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Research Article

Generalizations of 2-absorbing and 2-absorbing primary submodules

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Abstract

In this study, we introduce ϕ-2-absorbing and ϕ-2-absorbing primary submodules of modules over commutative rings generalizing the concepts of 2-absorbing and 2-absorbing primary submodules. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ denotes the set of all submodules of M and N a proper submodule of an R-module M. We will say that N is a *ϕ-2-absorbing submodule* of M if whenever $a, b \in R$, $m \in M$ with $abm \in N$ and $abm \notin \phi(N)$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$ and N is said to be a ϕ-2-absorbing primary submodule of M whenever if $a, b \in R$, $m \in M$ with $abm \in N$ and $abm \notin \phi(N)$, then $am \in M$-rad(N) or $bm \in M$-rad(N) or $ab \in (N :_R M)$. We investigate many properties of these new types of submodules and establish some characterizations for ϕ-2-absorbing and ϕ-2-absorbing primary submodules of multiplication modules.

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1. Introduction

Throughout this paper, R is a commutative ring with a nonzero identity and M denotes a unitary R-module. We will denote by $(N :_R M)$ *the residual of N by M*, that is, the set of all $r \in R$ such that $rM \subseteq N$. The annihilator of M which is denoted by $Ann_R(M)$ is $(0 :_R M)$. A *prime* (resp. *primary*) submodule is a proper submodule N of M with the property that for $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M)$ (resp. $m \in N$ or $a \in \sqrt{(N :_R M)}$). As prime ideals (submodules) have an important role in ring (module) theory, several authors generalized these concepts in different ways (see [3–10, 12], [14–26]). Weakly prime submodules were introduced by Ebrahimi et. al. in [8].

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A proper submodule N of M is *weakly prime* if for $a \in R$ and $m \in M$ with $0 \neq am \in N$, either $m \in N$ or $a \in (N :_R M)$. Behboodi and Koohi in [15] defined weakly prime submodules in a different way. In their paper, a proper submodule N of an R-module M is said to be *weakly prime* when $abm \in N$ for $a, b \in R$ and $m \in M$ implies that $am \in N$ or $bm \in N$. The concepts of ϕ-prime and ϕ-primary ideals are introduced in [4], [16], and the generalizations of these concepts are studied in [14]. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M. A proper submodule N of an R-module M is called ϕ-prime (resp. ϕ-primary) if $a \in R$, $m \in M$ with $am \in N$ and $am \notin \phi(N)$, then $a \in (N :_R M)$ or $m \in N$ (resp. $a \in \sqrt{(N :_R M)}$ or $m \in N$).

The concept of 2-absorbing ideal (resp. weakly 2-absorbing ideal) is introduced by Badawi in [9] (resp. Badawi and Darani in [10]) as a different generalization of prime ideal (resp. weakly prime ideal). According to [9] and [10], a nonzero proper ideal I of R is a *2-absorbing ideal* (resp. weakly 2-absorbing ideal) of R if whenever $a, b, c \in R$ and $abc \in I$ (resp. $0 \neq abc \in I$), then $ab \in I$ or $ac \in I$ or $bc \in I$. Then introducing 2-absorbing submodules (resp. weakly 2-absorbing submodules) of a module, Darani [17] generalized the concept of 2-absorbing ideals (resp. weakly 2-absorbing ideals) to submodules of a module over a commutative ring as following: Let N be a proper submodule of an R-module M. N is said to be a *2-absorbing submodule* (resp. weakly 2-absorbing submodule) of M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$ (resp. $0 \neq abm \in N$), then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. Badawi et. al. [12] introduced the concept of 2-absorbing primary ideals, where a proper ideal I of R is called *2-absorbing primary* if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Then ϕ-2-absorbing primary ideals of a commutative ring which are a generalization of 2-absorbing primary ideals are presented in [11]. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function where $S(R)$ is the set of all ideals of R. According to [11], a nonzero proper ideal I of R is called a *ϕ-2-absorbing primary ideal* of R if whenever $a, b, c \in R$ with $abc \in I$ and $abc \notin \phi(I)$ then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. The concept of 2-absorbing primary submodules is studied in [23] as a generalization of 2-absorbing primary ideals. A proper submodule N is said to be a *2-absorbing primary submodule* (resp. weakly 2-absorbing primary submodule) of M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$ (resp. $0 \neq abm \in N$), then $ab \in (N :_R M)$ or $am \in M\text{-}rad(N)$ or $bm \in M\text{-}rad(N)$.

