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Research Article

Some generalized numerical radius inequalities involving Kwong functions

Mojtaba Bakherad

Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

Abstract

We prove several numerical radius inequalities involving positive semidefinite matrices via the Hadamard product and Kwong functions. Among other inequalities, it is shown that if X is an arbitrary $n \times n$ matrix and A, B are positive semidefinite, then

$$\omega(H_{f,g}(A)) \leq k\,\omega(AX + XA),$$

which is equivalent to

$$\begin{aligned}&\omega(H_{f,g}(A,B) \pm H_{f,g}(B,A))\\ &\quad \leq k'\,\{\omega((A+B)X + X(A+B)) + \omega((A-B)X - X(A-B))\},\end{aligned}$$

where f and g are two continuous functions on $(0,\infty)$ such that $h(t) = \frac{f(t)}{g(t)}$ is Kwong, $k = \max\left\{\frac{f(\lambda)g(\lambda)}{\lambda} : \lambda \in \sigma(A)\right\}$ and $k' = \max\left\{\frac{f(\lambda)g(\lambda)}{\lambda} : \lambda \in \sigma(A) \cup \sigma(B)\right\}$.

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1. Introduction

Let $\mathcal{M}_n$ be the C^*-algebra of all $n \times n$ complex matrices and $\langle \cdot, \cdot \rangle$ be the standard scalar product in $\mathbb{C}^n$. A capital letter means an $n \times n$ matrix in $\mathcal{M}_n$. For Hermitian matrices A and B, we write $A \geq 0$ if A is positive semidefinite, $A > 0$ if A is positive definite, and $A \geq B$ if $A - B \geq 0$. The numerical radius of $A \in \mathcal{M}_n$ is defined by

$$\omega(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathbb{C}^n, \| x \| = 1\}.$$

It is well known that $\omega(\cdot)$ defines a norm on $\mathcal{M}_n$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $A \in \mathcal{M}_n$, $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|$; see [11]. For further information about numerical radius inequalities we refer the reader to [4, 11, 15, 16] and references therein. We use the notation J for the matrix whose entries are equal to one.

The Hadamard product (Schur product) of two matrices $A, B \in \mathcal{M}_n$ is the matrix $A \circ B$ whose (i, j) entry is $a_{ij}b_{ij}$ $(1 \leq i, j \leq n)$. The Schur multiplier operator S_A on $\mathcal{M}_n$ is

Email address: mojtaba.bakherad@yahoo.com
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defined by $S_A(X) = A \circ X$ ($X \in \mathcal{M}_n$). The induced norm of S_A with respect to the numerical radius norm will be denoted by

$$\|S_A\|_\omega = \sup_{X\neq 0} \frac{\omega(S_A(X))}{\omega(X)} = \sup_{X\neq 0} \frac{\omega(A\circ X)}{\omega(X)}.$$

A continuous real valued function f on an interval $(a,b) \subseteq \mathbb{R}$ is called operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for all Hermitian matrices $A, B \in \mathcal{M}_n$ with spectra in (a,b). Following [3], a continuous real-valued function f defined on an interval (a,b) with $a > 0$ is called a Kwong function if the matrix $K_f = \left(\frac{f(\lambda_i)+f(\lambda_j)}{\lambda_i+\lambda_j}\right)_{i,j=1,2,\cdots,n}$ is positive semidefinite for any (distinct) $\lambda_1, \cdots, \lambda_n$ in (a,b). It is easy to see that if f is a nonzero Kwong function, then f is positive and $\frac{1}{f}$ is Kwong. Kwong [13] showed that the set of all Kwong functions on $(0,\infty)$ is a closed cone and includes all non-negative operator monotone functions on $(0,\infty)$. Also, Audenaert [3] gave a characterization of Kwong functions by showing that, for given $0 \leq a < b$, a function f on an interval (a,b) is Kwong if and only if the function $g(x) = \sqrt{x}f(\sqrt{x})$ is operator monotone on (a^2, b^2).

