5.1}$$

where div y ∇ stand for the divergence and gradient operator respectively. In PDE theory, this equation is an elliptic equation of divergence type. If two functions f_1 and f_2 satisfy (5.1), then

$$Q(f_1) - Q(f_2) = L(f_1 - f_2) = 0,$$

where L is a *linear* elliptic operator. For linear equations, the Maximum principle asserts that the maximum of the function occurs at boundary points. A general reference in elliptic equations is [8]. This geometrically translates for CMC surfaces as follows:

Theorem 5.4 (Tangency principle)**.** *Let M_1 and M_2 two surfaces with the same (constant) mean curvature. Assume that M_1 and M_2 are tangent at some point p and both orientations agree at p. If one surface lies in one side of the other one, then $M_1 = M_2$ agree in an open set around p.*

See Figure 11. The proof of the Alexandrov theorem uses a method of reflection that allows us to compare the surface with itself and next, to apply the Tangency principle. One concludes that the surface has symmetries in each direction: see Figure 12

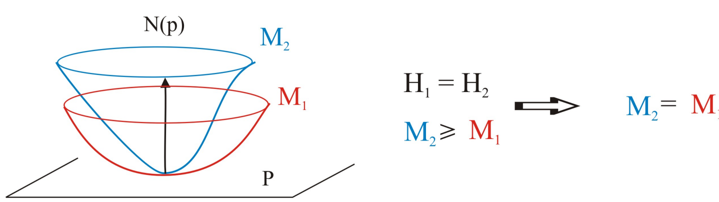


FIGURE 11. The Tangency principle.

6. THE EFFECT OF THE BOUNDARY IN THE SHAPE OF A CMC SURFACE

We now consider compact CMC surfaces with non-empty boundary. The simplest case of boundary is a round circle. If C is a circle of radius $r > 0$, we consider C included in a sphere $S(R)$ of radius R, $R \geq r$. The mean curvature of $S(R)$ is $H = 1/R$ with the inward orientation. Then C splits $S(R)$ in two spherical caps

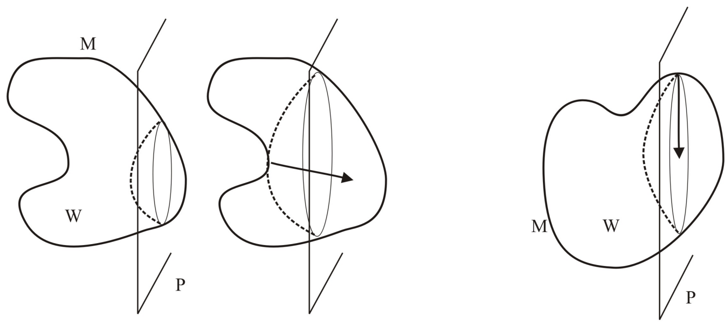


FIGURE 12. The Alexandrov reflection method.

with the same boundary C and constant mean curvature H. If $R = r$, both caps are hemispheres, and if $R > r$, there are two different caps named the small and the big spherical cap. On the other hand, the planar disc bounded by C is a compact surface with constant mean curvature $H = 0$. All these surfaces are the only umbilical examples of compact CMC surfaces bounded by C. See Figure 13.

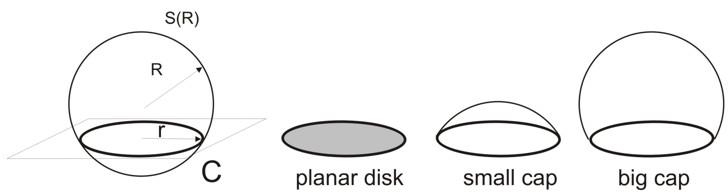


FIGURE 13. Planar disc and spherical caps.

It is natural to ask if planar discs and spherical caps are the only CMC compact surfaces bounded by a circle. If we compare with the closed case and with the Hopf and Alexandrov theorems, the natural hypothesis to consider is that M is a topological disc or that M is embedded. However, as yet it is unknown if the following two conjectures are true:

Conjeture 1. Planar discs and spherical caps are the only compact CMC surfaces bounded by a circle that are *topological discs*.
Conjeture 2. Planar discs and spherical caps are the only compact CMC surfaces bounded by a circle that are *embedded*.

However, we could do the following experiment. We take a drinking straw with circular cross-section and introduce one of its ends into a container with a soap solution. When we extract, it is formed a soap film that coincides with the flat disk whose boundary is the circular rim of the end. Now and carefully, we pump air into the straw from the other end and we see that the flat disk changes into a soap bubble attached to the border. On the end where we are putting air, we place a finger so that the air can not escape from the straw. The soap bubbles we almost always observe are spherical caps. Once that the spherical cap is formed, we can do small displacements of the straw without take off the finger and non destroying the soap bubble in such way that the bubble follows attached to the straw. Then the bubble perturbs and changes of shape, but when the soap bubble attains a new position of equilibrium, the surface formed is the original spherical cap again.

After these considerations, it is natural to ask: what is the shape of a *mathematical* soap bubble with circular boundary?

Surprisedly, and besides the spherical caps, there exist non-spherical mathematical soap bubbles and spanning circular boundaries. These surfaces were obtained in 1991 by Kapouleas founding other examples of CMC surfaces bounded by a circle [13].

This means that our knowledge about the structure of the space of CMC surfaces bounded by a circle is really small and only a few particular situations have been considered. Moreover, the proofs of Hopf and Alexandrov theorems can not carry to our context of non-empty boundary. This fact, together the lack of examples of CMC surfaces bounded by a circular circle, says that although the problems in the non-empty boundary case have the same flavor as in the closed one, the proofs are more difficult.

The theorems of Hopf and Alexandrov, even the Maximum principle, can see as results of uniqueness in the family of CMC surfaces. In the case of CMC compact surfaces with boundary, the simplest result is for graphs, that is, given a value of H and a boundary curve C, there exists a unique graph bounded by C with constant mean curvature H. We begin this section obtaining an *balancing formula* for CMC compact surfaces of $\mathbb{R}^3$. First, we precise the definition of boundary of an immersion. Let $x : M \to \mathbb{R}^3$ be an immersion from a compact surface and let $C \subset \mathbb{R}^3$ a closed curve. We say that C is the boundary of x if $x_{|\partial M} \to C$ is a diffeomorphism.