An R-module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R. In fact $I = (N :_R M)$ which is called a presentation ideal of N. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1I_2M$, see [13]. Then by [2, Theorem 3.4], the product of N and K is independent of presentations of N and K. Moreover, for $a, b \in M$, by ab, we mean the product of Ra and Rb. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$ (see [2]). Let N be a proper submodule of an R-module M. Then the M-radical of N, denoted by M-rad(N), is defined to be the intersection of all prime submodules of M containing N. If M has no prime submodule containing N, then we say M-rad$(N) = M$. It is shown in [20, Theorem 2.12] that if N is a proper submodule of a multiplication R-module M, then M-rad$(N) = \sqrt{(N :_R M)}M$.

In this work, our aim is to extend the concept of 2-absorbing submodules to ϕ-2-absorbing submodules in completely different way from [21] and also to extend 2-absorbing primary submodules to ϕ-2-absorbing primary submodules of modules over commutative rings. We discuss on the relations among the concepts which are defined above and ϕ-2-absorbing primary submodules, and investigate some characterizations of them in some special multiplication modules. We prove that a submodule N of an R-module M is a ϕ-2-absorbing (resp. ϕ-2-absorbing primary) submodule of M if and only if $N/\phi(N)$ is a weakly 2-absorbing (resp. weakly 2-absorbing primary) submodule of $M/\phi(N)$. Let M_1 be an R_1-module, M_2 be an R_2-module, and let $M = M_1 \times M_2$. Let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ $(i = 1, 2)$ be function, and let $\phi = \psi_1 \times \psi_2$. Suppose that $N = N_1 \times M_2$ for some proper submodule N_1 of M_1. Then we show that the following conditions hold:

(1) If $\psi_2(M_2) = M_2$, then N is a ϕ-2-absorbing submodule of M if and only if N_1 is a ψ_1-2-absorbing submodule of M_1.
(2) If $\psi_2(M_2) \neq M_2$, then N is a ϕ-2-absorbing submodule of M if and only if N_1 is a 2-absorbing submodule of M_1.

Also, it is shown that if N is a ϕ-2-absorbing primary submodule of an R-module M that is not 2-absorbing primary, then $(N :_R M)^2 N \subseteq \phi(N)$. Moreover, if M is multiplication, then $N^3 \subseteq \phi(N)$. Finally, we find conditions under which N is a ϕ-2-absorbing primary submodule of M if and only if $IJK \subseteq N$ and $IJK\ nsubseteq \phi(N)$ for some ideals I, J of R and a submodule K of M implies that either $IJ \subseteq (N :_R M)$ or $IK \subseteq M\text{-rad}(N)$ or $JK \subseteq M\text{-rad}(N)$.

2. ϕ-2-absorbing and ϕ-2-absorbing primary submodules

Definition 2.1. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M. Let N be a proper submodule of M.

(1) N is called a *ϕ-2-absorbing submodule* of M if whenever $a, b \in R$, $m \in M$ with $abm \in N$ and $abm \notin \phi(N)$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.
(2) N is called a *ϕ-2-absorbing primary submodule* of M if whenever $a, b \in R$, $m \in M$ with $abm \in N$ and $abm \notin \phi(N)$, then $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$ or $ab \in (N :_R M)$.

We can define the following special functions ϕ_α as follows: Let N be a ϕ_α-primary submodule of a multiplication R-module M. Then

$\phi_\emptyset(N) = \emptyset$	primary submodule
$\phi_0(N) = 0$	weakly primary submodule
$\phi_2(N) = N^2$	almost primary submodule
...	
$\phi_n(N) = N^n$	n-almost primary submodule
$\phi_\omega(N) = \bigcap_{n=1}^{\infty} N^n$	ω-primary submodule.

Moreover, let N be a ϕ_α-2-absorbing (resp. ϕ_α-2-absorbing primary) submodule of a multiplication R-module M. Then

$\phi_\emptyset(N) = \emptyset$	2-absorbing (resp. 2-absorbing primary) submodule
$\phi_0(N) = 0$	weakly 2-absorbing (resp.weakly 2-absorbing primary) submodule
$\phi_2(N) = N^2$	almost 2-absorbing (resp. almost 2-absorbing primary) submodule
...	
$\phi_n(N) = N^n$	n-almost 2-absorbing (resp. 2-absorbing primary) submodule
$\phi_\omega(N) = \bigcap_{n=1}^{\infty} N^n$	ω-2-absorbing (resp. ω-2-absorbing primary) submodule

Throughout this paper, ϕ denotes a function from $S(M)$ to $S(M) \cup \{\emptyset\}$. Since $N - \phi(N) = N - (N \cap \phi(N))$ for any submodule N of M, without loss generality throughout assume that $\phi(N) \subseteq N$. For any two functions ψ_1, $\psi_2 : S(M) \to S(M) \cup \{\emptyset\}$, we say $\psi_1 \leq \psi_2$ if $\psi_1(N) \subseteq \psi_2(N)$ for each $N \in S(M)$. Thus clearly we have the following order: $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq ... \leq \phi_{n+1} \leq \phi_n \leq ... \leq \phi_2 \leq \phi_1$.