The Heinz means are defined as $H_\nu(a,b) = \frac{a^{1-\nu}b^\nu + a^\nu b^{1-\nu}}{2}$ for $a, b > 0$ and $0 \leq \nu \leq 1$. These interesting means interpolate between the geometric and arithmetic means. In fact, the Heinz inequalities assert that $\sqrt{ab} \leq H_\nu(a,b) \leq \frac{a+b}{2}$, where $a, b > 0$ and $0 \leq \nu \leq 1$. There have been obtained several Heinz type inequalities for Hilbert space operators and matrices; see [5] and references therein.

For two continuous functions f and g on $(0,\infty)$ we denote

$$H_{f,g}(A,B) = f(A)Xg(B) + g(A)Xf(B)$$

and

$$H_{f,g}(A) = f(A)Xg(A) + g(A)Xf(A),$$

where $A, B, X \in \mathcal{M}_n$ such that A, B are positive semidefinite. In particular, for $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ ($\alpha \in [0,1]$), we get $H_\alpha(A,B) = A^\alpha XB^{1-\alpha} + A^{1-\alpha}XB^\alpha$ and $H_\alpha(A) = A^\alpha XA^{1-\alpha} + A^{1-\alpha}XA^\alpha$. A norm $|||\cdot|||$ on $\mathcal{M}_n$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathcal{M}_n$ and all unitary matrices $U, V \in \mathcal{M}_n$. Let $A, B, X \in \mathcal{M}_n$ such that A and B are positive semidefinite. In [14] it was conjectured a general norm inequality of the Heinz inequality $|||H_{f,g}(A,B)||| \leq |||AX + XB|||$, where f and g are two continuous functions on $(0,\infty)$ such that $f(t)g(t) \leq t$ and the function $h(t) = \frac{f(t)}{g(t)}$ is Kwong. In particular, if $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ ($\alpha \in [0,1]$), then we state a Heinz type inequality $|||H_\alpha(A,B)||| \leq |||AX + XB|||$, where $A, B, X \in \mathcal{M}_n$ such that A, B are positive semidefinite. For further information, we refer the reader to [5, 6] and references therein.

The numerical radius $\omega(\,\cdot\,)$ is a weakly unitarily invariant norm on $\mathcal{M}_n$, that is $\omega(U^*AU) = \omega(A)$ for every $A \in \mathcal{M}_n$ and every unitary $U \in \mathcal{M}_n$. In [1], the authors proved a Heinz type inequality for the numerical radius as follows

$$\omega(H_\alpha(A)) \leq \omega(AX + XA), \tag{1.1}$$

in which $A, X \in \mathcal{M}_n$ such that A is positive semidefinite. They also showed that the inequality $\omega(H_\alpha(A,B)) \leq \omega(AX + XB)$ is not true in general.

Our research aim is to show some numerical radius inequalities via the Hadamard product and Kwong functions. By using some ideas of [8, 10] and [14], we obtain some extensions and generalizations of inequality (1.1), which are generalizations of a Hienz type inequality for the numerical radius. For instance, we prove if $A, X \in \mathcal{M}_n$ such that A is positive semidefinite, then

$$\omega(H_{f,g}(A)) \leq k\,\omega(AX + XA),$$

where f and g are two continuous functions on $(0,\infty)$ such that $\frac{f(t)}{g(t)}$ is Kwong and $k = \max\left\{\frac{f(\lambda)g(\lambda)}{\lambda} : \lambda \in \sigma(A)\right\}$.

2. Main results

For our purpose we need the following lemmas.

Lemma 2.1 ([18, Theorem 3.4])**.** (*Spectral Decomposition*) *Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \cdots, \lambda_n$. Then A is normal if and only if there exists a unitary matrix U such that*

$$U^*AU = \operatorname{diag}(\lambda_1, \cdots, \lambda_n).$$

In particular, A is positive definite if and only if the λ_j $(1 \le j \le n)$ are positive.