We have seen that if C is a circle of radius r, the possible values of mean curvatures H for spherical caps bounded by C lies in the range $[-1/r, 1/r]$. Thus, the boundary C imposes restrictions to the possible values of mean curvature. This occurs for a general curved boundary. Consider M a compact CMC surface with boundary $\partial M = C$ and let Y be a variational field in $\mathbb{R}^3$. The first variation formula of the area $|A|$ of the surface M along Y gives

$$\delta_Y |A| = -\int_M 2H\langle N, Y\rangle \, \mathrm{d}M - \int_{\partial M} \langle Y, \nu\rangle \, \mathrm{d}s,$$

where N is an orientation of M, H is the mean curvature according to N, ν represents the inward unit vector along ∂M and $\mathrm{d}s$ is the arc length element of ∂M. Let us fix a vector $a \in \mathbb{R}^3$ and consider Y the vector field of translations in the direction of a. As Y generates isometries of $\mathbb{R}^3$, the first variation of A is 0. Because H is constant, we have

$$2H \int_M \langle N, a\rangle \, \mathrm{d}M + \int_{\partial M} \langle \nu, a\rangle \, \mathrm{d}s = 0. \tag{6.1}$$

The first integral changes into an integral on the boundary as follows. The divergence of the vector field $Z_p = (p \times a) \times N$, $p \in M$, has $\mathrm{div}(Z) = -2\langle N, a\rangle$. Here $\times$ denotes the cross product of $\mathbb{R}^3$. The divergence theorem, together with (6.1), yields

$$\int_{\partial M} \langle \nu, a\rangle \, \mathrm{d}s + H \int_{\partial M} \langle \alpha \times \alpha', a\rangle \, \mathrm{d}s = 0, \tag{6.2}$$

where α is a parametrization of ∂M such that $\alpha' \times \nu = N$. This equality is known as the "flux formula" or "balancing formula". This identity reflects the fact that the area (the potential) is invariant under the group of translations of Euclidean space. On the other hand, the formula can be viewed as the physical equilibrium between the forces of the surface tension of M that act along its boundary with the exterior pressure forces that act on the bounded domain D by C. See Figure 14. If the boundary C lies in the plane $P = \{x \in \mathbb{R}^3; \langle x, a\rangle = 0\}$, for $|a| = 1$, then

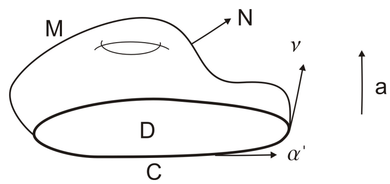


FIGURE 14. Scheme of the flux formula.

$\langle \alpha \times \alpha', a\rangle$ is the support function of the boundary. From (6.2),

$$2H\bar{A} = \int_{\partial M} \langle \nu, a\rangle \, \mathrm{d}s, \tag{6.3}$$

where $\bar{A}$ is the algebraic area of C. Given a closed curve $C \subset \mathbb{R}^3$ that bounds a domain D, and doing $\langle \nu, a\rangle \leq 1$, the value H of the possible mean curvature of M satisfies

$$|H| \leq \frac{\text{length}(C)}{2\ \text{area}(D)}. \tag{6.4}$$

In particular, *if C is a circle of radius $r > 0$, a necessary condition for the existence of a surface spanning C with constant mean curvature H is that $|H| \leq 1/r$.* The inequality 6.4 was proved in [9] for parametric surfaces.

Remark 6.1. It follows from the divergence theorem and from (5.1) that if M is the graph of $z = f(x, y)$ then

$$\begin{aligned} 2|H|\text{area}(D) &= \left|\int_D 2H \ \mathrm{d}D\right| = \left|\int_{\partial D} \langle \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}, \vec{n}\rangle \ \mathrm{d}s\right| \\ &\leq \int_{\partial D} \frac{|\nabla f|}{\sqrt{1+|\nabla f|^2}} \ \mathrm{d}s < \int_{\partial D} 1 \ \mathrm{d}s = \text{length}(C), \end{aligned}$$

where $\vec{n}$ is the unit normal vector to ∂D in P. Then,

$$|H| < \frac{\text{length(C)}}{2\ \text{area(D)}}.$$

Remark 6.2. (1) Assume that M is a compact surface and $x : M \to \mathbb{R}^3$ an immersion with non-necessarily with constant mean curvature. In the proof of balancing formula we have seen

$$2\int_M \langle N, a\rangle \ dM = \int_{\partial M} \langle \alpha \times \alpha', a\rangle \ \mathrm{d}s.$$

If M' is compact surface in $\mathbb{R}^3$ with $\partial M = \partial M'$, then

$$\int_M \langle N, a\rangle \ dM = \int_{M'} \langle N_{M'}, a\rangle \ dM'.$$

Thus

$$2H\int_{M'}\langle N_{M'},a\rangle\ dM'+\int_{\partial M}\langle \nu,a\rangle \mathrm{d}s=0, \tag{6.5}$$

where ν is the inner unit conormal along ∂M.

(2) There is other way to obtain the balancing formula. Let $x: M\to\mathbb{R}^3$ an immersion from a compact surface M. Given $a\in\mathbb{R}^3$, we define the differential 1-form

$$\omega_p(v)=\langle (Hx(p)+N)\times v,a\rangle.$$

Then

$$(d\omega)_p(u,v)=\langle\nabla H_p,u\rangle x\times v-\langle\nabla H_p,v\rangle x\times u,$$

and if $e_1,e_2\in T_pM$ is a positive orthonormal frame on the surface, then

$$(d\omega)_p(e_1,e_2)=x\times[N\times(dx)_p(\nabla H_p)].$$

If H is constant, ω is closed and one follows from Stokes formula that $0=\int_M d\omega=\int_{\partial M}w$. Then we obtain

$$\int_{\partial M}(H\alpha+N)\times\alpha'=0\Rightarrow H\int_{\partial M}\langle\alpha\times\alpha',a\rangle \mathrm{d}s+\int_{\partial M}\langle\nu,a\rangle\mathrm{d}s=0.$$

Theorem 6.1. *The only CMC compact surfaces bounded by a circle making a constant angle with the plane containing the boundary are planar discs and spherical caps* [21].

Proof. After a homothety, we assume the plane P is given by $\{(x,y,0)\in\mathbb{R}^3\}$ and C is a circle of radius 1 with center at the origin. Let M be a compact surface with constant mean curvature H and with boundary C. If $H=0$, M is a minimal surface and the Maximum Principle concludes the surface is the planar disc bounded by C. Next suppose that $H\neq 0$. Since the surface makes constant angle with P along Γ, the function $\langle\nu,a\rangle$ is constant, where ν is the inner conormal along C and a denotes the vector $(0,0,-1)$. We choose an orientation on C such that $\{\alpha',\nu,N\}$ and $\{\alpha,\alpha',a\}$ are positively oriented orthonormal basis, where x is the immersion of M in $\mathbb{R}^3$, and α' is a unit tangent field along C. The boundary ∂M is a line of curvature, because $\langle N,a\rangle$ is constant along ∂M. Then

$$0=\langle N',a\rangle\Rightarrow -\sigma(\alpha',\alpha')\langle\alpha',a\rangle-\sigma(\alpha',\nu)\langle\nu,a\rangle=-\sigma(\alpha',\nu)\langle\nu,a\rangle.$$

Therefore $\sigma(\alpha',\nu)=0$ along ∂M. Because the boundary is a circle and $\{\alpha,\alpha',a\}$ is a positive oriented basis of $\mathbb{R}^3$,

$$\langle N,\alpha\rangle=\langle\alpha'\times\nu,\alpha\rangle=\langle\alpha\times\alpha',\nu\rangle=\langle\nu,a\rangle.$$

Since $1=\langle\alpha,\alpha\rangle=\langle N,\alpha\rangle^2+\langle\nu,a\rangle^2$, and $\langle N,a\rangle$ is constant, the function $\langle\nu,a\rangle$ is constant too. From (6.2),

$$\langle\nu,a\rangle\int_{\partial M}1+H\int_{\partial M}1=0.$$

This implies that $\langle\nu,a\rangle=-H$ along ∂M. We calculate the normal curvature along ∂M.