Lemma 2.2. *Let N be a proper submodule of M and ψ_1, $\psi_2 : S(M) \to S(M) \cup \{\emptyset\}$ be two functions with $\psi_1 \leq \psi_2$. If N is a ψ_1-2-absorbing (resp. ψ_1-2-absorbing primary) submodule of M, then N is a ψ_2-2-absorbing (resp. ψ_2-2-absorbing primary) submodule of M.*

Proof. Suppose that N is a ψ_1-2-absorbing submodule of M and $a, b \in R, m \in M$ with $abm \in N$ and $abm \notin \psi_2(N)$. Then $abm \notin \psi_1(N)$. Since N is ψ_1-2-absorbing (resp. ψ_1-2-absorbing primary) submodule, we are done. □

Theorem 2.3. *Let N be a proper submodule of M. Then, the followings hold.*

(1) N is a ϕ-prime submodule of $M \Rightarrow N$ is a ϕ-2-absorbing submodule of $M \Rightarrow N$ is a ϕ-2-absorbing primary submodule of M.
(2) If M is multiplication and N is a ϕ-primary submodule of M, then N is a ϕ-2-absorbing primary submodule of M.
(3) Let M be a multiplication R-module. N is a 2-absorbing submodule of $M \Rightarrow N$ is a weakly 2-absorbing submodule of $M \Rightarrow N$ is a ω-2-absorbing submodule of $M \Rightarrow N$ is an $(n+1)$-almost 2-absorbing submodule of $M \Rightarrow N$ is an n-almost 2-absorbing submodule of M for all $n \geq 2 \Rightarrow N$ is an almost 2-absorbing submodule of M.
(4) Let M be a multiplication R-module. N is a 2-absorbing primary submodule of $M \Rightarrow N$ is a weakly 2-absorbing primary submodule of $M \Rightarrow N$ is a ω-2-absorbing primary submodule of $M \Rightarrow N$ is an $(n+1)$-almost 2-absorbing primary submodule of $M \Rightarrow N$ is an n-almost 2-absorbing primary submodule of M for all $n \geq 2 \Rightarrow$ N is an almost 2-absorbing primary submodule of M.
(5) Let M-$rad(N) = N$. Then N is a ϕ-2-absorbing primary submodule of M if and only if N is a ϕ-2-absorbing submodule of M.
(6) If N is an idempotent submodule of a multiplication R-module M, then N is a ω-2-absorbing submodule of M, and N is an n-almost 2-absorbing submodule of M for every $n \geq 2$.
(7) Let M be a multiplication R-module. Then N is an n-almost 2-absorbing (resp. n-almost 2-absorbing primary) submodule of M for all $n \geq 2$ if and only if N is a ω-2-absorbing (resp. ω-2-absorbing primary) submodule of M.

Proof. (1) It is obvious from Definition 2.1.

(2) Let $abm \in N \backslash \phi(N)$ for some $a, b \in R$ and some $m \in M$. Assume that $bm \notin M$-rad(N). Then $bm \notin N$ and so $a \in \sqrt{(N :_R M)}$ as N is a ϕ-primary submodule. Therefore $am \in \sqrt{(N :_R M)}M = M$-rad$(N)$. Consequently, N is ϕ-2-absorbing primary.

(3) and (4) are clear from Lemma 2.2.

(5) The claim is clear.

(6) Suppose that N is an idempotent submodule of M. Then $N = N^n$ for all $n > 0$, and so $\phi_\omega(N) = \cap_{n=1}^{\infty} N^n = N$. Thus N is an ω-2-absorbing submodule of M. By (3), we conclude that N is an n-almost 2-absorbing submodule of M for all $n \geq 2$.