Lemma 2.2 ([2, Corollary 4])**.** *Let $A = [a_{ij}] \in \mathcal{M}_n$ be positive semidefinite. Then*

$$\|S_A\|_\omega = \max_i a_{ii}.$$

Lemma 2.3. ([12]). *Let $X, Y \in \mathcal{M}_n$. Then*
(i) $\omega\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\{\omega(X), \omega(Y)\}$;
(ii) $\frac{\max(\omega(X+Y),\omega(X-Y))}{2} \le \omega\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \le \frac{\omega(X+Y)+\omega(X-Y)}{2}$.

Now, we are in position to demonstrate the first result of this section by using some ideas of [8, 10, 14].

Theorem 2.4. *Let $A, B \in \mathcal{M}_n$ be positive semidefinite, $X \in \mathcal{M}_n$, and let f, g be two continuous functions on $(0,\infty)$ such that $h(t) = \frac{f(t)}{g(t)}$ is Kwong. Then*

$$\omega(H_{f,g}(A)) \le k\,\omega(AX + XA), \tag{2.1}$$

where $k = \max_{\lambda\in\sigma(A)}\left\{\frac{f(\lambda)g(\lambda)}{\lambda}\right\}$.
Moreover, inequality (2.1) *is equivalent to the inequality*

$$\begin{aligned}&\omega(H_{f,g}(A,B) \pm H_{f,g}(B,A)) \\ &\qquad \le k'\,\{\omega((A+B)X + X(A+B)) + \omega((A-B)X - X(A-B))\}\,, \end{aligned}\tag{2.2}$$

where $k' = \max_{\lambda\in\sigma(A)\cup\sigma(B)}\left\{\frac{f(\lambda)g(\lambda)}{\lambda}\right\}$.

Proof. Assume that A is positive definite. Since the numerical radius is weakly unitarily invariant, we may assume that A is diagonal matrix with positive eigenvalues $\lambda_1, \cdots, \lambda_n$. It follows from $\frac{f}{g}$ is a Kwong function that

$$Z = [z_{ij}] = \Lambda\left(\frac{f(\lambda_i)g^{-1}(\lambda_j) + f(\lambda_j)g^{-1}(\lambda_i)}{\lambda_i + \lambda_j}\right)_{(i,j=1,\cdots,n)}\Lambda$$

is positive semidefinite, where $\Lambda = \operatorname{diag}(g(\lambda_1), \cdots, g(\lambda_n))$. It follows from Lemma 2.2 that

$$\|S_Z\|_\omega = \max_i z_{ii} = \max_i \frac{f(\lambda_i)g(\lambda_i)}{\lambda_i} \le k$$

or equivalently, $\frac{\omega(Z\circ X)}{\omega(X)} \le k$ $(0 \ne X \in \mathcal{M}_n)$. If we put $E = [\frac{1}{\lambda_i+\lambda_j}]$ and $F = [f(\lambda_i)g(\lambda_j) + f(\lambda_j)g(\lambda_i)] \in \mathcal{M}_n$, then

$$\omega(E \circ F \circ X) = \omega(Z \circ X) \le k\,\omega(X) \qquad (X \in \mathcal{M}_n).$$

Let the matrix C be the entrywise inverse of E, i.e., $C \circ E = J$. Thus

$$\omega(F \circ X) \le k\,\omega(C \circ X) \qquad (X \in \mathcal{M}_n)$$

or equivalently

$$\omega(H_{f,g}(A)) = \omega(f(A)Xg(A) + g(A)Xf(A)) \le k\,\omega(AX + XA). \tag{2.3}$$