$$\sigma(\alpha',\alpha')=-\langle N',\alpha'\rangle=-\langle N,\alpha\rangle=H.$$

Thus the boundary points are umbilical. Since the surface has constant mean curvature and the boundary is a set of non-isolated umbilical points, then Theorem 5.2 says that the surface is totally umbilical. Hence, M is a spherical cap. □

Corollary 6.1. *Let C be a circle of radius r. Then the only compact surfaces with constant mean curvature $|H| = 1/r$ bounded by C are the halfsphere of radius r [4].*

Proof. The balancing formula (6.2) gives

$$\int_{\partial M} \langle \nu, a\rangle = -H \int_{\partial M} \langle \alpha \times \alpha', a\rangle = -2\pi H r^2 = -2\pi r.$$

Then

$$2\pi r = \Big| \int_{\partial M} \langle \nu, a\rangle \Big| \leq \int_{\partial M} 1 = 2\pi r.$$

Thus $|\langle \nu, a\rangle| \equiv 1$ along ∂M and so the surface makes constant angle with this plane along the boundary and the above theorem ends the proof. □

7. Some uniqueness results on CMC surfaces

In this section we will obtain a type of control of the shape of a CMC surface with boundary using the Tangency Principle and the flux formula. We begin with CMC surfaces included in right-cylinders. The setting is the following. Let $\Omega \subset \mathbb{R}^2$ be a planar domain included in a plane P. We ask for those CMC surfaces with boundary $\partial\Omega$ and included in $\Omega \times \mathbb{R}$. For example, if the surface is a graph G, we know that G lies in $\Omega \times \mathbb{R}$. Moreover, it lies in one side of P. We have the following

Theorem 7.1. *Let M be an embedded CMC surface bounded by $\partial\Omega$. If $M \subset \Omega \times \mathbb{R}$, then M is a graph on Ω [26].*

Proof. The proof uses the Alexandrov reflection method with horizontal planes. Let $P_t = \{x \in \mathbb{R}^3; \langle x, a\rangle = t\}$, with $P_0 = P$ and $a = (0, 0, 1)$. Let $-m > 0$ sufficient big so that the reflection of M with respect to P lies above the plane P_m. Consider W the interior domain bounded by the closed, non-smooth, surface

$$M \cup (\partial\Omega \times [m, 0]) \cup (\Omega \times \{m\}).$$

We begin with planes for $t >> 0$. After the first intersection point, we go reflecting the part of M above P_t with respect to this plane until the first time that the reflection touches the part of M below P_t. If this point is a tangent point, Maximum Principle assures us that the plane is a plane of symmetry. Since M has boundary, this occurs only when $t = 0$, and then M is closed surface: contradiction. Therefore the Alexandrov method shows that we can arrive until $t = 0$ and not tangent points exist, that is, M is a graph on Ω. □

Consider now CMC surfaces not necessarily embedded.

Theorem 7.2. *Let Ω be a bounded domain in a plane P and let G be a graph on Ω with constant mean curvature H and bounded by $\Gamma = \partial\Omega$. Then, up reflections, G is the only compact surface immersed in Euclidean space with constant mean curvature H bounded by Γ and included in the cylinder $\Omega \times \mathbb{R}$ [26].*

Proof. Let M be a compact CMC surface with the same boundary than G and such that $M \subset \Omega \times \mathbb{R}$. Without loss of generality, assume that H is positive and that G lies above P. Thus the orientation N_G on G points downwards. Denote by N_M the orientation on M. First we prove that G lies above on M unless that $M = G$. On the contrary case, if M has points above G, we move up G until it does not intersect M. This is possible by the compactness of both surfaces. Next we move down G until the first contact point p with M. Then p is not a boundary point and the Maximum Principle gets a contradiction because the mean curvature of G is positive.

Thus we move down G until its original position. If at this moment there exists a tangent (interior or boundary) point, the Tangency Principle says that $M = G$. On the contrary, M lies strictly below G and the slope of M along $\partial\Omega$ is strictly less than the one of G.

We do a similar reasoning with the the reflection G^* of G with respect to P. As consequence, either $M = G$ of $M = G^*$ (and the proof finishes) or M lies in the domain determined by $G \cup G^*$. See Figure 15. Let ν_G, ν_M be the inner conormals of G and M respectively. Balancing formula (6.2) gives in each one of both surfaces:

$$2H\ \text{area}(\Omega) = \int_{\partial\Omega} \langle \nu_G, a\rangle, \qquad 2H\ \text{area}(\Omega) = \left| \int_{\partial\Omega} \langle \nu_M, a\rangle \right|. \tag{7.1}$$

But along Γ we have the strict inequality $|\langle \nu_M, a\rangle| < \langle \nu_G, a\rangle$. An integration along the boundary gives a contradiction with (7.1). □

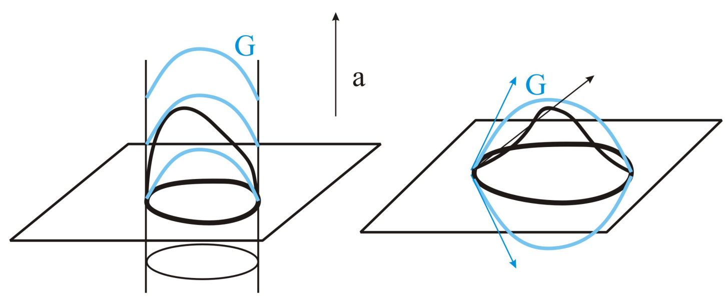


FIGURE 15. Scheme of the proof of Theorem .

It is known that given a domain Ω, there exists H_0 depending only on Ω such that there exist graphs on Ω with constant mean curvature H and boundary $\partial\Omega$ whenever H satisfies $|H| < H_0$. As conclusion, we obtain

Corollary 7.1. *Given a planar bounded domain Ω, there exists $H_0 > 0$ such that if $|H| \leq H_0$, there is a unique compact surface with constant mean curvature H included in $\Omega \times \mathbb{R}$ and with boundary $\partial\Omega$. Moreover this surface is a graph on Ω.*

We are going to obtain other consequence of Theorem 7.2. First, we recall the Plateau problem. Given a fixed Jordan curve $C \subset \mathbb{R}^3$, one asks for the existence of a CMC surface spanning C. For this, one restricts to consider immersions $X : \overline{D} \to \mathbb{R}^3$ from the closed unit disc $\overline{D} \subset \mathbb{R}^2$ such that $X : \partial D \to C$ is a diffeomorphism. Under certain conditions, and using techniques from the Functional Analysis, one can obtain a solution of the problem [32, 33]. The classical result is the following due to Hildebrandt [10]:

> Let C be a Jordan curve included in a ball B_R of radius $R > 0$. If $|H| \leq \frac{1}{R}$, there exists an immersion $X : D \to \mathbb{R}^3$ spanning C and with mean curvature H. Moreover, $X(D) \subset B_R$.