(7) Suppose that N is an n-almost 2-absorbing (resp. n-almost 2-absorbing primary) submodule of M for all $n \geq 2$. Let $a, b \in R$ and $m \in M$ with $abm \in N$ but $abm \notin \cap_{n=1}^{\infty} N^n$. Hence $abm \notin N^n$ for some $n \geq 2$. Since N is n-almost 2-absorbing (resp. n-almost 2-absorbing primary) for all $n \geq 2$, this implies either $ab \in (N :_R M)$ or $bm \in N$ or $am \in N$ (resp. $ab \in (N :_R M)$ or $bm \in M$-$rad(N)$ or $am \in M - rad(N)$), we are done. The converse is clear from (3) (resp. from (4)). □

Theorem 2.4. *Let N be a proper submodule of M. Then*

(1) *N is a ϕ-2-absorbing submodule of M if and only if $N/\phi(N)$ is a weakly 2-absorbing submodule of $M/\phi(N)$.*
(2) *N is a ϕ-2-absorbing primary submodule of M if and only if $N/\phi(N)$ is a weakly 2-absorbing primary submodule of $M/\phi(N)$.*
(3) *N is a ϕ-prime submodule of M if and only if $N/\phi(N)$ is a weakly prime submodule of $M/\phi(N)$.*
(4) *N is a ϕ-primary submodule of M if and only if $N/\phi(N)$ is a weakly primary submodule of $M/\phi(N)$.*

Proof. (1) If $\phi(N) = \emptyset$, then there is nothing to prove. Assume that $\phi(N) \neq \emptyset$. Let $a, b \in R$ and $m \in M$ such that $\phi(N) \neq ab(m + \phi(N)) = abm + \phi(N) \in N/\phi(N)$. Then $abm \in N$, but $abm \notin \phi(N)$. Hence either $ab \in (N :_R M)$ or $bm \in N$ or $am \in N$. So $ab \in (N/\phi(N) : M/\phi(N))$ or $b(m + \phi(N)) \in N/\phi(N)$ or $a(m + \phi(N)) \in N/\phi(N)$, so we are done.

Conversely, let $abm \in N$ and $abm \notin \phi(N)$ for some $a, b \in R$ and $m \in M$. Then $\phi(N) \neq ab(m + \phi(N)) \in N/\phi(N)$. Hence $ab \in (N/\phi(N) : M/\phi(N))$ or $b(m + \phi(N)) \in N/\phi(N)$ or $a(m + \phi(N)) \in N/\phi(N)$. So $ab \in (N :_R M)$ or $bm \in N$ or $am \in N$. Thus N is a ϕ-2-absorbing submodule of M.

(2) Let $\phi(N) \neq ab(m + \phi(N)) = abm + \phi(N) \in N/\phi(N)$. Then $abm \in N$, but $abm \notin \phi(N)$. Hence either $ab \in (N :_R M)$ or $bm \in M\text{-}rad(N)$ or $am \in M\text{-}rad(N)$. So $ab \in (N :_R M)/\phi(N)$ or $b(m + \phi(N)) \in M\text{-}rad(N)/\phi(N)$ or $a(m + \phi(N)) \in M\text{-}rad(N)/\phi(N)$. Since $M\text{-}rad(N)/\phi(N) = M/\phi(N)\text{-}rad(N/\phi(N))$, we are done. The converse can be easily shown with the previous manner.

Similarly, one can easily prove (3) and (4). □

Corollary 2.5. *Let N be a proper submodule of a multiplication R-module M and $n \geq 2$. Then*

(1) *N is a ϕ_n-2-absorbing submodule of M if and only if $N/\phi(N)$ is a weakly 2-absorbing submodule of M/N^n.*
(2) *N is a ϕ_n-2-absorbing primary submodule of M if and only if $N/\phi(N)$ is a weakly 2-absorbing primary submodule of M/N^n.*
(3) *N is a ϕ_n-prime submodule of M if and only if $N/\phi(N)$ is a weakly prime submodule of M/N^n.*
(4) *N is a ϕ_n-primary submodule of M if and only if $N/\phi(N)$ is a weakly primary submodule of M/N^n.*

Proof. Since $\phi_n(N) = N^n$, it is direct results of Theorem 2.4. □

Definition 2.6. Let N be a proper submodule of a multplication R-module M and $n \geq 2$.

(1) N is said to be n-potent 2-absorbing whenever if $a, b \in R$ and $m \in M$ with $abm \in N^n$, then $ab \in (N :_R M)$ or $bm \in N$ or $am \in N$.
(2) N is said to be n-potent 2-absorbing primary whenever if $a, b \in R$ and $m \in M$ with $abm \in N^n$, then $ab \in (N :_R M)$ or $bm \in M\text{-}rad(N)$ or $am \in M\text{-}rad(N)$.

Proposition 2.7. *Let M be a multiplication R-module. Then the following statements are satisfied:*

(1) Let N be an n-almost 2-absorbing primary submodule of M for some $n \geq 2$. If N is k-potent 2-absorbing primary for some $k \leq n$, then N is a 2-absorbing primary submodule of M.
(2) Let N be an n-almost 2-absorbing submodule of M for some $n \geq 2$. If N is k-potent 2-absorbing for some $k \leq n$, then N is a 2-absorbing submodule of M.