Now, if A is positive semidefinite, we may assume that $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, where $A_1 \in \mathcal{M}_k\,(k < n)$ is a positive definite matrix. Let $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$, where $X_1 \in \mathcal{M}_k$ and $X_4 \in \mathcal{M}_{n-k}$. Then we have

$$\begin{aligned}
\omega(H_{f,g}(A)) &= \omega\left(\begin{bmatrix} f(A_1)X_1g(A_1) + g(A_1)X_1f(A_1) & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\qquad \text{(by Lemma 2.3(i))} \\
&\le k\,\omega\left(\begin{bmatrix} A_1X_1 + X_1A_1 & 0 \\ 0 & 0 \end{bmatrix}\right) \qquad \text{(by (2.3))} \\
&= k\,\omega(A_1X_1 + X_1A_1) \qquad \text{(by Lemma 2.3(i))} \\
&\le k\,\omega(AX + XA) \qquad \text{(by [7, Lemma 2.1])}.
\end{aligned} \tag{2.4}$$

Hence, we reach inequality (2.1). Moreover, if we replace A and X by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}$ in inequality (2.1), respectively, then

$$\omega\left(\begin{bmatrix} 0 & H_{f,g}(A,B) \\ H_{f,g}(B,A) & 0 \end{bmatrix}\right) \le k'\,\omega\left(\begin{bmatrix} 0 & AX + XB \\ XA + BX & 0 \end{bmatrix}\right),$$

whence

$$\begin{aligned}
&\max\left\{\omega(H_{f,g}(A,B) \pm H_{f,g}(B,A))\right\} \\
&\le 2\,\omega\left(\begin{bmatrix} 0 & f(A)Xg(B) + g(A)Xf(B) \\ g(B)Xf(A) + f(B)Xg(A) & 0 \end{bmatrix}\right) \\
&\qquad \text{(by Lemma 2.3(ii))} \\
&\le 2k'\,\omega\left(\begin{bmatrix} 0 & AX + XB \\ XA + BX & 0 \end{bmatrix}\right) \qquad \text{(by (2.4))} \\
&\le k'\,(\omega(AX + XB + XA + BX) + \omega(AX + XB - XA - BX)) \\
&\qquad \text{(by Lemma 2.3(ii))}.
\end{aligned}$$

Thus, we have inequality (2.2). Also, if we put $B = A$ in inequality (2.2), then we reach inequality (2.1). □

If we take $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ in Theorem 2.4 for each $0 \le \alpha \le 1$, then we get the next result.

Corollary 2.5 ([1, Theorem 2.4])**.** *Let $A, B \in \mathcal{M}_n$ be positive semidefinite, $X \in \mathcal{M}_n$, and let $0 \le \alpha \le 1$. Then*

$$\omega(H_\alpha(A)) \le \omega(AX + XA). \tag{2.5}$$

Moreover, inequality (2.5) is equivalent to the inequality

$$\begin{aligned}
&\omega(H_\alpha(A,B) \pm H_\alpha(B,A)) \\
&\qquad \le \omega((A+B)X + X(A+B)) + \omega((A-B)X - X(A-B)).
\end{aligned}$$

Corollary 2.6. *Let $A, B \in \mathcal{M}_n$ be positive semidefinite, $X \in \mathcal{M}_n$, and let f be a non-negative operator monotone function on $[0, \infty)$ such that $f'(0) = \lim_{x \to 0^+} f'(x) < \infty$ and $f(0) = 0$. Then*

$$\omega(f(A)X + Xf(A)) \leq f'(0)\, \omega(AX + XA). \tag{2.6}$$

Moreover, inequality (2.6) is equivalent to the inequality

$$\begin{aligned} &\omega(X(f(A) + f(B)) + (f(A) + f(B))X) \\ &\leq f'(0) \Big(\omega((A + B)X + X(A + B)) + \omega((A - B)X - X(A - B)) \Big). \end{aligned}$$

Proof. A function g is non-negative operator increasing on $[0, \infty)$ if and only if $\frac{t}{g(t)}$ is non-negative operator increasing on $[0, \infty)$; see [9]. Hence $\frac{t}{f(t)}$ is operator increasing. Then $\frac{f(t)}{t}$ is decreasing. If $0 \leq x \leq t$, then $\frac{f(t)}{t} \leq \frac{f(x)}{x}$. Now, by taking $x \to 0^+$ we have $\frac{f(t)}{t} \leq f'(0)$. If we put $g(t) = 1\,(t \in [0, \infty))$ in Theorem 2.4, it follows from $k = k' \leq f'(0)$ that we get the required result. □

We first cite the following lemma due to Fujii et al. [10], which will be needed in the next theorem.