We mention that the Hildebrandt result is the best possible as one can see if C is a circle. As consequence of this result together Theorem 7.2, we have

Theorem 7.3. *Let C be a convex curve and denote Ω de bounded domain bounded by C. Then there exists $H_1 > 0$ depending only on C such that if $|H| < H_1$, any Hildebrandt solution bounded by C and with mean curvature H is a graph.*

Proof. Consider H_0 the positive number such that for any H, $|H| < H_0$, there exists a graph on Ω spanning C. On the other hand, because C is a convex curve, let $R_0 > 0$ such that if $R > R_0$, any circle of radius R containing C in its interior, can roll in such way that the circle touches every point of C. This follows if $R_0 = 1/\min\kappa$, with κ the curvature of C. Let $H_1 = \min\{H_0, \frac{1}{R_0}\}$.

Let $R_1 = 1/H_1$. Then $R_1 \geq R_0$ and so, C is included in the ball B_{R_1}. Consider $|H| < H_1$ and $R = 1/|H|$. Then C is included in B_R because $B_{R_1} \subset B_R$. Let M be the Hildebrandt solution corresponding to (H, B_R). Since the mean curvature of the sphere B_R is $|H|$, we can move B_R in any direction being not possible to have a tangent point with M, unless that the point lies in the boundary of the surface. Moreover, the radius of B_R is $R > 1/H_1 \geq R_0$. Thus we can place B_R such that C lies in the domain determined by an equator of B_R. Then it is possible to roll B_R such that touches each point of C. This shows that M lies in $\Omega \times \mathbb{R}$. Theorem 7.2 proves that M must be a graph. □

We end this section with the next result ([3]):

Theorem 7.4. *Let M be a compact surface with constant mean curvature H and bounded by a circle C. If M lies in a ball of radius $1/|H|$, then M is a spherical cap.*

Proof. By basic geometry, the radius of C is r, with $r \leq 1/|H|$. The mean curvature of the ball B of radius $1/|H|$ is $|H|$ with the unit normal pointing inside. The Tangency Principle shows that if we move B in any direction, it can not exist a tangent point with the surface, unless that M is a spherical cap.

On the contrary, M lies between two small spherical caps (and included en B) with the same mean curvature as M. Now we apply Theorem 7.2 since there are graphs (spherical caps): contradiction. □

8. Embedded CMC surfaces

We consider *embedded* surfaces with constant mean curvature and with non-empty boundary C. Assume that C is a planar curve contained in a plane P. Now we have

Theorem 8.1. *Let M be a CMC embedded surface bounded by a circle. If M lies in one side of P, then M is a spherical cap.*

Proof. We use the same technique of Alexandrov as in Theorem 5.3. We attach to the surface the disc bounded by C, obtaining a domain W. We have to show that M is rotationally symmetric with respect to the line L orthogonal to P containing the center of C. Now we take v a orthogonal direction to L. See Figure 16. We consider the uniparametric family of planes orthogonal to v. Coming from $t = \infty$, we arrive to M and we begin with the reflection process until the first time of contact. If this time is before to arrive to L, then the contact point must be interior, because the surface lies over P. This is impossible by the Maximum Principle. Thus, we arrive until the position that the plane contains L. If there is a contact point, then the plane is a plane of symmetry. On the contrary, we begin from $t = -\infty$. Then

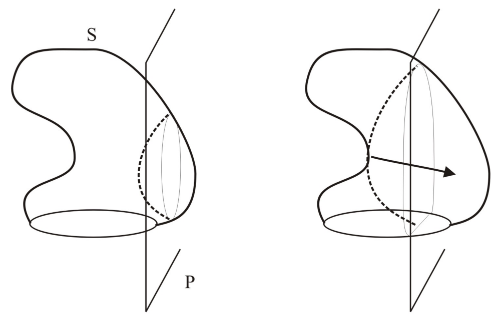


FIGURE 16. The Alexandrov reflection method in Theorem 8.1.

it must be a contact (interior) point, which it is impossible. This shows that the process finishes when the plane contains L and this plane is a plane of symmetry. Because v is an arbitrary vector, we conclude that the surface is rotational. □

Thus we ask for those hypothesis that assure that an embedded CMC surface lies in one side of the boundary plane.

Corollary 8.1. *Let C be a closed curve contained in a plane P. Assume that C is symmetric with respect to a straight-line $L \subset P$ and that each piece of C that L divides is a graph on L. If M is an embedded CMC surface with boundary C and M lies in one side of P, then M is symmetric with respect to the plane Π orthogonal to P with $P \cap \Pi = L$.*

One can extend Theorem 8.1 by considering that the boundary is formed by two coaxial circles.

Theorem 8.2. *Consider $C_1 \cup C_2$ two coaxial circles in parallel planes P_i. Assume that M is an embedded CMC surface bounded by $C_1 \cup C_2$. If M lies in the slab determined by P_1 and P_2, then M is a surface of revolution.*

On the other hand, there exist pieces of nodoids spanning two coaxial circles, but the surface is not included in the slab determined by the boundary. See Figure 17

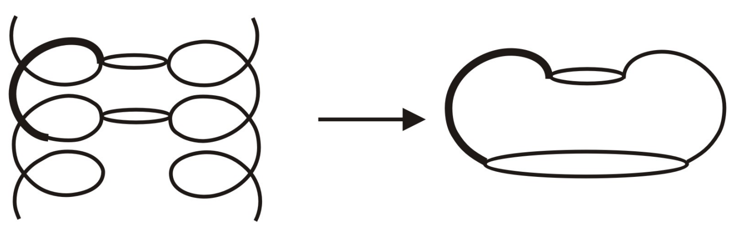


FIGURE 17. Pieces of nodoids not included in the slab determined by the boundary.

We present two results that inform us how the geometry of the boundary has an effect on the whole surface. We give a quick proof of them without going into details.

Theorem 8.3. *Let C be a closed curve included in a plane P and denote D the domain bounded by C. Assume that M is a CMC embedded surface bounded by C. If $M \cap ext(D) = \emptyset$, then M lies in one side of P* [15].

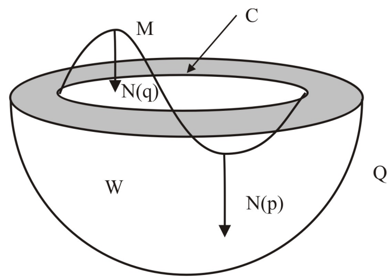


FIGURE 18. The contradiction in Theorem 8.3

Proof. We consider a sufficiently big hemisphere Q such that M together Q and an annulus of P defines a closed embedded surface. Then M, Q and the annulus determine an interior domain, called $W \subset \mathbb{R}^3$. If M has points in both sides of P, we will arrive to a contradiction: see Figure 18. We orient M so that the Gauss map points towards W. In the highest point of M, the normal vector points down. The Maximum Principle asserts that $H > 0$. On the other hand, in the lowest point, the normal points down again (towards W). But the Maximum Principle comparing M with a horizontal plane at p yields a contradiction. □

Theorem 8.4. *Let C be a convex curve included in a plane P. Let M be a CMC embedded surface spanning C. If M is transverse to P along C, then M lies in one side of P* [5].

Proof. We only present two cases that illustrate the proof of Theorem 8.4. In Figure 19 (left) the surface meets the plane P in a nullhomotopic closed curve G in $P \setminus D$. We use the Alexandrov reflection method by vertical planes arriving from infinity. Since C is convex, we would have an interior contact point, proving that M has a symmetry by a vertical plane that does not intersect C, which it is impossible.