Proof. (1) Suppose that N is an n-almost 2-absorbing primary submodule. Let $abm \in N$ for some $a, b \in R$, $m \in M$. If $abm \notin N^k$, then $abm \notin N^n$. So we are done as N is an n-almost 2-absorbing primary submodule. So assume that $abm \in N^k$. Hence we get either $ab \in (N :_R M)$ or $bm \in M\text{-}rad(N)$ or $am \in M\text{-}rad(N)$ as N is a k-potent 2-absorbing primary submodule of M.

(2) The proof can be obtained by a similar argument in (1). □

Lemma 2.8 ([22, Corollary 1.3])**.** *Let M and M' be R-modules with $f : M \to M'$ an R-module epimorphism. If N is a submodule of M containing $Ker(f)$, then $f(M\text{-}rad(N)) = M'\text{-}rad(f(N))$.*

Theorem 2.9. *Let $f : M \to M'$ be an epimorphism of R-modules M and M' and let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ and $\phi' : S(M') \to S(M') \cup \{\emptyset\}$ be functions. Then the following statements hold:*

(1) If N' is a ϕ'-2-absorbing primary submodule of M' and $\phi(f^{-1}(N')) = f^{-1}(\phi'(N'))$, then $f^{-1}(N')$ is a ϕ-2-absorbing primary submodule of M.
(2) If N is a ϕ-2-absorbing primary submodule of M containing $Ker(f)$ and $\phi'(f(N)) = f(\phi(N))$, then $f(N)$ is a ϕ'-2-absorbing primary submodule of M'.
(3) If N' is a ϕ'-2-absorbing submodule of M' and $\phi(f^{-1}(N')) = f^{-1}(\phi'(N'))$, then $f^{-1}(N')$ is a ϕ-2-absorbing submodule of M.
(4) If N is a ϕ-2-absorbing submodule of M containing $Ker(f)$ and $\phi'(f(N)) = f(\phi(N))$, then $f(N)$ is a ϕ'-2-absorbing submodule of M'.

Proof. (1) Since f is epimorphism, $f^{-1}(N')$ is a proper submodule of M. Let $a, b \in R$ and $m \in M$ such that $abm \in f^{-1}(N')$ and $abm \notin f^{-1}(\phi'(N'))$. Since $abm \in f^{-1}(N')$, $abf(m) \in N'$. Also, $\phi(f^{-1}(N')) = f^{-1}(\phi'(N'))$ implies that $abf(m) \notin \phi'(N')$. Thus $abf(m) \in N' \backslash \phi'(N')$. Then $ab \in (N' :_R M')$ or $af(m) \in M'$-rad(N') or $bf(m) \in M'$-rad(N'). Thus $ab \in (f^{-1}(N') :_R M)$ or $am \in f^{-1}(M'$-rad$(N'))$ or $bm \in f^{-1}(M'$-rad$(N'))$. Since $f^{-1}(M'$-rad$(N')) \subseteq M$-rad$(f^{-1}(N'))$, we conclude that $f^{-1}(N')$ is a ϕ-2-absorbing primary submodule of M.
(2) Let $a, b \in R$ and $m' \in M'$ such that $abm' \in f(N) \backslash \phi'(f(N))$. Since f is epimorphism, there exists $m \in M$ such that $m' = f(m)$. Therefore $f(abm) \in f(N)$ and so $abm \in N$ as Ker$(f) \subseteq N$. Since $\phi'(f(N)) = f(\phi(N))$, we have $abm \notin \phi(N)$. Hence $abm \in N \backslash \phi(N)$. It implies that $ab \in (N :_R M)$ or $am \in M$-rad(N) or $bm \in M$-rad(N). Thus $ab \in (f(N) :_R M')$ or $am' \in f(M$-rad$(N))$ or $bm' \in f(M$-rad$(N))$. From Lemma 2.8, we are done.
(3), (4) can be easily obtained similar to (1) and (2). □

Corollary 2.10. *Let K, N be submodules of a multiplication R-module M with $K \subseteq N$ and $n \geq 2$.*

(1) If N is a ϕ_n-2-absorbing primary submodule of M, then N/K is a ϕ_n-2-absorbing primary submodule of M/K.
(2) If N is a ϕ_n-2-absorbing submodule of M, then N/K is a ϕ_n-2-absorbing submodule of M/K.
(3) If N is a ϕ_ω-2-absorbing primary submodule of M, then N/K is a ϕ_ω-2-absorbing primary submodule of M/K.
(4) If N is a ϕ_ω-2-absorbing submodule of M, then N/K is a ϕ_ω-2-absorbing submodule of M/K.