Lemma 2.7 ([10, Lemma 3.1])**.** *Let $\lambda_1, \cdots, \lambda_n$ be any positive real numbers and $-2 < t \leq 2$. If f and g are two continuous functions on $(0, \infty)$ such that $\frac{f(t)}{g(t)}$ is Kwong, then the $n \times n$ matrix*

$$Y = \left(\frac{f(\lambda_i)g^{-1}(\lambda_j) + f(\lambda_j)g^{-1}(\lambda_i)}{\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2} \right)_{i,j=1,\cdots,n}$$

is positive semidefinite.

Theorem 2.8. *Let $A, B \in \mathcal{M}_n$ be positive semidefinite, $X \in \mathcal{M}_n$, f, g be two continuous functions on $(0, \infty)$ such that $\frac{f(t)}{g(t)}$ is Kwong, and let $-2 < t \leq 2$. Then*

$$\omega(A^{\frac{1}{2}}(H_{f,g}(A))A^{\frac{1}{2}}) \leq \frac{2k}{t+2}\, \omega(A^2X + tAXA + XA^2), \tag{2.7}$$

where $k = \max_{\lambda \in \sigma(A)} \left\{ \frac{f(\lambda)g(\lambda)}{\lambda} \right\}$.
Moreover, inequality (2.7) is equivalent to the inequality

$$\omega(A^{\frac{1}{2}}(H_{f,g}(A, B))B^{\frac{1}{2}}) \leq \frac{4k'}{t+2}\, \omega(A^2X + tAXB + XB^2), \tag{2.8}$$

where $k' = \max_{\lambda \in \sigma(A) \cup \sigma(B)} \left\{ \frac{f(\lambda)g(\lambda)}{\lambda} \right\}$.

Proof. First, we show inequality (2.7). It is enough to show the inequality in the case A is positive definite. Since the numerical radius is weakly unitarily invariant, we may assume that A is diagonal matrix with positive eigenvalues $\lambda_1, \cdots, \lambda_n$. Let $\Sigma = \operatorname{diag}\left(\lambda_1^{\frac{1}{2}} g(\lambda_1), \cdots, \lambda_n^{\frac{1}{2}} g(\lambda_n) \right)$. It follows from Lemma 2.7 that

$$Z = [z_{ij}] = \Sigma \Big(\frac{(t+2)\left(f(\lambda_i)g^{-1}(\lambda_j) + f(\lambda_j)g^{-1}(\lambda_j)\right)}{2(\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2)} \Big)_{i,j=1,\cdots,n} \Sigma$$

is positive semidefinite for $-2 < t \leq 2$. In addition, all diagonal entries of Z are no more than k. Therefore,

$$\|S_Z\|_\omega = \max_i z_{ii} = \max_i \frac{f(\lambda_i)g(\lambda_i)}{\lambda_i} \leq k,$$

whence $\frac{\omega(Z\circ X)}{\omega(X)} \le k$ $(0 \ne X \in \mathcal{M}_n)$. Now, let $M = \left[\frac{1}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2}\right]_{i,j=1,\cdots,n}$ and $P = \left[\frac{t+2}{2}\lambda_i^{\frac{1}{2}} f(\lambda_i)g(\lambda_j) + f(\lambda_j)g(\lambda_i)\lambda_j^{\frac{1}{2}}\right]_{i,j=1,\cdots,n}$. Then

$$\omega(M\circ P\circ X) = \omega(Z\circ X) \le k\,\omega(X) \qquad (0\ne X\in\mathcal{M}_n).$$

Let the matrix N be the entrywise inverse of M, i.e., $M\circ N = J$. Hence

$$\omega(P\circ X) \le k\,\omega(N\circ X) \qquad (0\ne X\in\mathcal{M}_n)$$

or equivalently

$$\omega(A^{\frac{1}{2}}\,(H_{f,g}(A))\,A^{\frac{1}{2}}) \le \frac{2k}{t+2}\,\omega(A^2X+tAXA+XA^2),$$

where $X\in\mathcal{M}_n$, $-2<t\le 2$ and $k=\max\left\{\frac{f(\lambda)g(\lambda)}{\lambda} : \lambda\in\sigma(A)\right\}$. Hence we have inequality (2.7).