The second case that we analyze appears in Fig. 19 (right). Again, this surface is impossible. In this situation, we use the balancing formula as follows. The surface M together with D encloses a domain W and we orient M by N pointing towards W. Here the mean curvature H is positive. Set $a = (0, 0, 1)$. In the expression (6.2), $\langle \nu, a \rangle$ is positive, since the surface is transverse to P along ∂M. Since $\alpha' = \nu \times N$, the term $\langle \alpha \times \alpha', a \rangle$ is also positive, which it is a contradiction in (6.2).

□

It is possible to add more information in Theorem 8.4.

Theorem 8.5. *Let C be a Jordan curve included in a plane P and let D be the domain that bounds in P. Let M be an embedded CMC surface spanning C. If M is a graph over D around the boundary C, then the surface is a graph* [20].

Proof. We use the Alexandrov reflection method but with horizontal planes. See Figure 20. If the surface is not a graph on D, there would be a horizontal plane Q

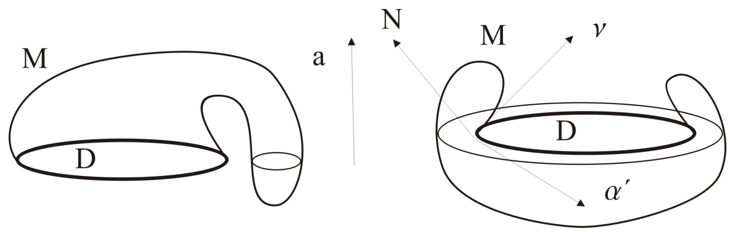


FIGURE 19. CMC surfaces that do no exist by Theorem 8.4

such that the reflection of the part of M over Q would contact with the below part. Moreover, since M is a graph around C, this contact point must be an interior point. Now the Maximum Principle implies that Q is a plane of symmetry of the surface. Since ∂M lies below Q, we get a contradiction.

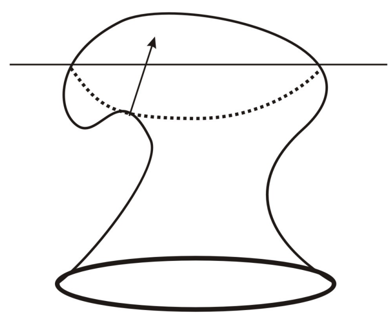


FIGURE 20. CMC surface that does no exist by Theorem 8.5

□

9. CMC surfaces with circular boundary

Let M be a *topological disc* and let $x : M \to \mathbb{R}^3$ an immersion with constant mean curvature and bounded by a circle C (for instance of radius 1). A result due to Barbosa and do Carmo assures that the area of A satisfies $A \leq A_-$ or $A \geq A_+$, where A_- and A_+ denote the area of the small and big spherical cap with the same mean curvature and boundary than M, respectively. An explicit computation of A_- and A_+ gives

$$A_- = \frac{2\pi}{H^2}(1 - \sqrt{1-H^2}), \qquad A_+ = \frac{2\pi}{H^2}(1 + \sqrt{1-H^2}).$$

Moreover the equality holds if and only if M is a spherical cap.

Theorem 9.1. *If the $A \leq A_-$, then M is a (small) spherical cap* [25].

Proof. Using the Gauss-Bonnet theorem and that M is a topological disc, we have

$$2\pi = \int_M K\, dM + \int_C \kappa_g \leq \int_M H^2\, dM + \int_C \kappa_g = AH^2 + \int_C \kappa_g, \tag{9.1}$$

where κ_g is the geodesic curvature along C. We have used that $K \leq H^2$, and equality holds iff M is umbilical, that is, M is a spherical cap. We write the flux formula (6.2) with $a = (0, 0, -1)$:

$$-2\pi H = \int_C \langle \nu, a \rangle. \tag{9.2}$$

This equality together the Cauchy-Schwarz inequality imply

$$\int_C \langle \nu, a \rangle^2 \, ds \geq \frac{1}{2\pi} \left(\int_C \langle \nu, a \rangle \, ds \right)^2 = 2\pi H^2. \tag{9.3}$$

We compute the geodesic curvature. Since C is a circle of radius 1, $\kappa_n^2 + \kappa_g^2 = 1$, where κ_n is the normal curvature. In our case,

$$\kappa_n = -\langle N', \alpha' \rangle = \langle N, \alpha'' \rangle = -\langle N, \alpha \rangle.$$

Thus

$$|\kappa_g| = \sqrt{1 - \langle N, \alpha \rangle^2} = |\langle N, a \rangle|.$$

Using the hypothesis on the area, and the value of κ_g, we have from (9.1)

$$2\pi \leq 2\pi(1 - \sqrt{1 - H^2}) + \int_C |\langle N, a \rangle|.$$

Squaring,

$$\begin{aligned} 4\pi^2(1 - H^2) &\leq \left(\int_C \langle N, a \rangle \right)^2 \leq 2\pi \int_C \langle N, a \rangle^2 \\ &= 2\pi \left(\int_C 1 - \langle \nu, a \rangle^2 \right) = 4\pi^2 - 2\pi \int_C \langle \nu, a \rangle^2. \end{aligned}$$

This implies $\int_C \langle \nu, a \rangle^2 \leq 2\pi H^2$ and together (9.3) we obtain that the surface is umbilical. □

We report here recent results that characterize spherical caps in the family of CMC surfaces with circular boundary.

Theorem 9.2. *Let C be a circle of radius R and let M be a compact surface spanning C and with constant mean curvature H. Then M is a spherical cap if one of the following conditions holds:*

(1) *The surface is embedded and lies over the plane containing C.*
(2) *The surface is embedded and does not intersect the exterior domain of the circle in the boundary plane.*
(3) *The mean curvature satisfies $|H| = 1/R$.*
(4) *The surface is included in a closed ball of radius $1/|H|$.*
(5) *The surface is embedded and transverse to the boundary plane along the boundary.*
(6) *The surface is a minimizer surface* [16].
(7) *The surface is stable with free boundary supported in a plane* [17].
(8) *The surface is a topological disc with area less than the area of the small spherical cap with mean curvature H and boundary C.*
(9) *The volume is less than the volume of a hemisphere with the same mean curvature* [26].
(10) *The surface is embedded and included in a slab of width $1/|H|$* [19].

(11) *The surface is a stable topological disc* [2].
(12) *The surface is a topological disc that makes a constant contact angle along the boundary.*

10. Two equations for CMC surfaces

Let $x : M \to \mathbb{R}^3$ be an immersion of an orientable surface in Euclidean space $\mathbb{R}^3$, let $\mathcal{X}(M)$ be the set of vector fields on M and by N the Gauss map of the immersion. Recall the fundamental equations for x:

$$\nabla^0_X Y = \nabla_X Y + \sigma(X,Y)N, \quad X,Y \in \mathfrak{X}(M). \tag{10.1}$$

$$\nabla^0_X N = -A_N X = -AX, \quad \sigma(X,Y) = \langle AX, Y\rangle. \tag{10.2}$$

Here ∇^0 is the usual connection on $\mathbb{R}^3$, $\sigma : T_pM \times T_pM \to \mathbb{R}$ is the second fundamental form, $A : T_pM \to T_pM$, $A = -(dN)$, is the Weingarten map.