Proof. Since the canonical epimorphism $f : M \to M/K$ satisfies the equalities $\phi_n(f(N)) = \phi_n(N/K) = (N/K)^n = N^n/K = \phi_n(N)/K = f(\phi_n(N))$, and $\phi_\omega(f(N)) = \cap_{n=1}^{\infty}(N/K)^n = (\cap_{n=1}^{\infty} N^n)/K = f(\phi_\omega(N))$, we are done. □

Let $\mathcal{S}$ be a multiplicatively closed subset of R. It is well-known that each submodule of $\mathcal{S}^{-1}M$ is of the form $\mathcal{S}^{-1}N$ for some submodule N of M. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and define $\phi_{\mathcal{S}} : S(\mathcal{S}^{-1}M) \to S(\mathcal{S}^{-1}M) \cup \{\emptyset\}$ by $\phi_{\mathcal{S}}(\mathcal{S}^{-1}N) = \mathcal{S}^{-1}\phi(N)$ (and $\phi_{\mathcal{S}}(\mathcal{S}^{-1}N) = \emptyset$ when $\phi(N) = \emptyset$) for every submodule N of M. We also know that if N is a 2-absorbing primary submodule of M, then $\mathcal{S}^{-1}N$ is a 2-absorbing primary submodule of $\mathcal{S}^{-1}M$ by Theorem 2.11 of [23]. In the next theorem, we want to generalize this fact to ϕ-2-absorbing primary submodules of M.

Theorem 2.11. *Let $\mathcal{S}$ be a multiplicatively closed subset of R.*

(1) If N is a ϕ-2-absorbing primary submodule of M and $\mathcal{S}^{-1}N \neq \mathcal{S}^{-1}M$, then $\mathcal{S}^{-1}N$ is a $\phi_{\mathcal{S}}$-2-absorbing primary submodule of $\mathcal{S}^{-1}M$.

(2) If N is a ϕ-2-absorbing submodule of M and $\mathcal{S}^{-1}N \neq \mathcal{S}^{-1}M$, then $\mathcal{S}^{-1}N$ is a $\phi_{\mathcal{S}}$-2-absorbing submodule of $\mathcal{S}^{-1}M$.

Proof. (1) Let $\frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s} \in \mathcal{S}^{-1}N$ and $\frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s} \notin \phi_{\mathcal{S}}(\mathcal{S}^{-1}N)$. Since $\phi_{\mathcal{S}}(\mathcal{S}^{-1}N) = \mathcal{S}^{-1}\phi(N)$, we have $ua_1a_2m \in N$ and $ua_1a_2m \notin \phi(N)$ for some $u \in \mathcal{S}$. Hence $ua_1m \in M$-rad(N) or $ua_2m \in M$-rad(N) or $a_1a_2 \in (N :_R M)$, so we conclude that $\frac{a_1}{s_1}\frac{m}{s} = \frac{ua_1m}{us_1s} \in \mathcal{S}^{-1}(M$-rad$(N)) \subseteq \mathcal{S}^{-1}M$-rad$(\mathcal{S}^{-1}N)$ or $\frac{a_2}{s_2}\frac{m}{s} = \frac{ua_2m}{us_2s} \in \mathcal{S}^{-1}M$-rad$(\mathcal{S}^{-1}N)$ or $\frac{a_1}{s_1}\frac{a_2}{s_2} = \frac{a_1a_2}{s_1s_2} \in \mathcal{S}^{-1}(N :_R M) \subseteq (\mathcal{S}^{-1}N :_{\mathcal{S}^{-1}R} \mathcal{S}^{-1}M)$.

(2) Similar to (1), it is easily obtained. □

Definition 2.12. Let N be a proper submodule of M and $a, b \in R$, $m \in M$.

(1) If N is a ϕ-2-absorbing submodule, $abm \in \phi(N)$, $ab \notin (N :_R M), am \notin N$ and $bm \notin N$, then (a, b, m) is called a *ϕ-triple-zero of N*.

(2) If N is a ϕ-2-absorbing primary submodule, $abm \in \phi(N)$, $ab \notin (N :_R M), am \notin M$-*rad*$(N)$ and $bm \notin M$-*rad*(N), then (a, b, m) is called a *ϕ-primary triple-zero of N*.