Now, if we replace A and X by $\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}$ and $\begin{pmatrix} 0 & X\\ 0 & 0\end{pmatrix}$ inequality (2.7), respectively, then

$$\omega\left(\begin{bmatrix} 0 & A^{\frac{1}{2}}\,(H_{f,g}(A,B))\,B^{\frac{1}{2}}\\ 0 & 0\end{bmatrix}\right) \le \frac{2k'}{t+2}\,\omega\left(\begin{bmatrix} 0 & A^2X+tAXB+XB^2\\ 0 & 0\end{bmatrix}\right).$$

Hence

$$\begin{aligned}\frac{1}{2}\omega(A^{\frac{1}{2}}(H_{f,g}(A,B))B^{\frac{1}{2}}) &\le \omega\left(\begin{bmatrix} 0 & A^{\frac{1}{2}}(H_{f,g}(A,B))B^{\frac{1}{2}}\\ 0 & 0\end{bmatrix}\right)\\ &\qquad\text{(by Lemma 2.3)}\\ &\le \frac{2k'}{t+2}\,\omega\left(\begin{bmatrix} 0 & A^2X+tAXB+XB^2\\ 0 & 0\end{bmatrix}\right)\\ &\le \frac{2k'}{t+2}\,\omega(A^2X+tAXB+XB^2)\\ &\qquad\text{(by Lemma 2.3).}\end{aligned}$$

Thus, we reach inequality (2.8). Also, if we put $B=A$ in inequality (2.7), then we get inequality (2.8). □

Corollary 2.9. *Let $A\in\mathcal{M}_n$ be positive semidefinite. If f is a positive operator monotone function on $(0,\infty)$, then*

$$\begin{aligned}&\omega(A^{\frac{1}{2}}f(A)Xf(A)^{-1}A^{\frac{3}{2}} + A^{\frac{3}{2}}f(A)^{-1}Xf(A)A^{\frac{1}{2}})\\ &\qquad\le \frac{4}{t+2}\,\omega(A^2X+tAXA+XA^2),\end{aligned}$$

where $X\in\mathcal{M}_n$ and $-2<t\le 2$.

Proof. Since f positive operator monotone on $(0,\infty)$, then $g(t)=\frac{t}{f(t)}$ is operator monotone on $(0,\infty)$ and also $\frac{f(t)}{g(t)} = tf^2(t)$ is Kwong function [14]. So f and g satisfy the conditions of Theorem 2.8. Hence we have the desired inequality. □

Example 2.10. The function $f(t)=\log(1+t)$ is operator monotone on $(0,\infty)$; see [9]. If we put $g(t)=1$, then $\frac{f(t)}{g(t)} = \log(1+t)$ is Kwong [13]. Using Theorem 2.4 we have

$$\begin{aligned}&\omega(A^{\frac{1}{2}}(\log(I+A)X + X\log(I+A))A^{\frac{1}{2}})\\ &\qquad\le \frac{2}{t+2}\,\omega(A^2X+tAXA+XA^2),\end{aligned}$$

where $A,X\in\mathcal{M}_n$ such that A is positive semidefinite and $-2<t\le 2$.

Now, we infer the following lemma due to Zhan [17], which will be needed in the next theorem.

Lemma 2.11 ([17, Lemma 5]). *Let $\lambda_1, \cdots, \lambda_n$ be any positive real numbers, $r \in [-1, 1]$ and $-2 < t \leq 2$. Then the $n \times n$ matrix*

$$L = \left(\frac{\lambda_i^r + \lambda_j^r}{\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2} \right)_{i,j=1,\cdots,n}$$

is positive semidefinite.