Given a (smooth) function f on M, we define the gradient of f, ∇f, as the vector field given by

$$\langle (\nabla f)_p, v\rangle = (df)_p(v), \qquad \forall v \in T_pM.$$

If $\{e_1, e_2\}$ is an orthonormal basis on T_pM, then $(\nabla f)_p = \sum_{i=1}^2 (df)_p(e_i)e_i$.

If X is a vector field on M, the divergence of X is the function $\mathrm{div}(X)$ defined as

$$\mathrm{div}(X)(p) = \mathrm{trace}\Big(v \longmapsto \nabla_v X\Big),$$

where ∇ is the Levi-Civitta connection on M. Finally, the Laplacian of f is defined by

$$\Delta f = \mathrm{div}(\nabla f).$$

If we fix $\in M$, there exists an orthonormal basis on T_pM, $\{e_1, e_2\}$ such that $(\nabla_{e_i} e_j)_p = 0$. Then $(\nabla_{e_i} e_j)_p$ is orthogonal to M at p and then

$$\begin{aligned} \nabla f(p) &= \sum \langle e_i, \nabla_{e_i}\nabla f\rangle = \sum e_i\langle e_i, \nabla f\rangle - \sum \langle \nabla^0_{e_i} e_i, \nabla f\rangle \\ &= \sum e_i\langle e_i, \nabla f\rangle = \sum e_i(e_i(f)). \end{aligned} \tag{10.3}$$

In this section, we compute the Laplacian of two functions defined in a CMC surface and we will obtain geometric consequences. For this, we use the Maximum Principle to the Laplacian operator: if M is a compact surface with $\partial M \neq \emptyset$, and if $\Delta f \geq 0$, then $f \leq \max_{\partial M} f$.

The next result holds for any immersion independently if H is constant.

Theorem 10.1. *Let $x : M \to \mathbb{R}^3$ an immersion of an orientable surface M. If $a \in \mathbb{R}^3$, then*

$$\Delta\langle x, a\rangle = 2H\langle N, a\rangle. \tag{10.4}$$

Proof. Fix $p \in M$ and take $\{e_1, e_2\} \subset T_pM$ such that $(\nabla_{e_i} e_j)_p = 0$. Using (10.1) and (10.3)

$$\begin{aligned} \Delta\langle x, a\rangle &= \sum_i e_i e_i \langle x, a\rangle = \sum_i e_i\langle e_i, a\rangle \\ &= \sum_i \langle \nabla^0_{e_i} e_i, a\rangle = \sum_i \sigma(e_i, e_i)\langle N, a\rangle = 2H\langle N, a\rangle. \end{aligned}$$

□

Theorem 10.2. *Let $x : M \to \mathbb{R}^3$ an immersion with constant mean curvature. Then*

$$\Delta\langle N, a\rangle + |\sigma|^2\langle N, a\rangle = 0, \qquad |\sigma|^2 = 4H^2 - 2K. \tag{10.5}$$

Proof. Using (10.1) and (10.2) and because $2H = \mathrm{trace}(A)$, we know that

$$2H = \sum\langle Ae_i, e_i\rangle = \sum \sigma(e_i, e_i) = \langle\nabla^0_{e_i} N, e_i\rangle$$

is constant. Thus for each $j \in \{1, 2\}$,

$$\begin{aligned} 0 &= e_j\langle\sum_i \nabla^0_{e_i} N, e_i\rangle = -\sum_i\langle\nabla^0_{e_j}\nabla^0_{e_i} N, e_i\rangle + \sum_i\langle\nabla^0_{e_i} N, \nabla^0_{e_j} e_i\rangle \\ &= \sum_i\langle\nabla^0_{e_i}\nabla^0_{e_j} N, e_i\rangle = \sum_i e_i\langle\nabla^0_{e_j} N, e_i\rangle. \end{aligned} \tag{10.6}$$

Now we compute the Laplacian of $\langle N, a\rangle$.

$$\begin{aligned} \Delta\langle N, a\rangle &= \sum e_i\langle\nabla^0_{e_i} N, a\rangle) = \sum_{i,j} e_i\Big(\langle\nabla^0_{e_i} N, e_j\rangle\langle a, e_j\rangle\Big) \stackrel{*}{=} \sum_{i,j} e_i\Big(\langle\nabla^0_{e_j} N, e_i\rangle\langle a, e_j\rangle\Big) \\ &= \sum_j\sum_i e_i\Big(\langle\nabla^0_{e_j} N, e_i\rangle\Big)\langle a, e_j\rangle + \sum_j\Big(\langle\nabla^0_{e_j} N, e_i\rangle\sum_i e_i\langle a, e_j\rangle\Big) \\ &\stackrel{**}{=} \sum_j\Big(\langle\nabla^0_{e_j} N, e_i\rangle\sum_i e_i\langle a, e_j\rangle\Big) \\ &= \sum_j\Big(\langle\nabla^0_{e_j} N, e_i\rangle\sum_i\langle a, \nabla^0_{e_i} e_j\rangle\Big) = -\sum_{i,j}\sigma(e_i, e_j)^2\langle N, a\rangle \\ &= -|\sigma|^2\langle N, a\rangle. \end{aligned}$$

In (*) we have used that $Av = -\nabla^0_v N$ is a self-adjoint endomorphism: $\langle\nabla^0_{e_i} N, e_j\rangle = \langle\nabla^0_{e_j} N, e_i\rangle$; in (**) we use (10.6). $\square$

The next result appears in [27] and generalizes one given in [30]. More generalizations, see [24].

Theorem 10.3. *Let M be a graph with constant mean curvature H. Assume that the boundary of M lies in a plane P. Then the height of M with respect to P is less than $1/|H|$. In the general case that the boundary is not planar, then $|\langle x, a\rangle| \leq \max_{\partial M}|\langle x, a\rangle| + 1/|H|$.*

Proof. Because M is a graph, Corollary 4.1 says that M lies in one side of P. Assume that M lies over P. We choose the orientation such that $\langle N, a\rangle \leq 0$, with $a = (0, 0, 1)$. This means that $H > 0$ (by Corollary 4.1 again). Using (10.4)-(10.5) and the fact that $|\sigma|^2 = 4H^2 - 2K \geq 2H^2$, we obtain

$$\Delta(H\langle x, a\rangle + \langle N, a\rangle) = (2H^2 - |\sigma|^2)\langle N, a\rangle \geq 0.$$

We are in position to apply the Maximum Principle. Since $\langle x, a\rangle = 0$ on ∂M, we have

$$\begin{aligned} H\langle x, a\rangle + \langle N, a\rangle &\leq \max_{\partial M}(H\langle x, a\rangle + \langle N, a\rangle) \\ &= \max_{\partial M}(\langle N, a\rangle \leq 0 \end{aligned}$$

Thus

$$\langle x, a\rangle \leq \frac{-\langle N, a\rangle}{H} \leq \frac{1}{H}.$$

□

Corollary 10.1. *Let M be a CMC embedded surface with boundary ∂M included in a plane P. Then the height of M with respect to P is $2/|H|$.*

Proof. We use the Alexandrov method with respect to horizontal planes. Because there is not a contact interior point, in the reflection process we descend until a height h_o with $h_o \leq h/2$, being h the height of M. Moreover, the part of the surface over the height h_o is a graph and its height $h - h_o$ is less than $1/|H|$. Thus $h/2 < 1/|H|$. □

Using the same idea as in Theorem 9.1, we have the next result on uniqueness of graphs without the use of the Maximum Principle.