Remark 2.13. Note that if N is a ϕ-2-absorbing (resp. ϕ-2-absorbing primary) submodule of M which is not 2-absorbing (resp. 2-absorbing primary), then there exists (a, b, m) a ϕ-triple-zero (resp. ϕ-primary triple-zero) of N for some $a, b \in R$, $m \in M$.

Proposition 2.14. *Let N be a ϕ-2-absorbing submodule of M and $a, b \in R$, $m \in M$. Then (a, b, m) is a ϕ-triple-zero of N if and only if $(a, b, m + \phi(N))$ is a triple-zero of $N/\phi(N)$.*

Proof. Suppose that (a, b, m) is a ϕ-triple-zero of N. Hence $abm \in \phi(N)$ but $ab \notin (N :_R M)$, $am \notin N$ and $bm \notin N$. It implies that $ab \notin (N/\phi(N) :_R M/\phi(N))$, $a(m + \phi(N)) \notin N/\phi(N)$ and $b(m + \phi(N)) \notin N/\phi(N)$. Since $N/\phi(N)$ is a weakly 2-absorbing primary submodule of M by Theorem 2.4, so we conclude that $ab(m + \phi(N)) = \phi(N)$. Thus $(a, b, m + \phi(N))$ is a triple-zero of $N/\phi(N)$. The converse part is easily obtained by the same argument. □

Proposition 2.15. *Let N be a ϕ-2-absorbing primary submodule of M and $a, b \in R$, $m \in M$. Then (a, b, m) is a ϕ-primary triple-zero of N if and only if $(a, b, m + \phi(N))$ is a triple-zero of $N/\phi(N)$.*

Proof. One can easily verify similar to the proof of Proposition 2.14. □

Theorem 2.16. *Let M_1 be an R_1-module, M_2 be an R_2-module, and let $M = M_1 \times M_2$. Let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ $(i = 1, 2)$ be function, and let $\phi = \psi_1 \times \psi_2$. Suppose that $N = N_1 \times M_2$ for some proper submodule N_1 of M_1.*

(1) If $\psi_2(M_2) = M_2$, then N is a ϕ-2-absorbing submodule of M if and only if N_1 is a ψ_1-2-absorbing submodule of M_1.

(2) If $\psi_2(M_2) \neq M_2$, then N is a ϕ-2-absorbing submodule of M if and only if N_1 is a 2-absorbing submodule of M_1.

Proof. (1) Suppose that N is a ϕ-2-absorbing submodule of M. First we show that N_1 is a ψ_1-2-absorbing submodule of M_1 independently whether $\psi_2(M_2) = M_2$ or $\psi_2(M_2) \neq M_2$. Let $a_1b_1m_1 \in N_1\ \psi_1(N_1)$ for some $a_1, b_1 \in R_1$ and $m_1 \in M_1$. Then $(a_1, 1)(b_1, 1)(m_1, m) \in (N_1 \times M_2)\backslash(\psi_1(N_1) \times \psi_2(M_2)) = N\backslash\phi(N)$ for any $m \in M_2$. Since N is a ϕ-2-absorbing submodule of M, we get either $(a_1, 1)(b_1, 1) \in ((N_1 \times M_2) : M_1 \times M_2)$ or $(a_1, 1)(m_1, m) \in (N_1 \times M_2)$ or $(b_1, 1)(m_1, m) \in (N_1 \times M_2)$. So clearly we conclude that $a_1b_1 \in (N_1 : M_1)$ or $a_1m_1 \in N_1$ or $b_1m_1 \in N_1$. Therefore, N_1 is obtained as a ψ_1-2-absorbing submodule of M_1. Conversely, suppose that N_1 is ψ_1-2-absorbing submodule and $\psi_2(M_2) = M_2$. Let $a = (a_1, a_2)$, $b = (b_1, b_2) \in R_1 \times R_2$ and $m = (m_1, m_2) \in M$ such that $abm \in N\backslash\phi(N)$. Since $\psi_2(M_2) = M_2$, we get $a_1b_1m_1 \in N_1\backslash\psi_1(N_1)$ and this implies that either $a_1b_1 \in (N_1 : M_1)$ or $a_1m_1 \in N_1$ or $b_1m_1 \in N_1$. Thus either $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$.