Now, we shall show the following result related to [10].

Proposition 2.12. *Let $A, X \in \mathcal{M}_n$ such that A is positive semidefinite, $\beta > 0$ and $1 \leq 2r \leq 3$. Then*

$$\begin{aligned} &\omega(A^r X A^{2-r} + A^{2-r} X A^r) \\ &\leq \omega\left(2(1-2\beta+2\beta r_0)AXA + \frac{4\beta(1-r_0)}{t+2}(A^2X + tAXA + XA^2)\right), \end{aligned}$$

where $-2 < t \leq 2\beta - 2$ and $r_0 = \min\{\frac{1}{2} + |1-r|, 1 - |1-r|\}$.

Proof. Since the numerical radius is weakly unitarily invariant, we may assume that A is diagonal matrix with positive eigenvalues $\lambda_1, \cdots, \lambda_n$. Since $1 \leq 2r \leq 3$, then $\frac{1}{2} \leq r_0 \leq \frac{3}{4}$. Let $t_0 = \frac{1-2\beta+2\beta r_0}{2\beta(1-r_0)}(t+2) + t$. It follows from $-2 < t \leq 2\beta - 2$ and $\frac{1}{4} \leq 1 - r_0 \leq \frac{1}{4}$, that $\frac{t+2}{4\beta(1-r_0)} > 0$ and $-2 < t_0 \leq 2$, where $t_0 = \frac{t}{2\beta(1-r_0)} + \frac{1}{\beta(1-r_0)} - 2$. Hence, by using Lemma 2.11, the $n \times n$ matrix

$$W = [w_{ij}] = \frac{t+2}{4\beta(1-r_0)} \Lambda^r \Big(\frac{\lambda_i^{2-2r} + \lambda_j^{2-2r}}{\lambda_i^2 + t_0\lambda_i\lambda_j + \lambda_j^2} \Big)_{i,j=1,\cdots,n} \Lambda^r$$

is positive semidefinite for $\frac{1}{2} \leq r \leq \frac{3}{2}$, where $\Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$. Therefore,

$$\|S_W\|_\omega = \max_i w_{ii} = \max_i \frac{(t+2)\lambda_i^r(2\lambda_i^{2-2r})\lambda_i^r}{4\beta(1-r_0)(t_0+2)\lambda_i^2} = 1,$$

whence $\frac{\omega(W \circ X)}{\omega(X)} \leq 1$ $(0 \neq X \in \mathcal{M}_n)$. Now, let $O = \left[\lambda_i^2 + t_0\lambda_i\lambda_j + \lambda_j^2\right]_{i,j=1,\cdots,n}$ and

$$M = \left[\frac{1}{2(1-2\beta+2\beta r_0)\lambda_i\lambda_j + \frac{4\beta(1-r_0)}{t+2}(\lambda_i^2 X + t\lambda_i\lambda_j + \lambda_j^2)} \right]_{i,j=1,\cdots,n}.$$

Then

$$\omega(O \circ M \circ X) = \omega(W \circ X) \leq \omega(X) \qquad (0 \neq X \in \mathcal{M}_n).$$

Let the matrix N be the entrywise inverse of M, i.e., $M \circ N = J$. Hence

$$\omega(O \circ X) \leq \omega(N \circ X) \qquad (0 \neq X \in \mathcal{M}_n)$$

or equivalently

$$\begin{aligned} &\omega(A^r X A^{2-r} + A^{2-r} X A^r) \\ &\leq \omega\left(2(1-2\beta+2\beta r_0)AXA + \frac{4\beta(1-r_0)}{t+2}(A^2X + tAXA + XA^2)\right), \end{aligned}$$

where $-2 < t \leq 2\beta - 2$ and $r_0 = \min\{\frac{1}{2} + |1-r|, 1 - |1-r|\}$. $\square$

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