Theorem 10.4. *Let M be a compact CMC surface bounded by a round circle C. If M is a graph, then M is a planar disc or a spherical cap.*

Proof. We assume that the radius of the circle is 1 and we use the same notation as in Theorem 9.1. We apply the divergence theorem in Equation (10.5) and we get

$$\int_M |\sigma|^2 \langle N, a\rangle \ \mathrm{d}M = \int_C \langle dN\nu, a\rangle \ \mathrm{d}s. \tag{10.7}$$

We study each side of (10.7) and we begin with the left-hand side. We know that

$$2H\int_M \langle N, a\rangle = -\int_C \langle \nu, a\rangle = 2\pi H \Rightarrow \int_M \langle N, a\rangle = \pi.$$

Thus $\langle N, a\rangle$ is positive, since M is a graph. As $K \leq H^2$, then $K\langle N, a\rangle \leq H^2\langle N, a\rangle$. Since $|\sigma|^2 = 4H^2 - 2K$, we have

$$\begin{aligned} \int_M |\sigma|^2 \langle N, a\rangle \ \mathrm{d}M &= 4H^2\int_M \langle N, a\rangle \ \mathrm{d}M - 2\int_M K\langle N, a\rangle \ \mathrm{d}M \\ &\geq 2H^2\int_M \langle N, a\rangle \ \mathrm{d}M = 2\pi H^2. \end{aligned} \tag{10.8}$$

Let the right-hand side of (10.7). First,

$$dN\nu = -\sigma(\alpha', \nu)\alpha' - \sigma(\nu, \nu)\nu.$$

Because $\langle N, \alpha\rangle = \langle \nu, a\rangle$ and the radius of C is 1 ($\alpha'' = -\alpha$), we have

$$\begin{aligned} \sigma(\nu, \nu) &= 2H - \sigma(\alpha', \alpha') = 2H + \langle dN\alpha', \alpha'\rangle \\ &= 2H - \langle N, \alpha''\rangle = 2H + \langle N, \alpha\rangle = 2H + \langle \nu, a\rangle. \end{aligned} \tag{10.9}$$

Because $\langle \alpha', a\rangle = 0$, and using (9.2) and (10.9), we have

$$\begin{aligned} \int_C \langle dN\nu, a\rangle \ \mathrm{d}s &= -\int_C \sigma(\nu, \nu)\langle \nu, a\rangle \ \mathrm{d}s = -\int_C \Big(2H + \langle \nu, a\rangle\Big)\langle \nu, a\rangle \ \mathrm{d}s \\ &= 4\pi H^2 - \int_C \langle \nu, a\rangle^2 \ \mathrm{d}s. \end{aligned} \tag{10.10}$$

Then (9.3) and (10.10) imply

$$\int_C \langle dN\nu, a\rangle \ \mathrm{d}s \leq 2\pi H^2. \tag{10.11}$$

Finally, from (10.8) and (10.11), we obtain equalities in (10.7) and so $|\sigma|^2 = 2H^2$ on M. Then M is an umbilical surface. □

11. The Dirichlet problem of the CMC equation

We look for CMC graphs with constant mean curvature. If $z = u(x, y)$ is a graph M on $\Omega \subset \mathbb{R}^2$ with constant mean curvature H, then u satisfies

$$\text{div }(Tu) = 2H \quad \text{in } \Omega, \qquad Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \tag{11.1}$$

$$u = \phi \quad \text{along } \partial\Omega. \tag{11.2}$$

The orientation N assumed on the graph points upwards, that is, $N = (-\nabla f, 1)/\sqrt{1+|\nabla f|^2}$. If $\phi = 0$, then the boundary of the surface is the curve $\partial\Omega$.

The general technique employed in the solvability of the Dirichlet problem is the method of continuity. We briefly explain this technique with some details (see [8]). Consider H a fixed real number. For each $t \in [0, 1]$, we pose the family of Dirichlet problems

$$(P_t): \qquad \begin{cases} \text{div }(Tu_t) = -2tH & \text{in } \Omega \\ u = \phi & \text{along } \partial\Omega. \end{cases}$$

The solutions of (P_t) are graphs on Ω with constant mean curvature tH and all these graphs have the same boundary. Define the set

$$J = \{t \in [0, 1]; \text{there exists a solution } u_t \text{ of } (P_t)\}.$$

In this setting, a solution of (11.1)-(11.2) exists provided one shows that $1 \in J$. For this purpose, we shall prove that J is a non-empty, open and closed subset of $[0, 1]$, and hence, $J = [0, 1]$. Let us prove first that $0 \in J$, that is, there exists a minimal surface with the same boundary value ϕ on $\partial\Omega$. In general, this solution is obtained from the theory of minimal surfaces and, at first, this difficulty is not easily overcome. In our case if $\phi = 0$, then $u = 0$ is an immediate solution. In a second step, we prove that J is open in $[0, 1]$. Consider $\tau \in J$ and we will see that the Dirichlet problem (P_t) can be solved for each t in a certain interval around τ. Denote by Σ_t the graph corresponding to u_t and define a map $h : C_0^{2,\alpha}(\Sigma_\tau) \to C_0^{\alpha}(\Sigma_\tau)$ taking each v onto the mean curvature function of the normal graph on Σ_τ corresponding to the function v:

$$h(v) = \text{mean curvature of } (p \longmapsto x(p) + v(p)N(p)),$$

where $x : \Sigma_\tau \to \mathbb{R}^3$ is the inclusion map. The linearisation of h is the Jacobi operator of Σ_τ, namely,

$$L(v) = \Delta v + |\sigma|^2 v,$$

where Δ is the Laplace-Beltrami operator in Σ_τ and σ is its second fundamental form. Here L is a self-adjoint linear elliptic operator with trivial kernel, since

$$L\langle N, a\rangle = 0 \quad \text{and} \quad \langle N, a\rangle < 0, \tag{11.3}$$

where N is the orientation on Σ_t and $a = (0, 0, 1)$. Hence, and using the implicit function theorem for Banach spaces, h is locally invertible. This shows that there exists a solution u_t of (P_t) for values of t around τ.

Finally, it remains to be proved that J is closed in $[0, 1]$. The Schauder theory reduces the question to establish a priori C^0 and C^1 estimates of each solution u_t of (P_t) independent of t, that is, it suffices to prove that there exists a constant M independent of t such that

$$\sup_\Omega |u_t| \le M, \quad \sup_\Omega |\nabla u_t| \le M.$$

The value of $|u_t|$, that is, the height of Σ_t is controlled by a universal constant. Exactly, the height of Σ_t is less than the one of Σ because $tH < H$: see Corollary 4.3. Finally, the height of Σ is given by Theorem 10.3. Then (putting $\phi = 0$) $u_0 < u_t < u_1 \leq 1/|H|$.

Now we seek a priori estimates for $|\nabla u_t|$. By the expression of N in terms of ∇u_t, we know

$$\langle N, a\rangle = \frac{1}{\sqrt{1+|\nabla u_t|^2}}. \tag{11.4}$$

But Equation (11.3) tells us that $\Delta\langle N, a\rangle \leq 0$ and so, the minimum of $\langle N, a\rangle$ is attained at a boundary point of $\partial\Omega$. By combining with (11.4), we conclude

$$\sup_{\Omega} |\nabla u_t| = \sup_{\partial\Omega} |\nabla u_t|.$$

At this moment, and for each particular case of domain Ω, we shall need suitable surfaces as barriers to compare the slope of the graph of u_t along its boundary.

Theorem 11.1. *The Dirichlet problem (11.1)-(11.2) for $\phi = 0$ has a solution if one can establish C^1-a priori estimates of a solution along the boundary $\partial\Omega$.*

When Ω is a bounded convex domain, the classical result of the existence for Equation (11.1) is due to Serrin [31]:

> *If the curvature κ of $\partial\Omega$ with respect to the inner orientation satisfies $0 < 2|H| < \kappa$, then for an arbitrary smooth function ϕ on $\partial\Omega$, there exists a unique solution of (11.1) with $u = \phi$ along $\partial\Omega$.*

However, if $\phi = 0$ on $\partial\Omega$, one expects that the range of possible H is bigger, as it happens when Ω is a round disc: in this case, if the radius is 1, $\kappa = 1$, and there exist graphs for $|H| < 1$.

Theorem 11.2. *Let Ω be a bounded convex domain. If one of the following assumptions holds, then there is a solution of (11.1) for $u = 0$ along $\partial\Omega$:*

(1) $0 < |H| < \kappa$, *where κ is the curvature of $\partial\Omega$* [21].
(2) Ω *is included in a strip of width* $1/|H|$ [22].

In this theorem, pieces of spheres and cylinders are used as barriers since they fit well with the convexity of the domain Ω.

We explicit the proof for the case (2) in Theorem 11.2. First, we claim: there exists a number $\delta > 0$, $\delta = \delta(\Omega, H)$, such that if M is a graph on Ω with $\partial M = \partial\Omega$, then the height h of M satisfies

$$h < \frac{1}{2|H|} - \delta. \tag{11.5}$$

Proof of the claim. Because Ω is included in a strip of width $1/|H|$, we can put Ω between two parallel lines $L_1' \cup L_2'$ such that $dist(L_1', L_2') < 1/|H|$. Let M be a graph on Ω with constant mean curvature $H > 0$ with the downward orientation. This means that $M \subset P^+$. Let S_H be a half-cylinder of radius $1/(2|H|)$ such that $S_H \subset P^+$, $\partial S_H \subset P$. Let remark that ∂S_H are two parallel lines $L_1 \cup L_2$ whose distance is $1/|H|$. Let D_H be strip bounded by ∂S_H. We place S_H such that $L_1' \cup L_2' \subset D_H$ (and then the four straight-lines are parallel) and $dist(L_1, L_1') = dist(L_2, L_2')$. We prove that it is possible to descend S_H until to arrive the position $L_i = L_i'$ and the whole surface M lies below S_H. For this, we move up S_H until it does not touch M. See Figure 21. Next, we move down. Because M and S_H

have the same (constant) mean curvature with the downward orientation, it is not possible to have a contact point until that S_H arrives its original position. For the same reason, we can move down S_H until that $S_H \cap P = L_1' \cup L_2'$. Thus M lies below S_H and so, the height h of M is less than the one of S_H in the last position. Now the height of S_H is $1/(2|H|) - \delta$, where δ depends only on $L_1' \cup L_2'$, that is, on Ω, and the cylinder S_H. We point out that we have not used that Ω is a convex domain.

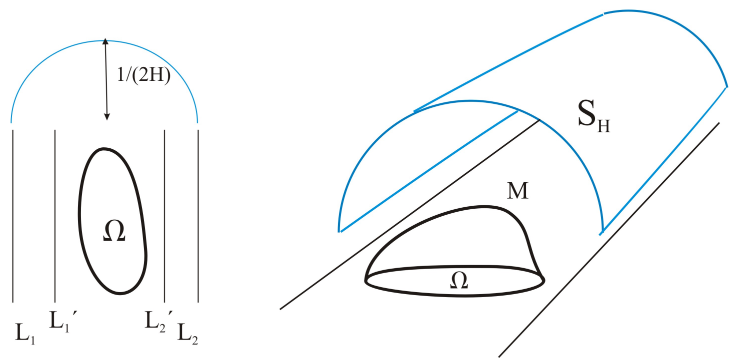


FIGURE 21. Proof of Theorem 11.2: comparison between M and S_H.

Once proved the estimate (11.5), we show the existence of a priori C^1-estimates along the boundary. We consider Q_H quarters of cylinders of radius $1/(2|H|)$ whose boundary is formed by two parallel lines to the axis A of the cylinder, $\partial Q_H = R_1 \cup R_2$. We place Q_H such that $Q_H \subset P^+$, $R_1 \cup A \subset P$. Thus the height of Q_H is exactly $1/(2H)$. Consider a parallel direction v to P and we displace Q_H so does not intersect M, A is orthogonal to v and M lies in the convex side of Q_H. We descend Q_H until that the height of Q_H is $1/(2H) - \frac{\delta}{2}$, where δ is the number given in (11.5). We call Q_H again the portion of Q_H in P^+. We displace Q_H to M orthogonally to v until that Q_H touch M: see Figure 22. Because both surfaces have the same constant mean curvature (with the same downward orientation), it is not possible a contact point between M and Q_H. Thus we can place Q_H until that R_1 touches $\partial\Omega$ and some point $p \in \partial\Omega$. At this point of M, the slope of M is less than the one of Q_H. But the slope of Q_H is constant and independently on M.

Finally, doing this with any parallel direction to P together the fact that Ω is convex, we can place Q_H *at each point* of ∂M. This shows the existence of a priori estimate of the slope of M along its boundary, proving the result.

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Final remark. The purpose of this paper has been to present some problems on CMC surfaces related with the author's work. For this reason, this manuscript is not a survey of the theory of CMC surfaces and this has not been the motivation of this work. For example, the number of references in the text is small and we only cite some recent papers related with enounced results. For the interested reader, we refer to [14, 23]. For minimal surfaces, the classical text [29] is a good guide

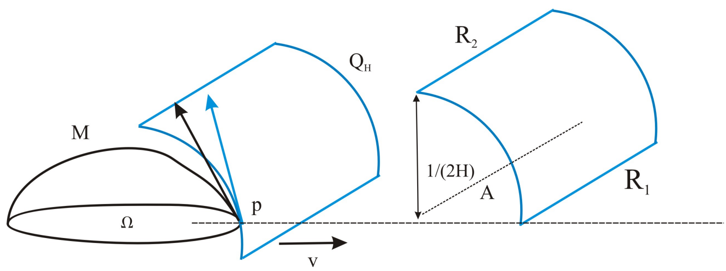

FIGURE 22. Proof of Theorem 11.2: comparison between M and Q_H.

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