(2) Suppose that $\psi_2(M_2) \neq M_2$ and N is a ϕ-2-absorbing submodule of M. Then there is an element $m_2 \in M_2 \backslash \psi_2(M_2)$. Assume that N_1 is not a 2-absorbing submodule of M_1. As it is shown in part (1), N_1 is a ψ_1-2-absorbing submodule of M_1. Hence there is a ψ_1-triple-zero (a_1, b_1, m_1) for some $a_1, b_1 \in R_1$ and $m_1 \in M_1$ by Remark 2.13. So $(a_1, 1)(b_1, 1)(m_1, m_2) \in (N_1 \times M_2)\backslash(\psi_1(N_1) \times \psi_2(M_2)) = (N_1 \times M_2)\backslash\phi(N_1 \times M_2)$ which clearly implies $a_1b_1 \in (N_1 : M_1)$ or $a_1m_1 \in N_1$ or $b_1m_1 \in N_1$, a contradiction. Thus N_1 is a 2-absorbing submodule of M_1. Conversely, if N_1 is a 2-absorbing submodule of M_1, then $N = N_1 \times M_2$ is a 2-absorbing submodule of M. Hence N is a ϕ-2-absorbing submodule of M for any ϕ. □

Theorem 2.17. *Let M_1 be an R_1-module, M_2 be an R_2-module, and let $M = M_1 \times M_2$. Let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ $(i = 1, 2)$ be function, and let $\phi = \psi_1 \times \psi_2$. Suppose that $N = N_1 \times M_2$ for any proper submodule N_1 of M_1.*

1. If $\psi_2(M_2) = M_2$, then N is a ϕ-2-absorbing primary submodule of M if and only if N_1 is a ψ_1-2-absorbing primary submodule of M_1.
2. If $\psi_2(M_2) \neq M_2$, then N is a ϕ-2-absorbing primary submodule of M if and only if N_1 is a 2-absorbing primary submodule of M_1.

Proof. (1) It can be easily shown by using a similar argument in the proof of Theorem 2.16.

(2) Assume that N_1 is a 2-absorbing primary submodule of M_1. Then $N = N_1 \times M_2$ is a 2-absorbing primary submodule of M by Theorem 2.28 of [23]. Hence N is a ϕ-2-absorbing submodule of M for any ϕ. The remaining of this proof is similar to Theorem 2.16. □

Proposition 2.18. *Let M be a multiplication R-module and let a be an element of R such that $aM \neq M$. Suppose that $(0 :_M a) \subseteq aM$. Then aM is an almost 2-absorbing primary submodule of M if and only if it is a 2-absorbing primary submodule of M.*

Proof. Assume that aM is an almost 2-absorbing primary submodule of M. Let $x, y \in R$ and $m \in M$ such that $xym \in aM$. We show that $xm \in M\text{-rad}(aM)$ or $ym \in M\text{-rad}(aM)$ or $xy \in (aM :_R M)$. If $xym \notin a^2M$, then there is nothing to prove since aM is almost 2-absorbing primary. So assume that $xym \in a^2M$. Note that $(x+a)ym \in aM$. If $(x+a)ym \notin a^2M$, then $(x+a)m \in M\text{-rad}(aM)$ or $ym \in M\text{-rad}(aM)$ or $(x+a)y \in (aM :_R M)$. Hence $xm \in M\text{-rad}(aM)$ or $ym \in M\text{-rad}(aM)$ or $xy \in (aM :_R M)$. Therefore,assume that $(x+a)ym \in a^2M$. Hence $xym \in a^2M$ gives $aym \in a^2M$. Then, there exists $m' \in M$ such that $aym = a^2m'$, and so $am' - ym \in (0 :_M a) \subseteq aM$. Consequently, $ym \in aM$ which shows that aM is 2-absorbing primary. □

A commutative ring R is called a *von Neumann regular ring (or an absolutely flat ring)* if for any $a \in R$ there exists an $x \in R$ with $a^2x = a$, equivalently, $I = I^2$ for every ideal I of R.

Proposition 2.19. *Let R be a von Neumann regular ring, M an R-module and N be a submodule of M.*

1. N is a ϕ-2-absorbing primary submodule of M if and only if $e_1e_2m \in N\backslash\phi(N)$ for some idempotent elements $e_1, e_2 \in R$ and some $m \in M$ implies that either $e_1m \in M\text{-rad}(N)$ or $e_2m \in M\text{-rad}(N)$ or $e_1e_2 \in (N :_R M)$.
2. If M is multiplication, then N is a ω-2-absorbing (ω-2-absorbing primary) submodule of M.

Proof. (1) Notice the fact that any principal (finitely generated) ideal of a von Neumann regular ring R is generated by an idempotent element. On the other hand N is 2-absorbing primary if and only if $(Ra)(Rb)m \subseteq N$ for some $a, b \in R$ and $m \in M$ implies that $(Ra)m \subseteq M\text{-rad}(N)$ or $(Rb)m \subseteq M\text{-rad}(N)$ or $(Ra)(Rb) \subseteq (N :_R M)$.

(2) It is clear that N is idempotent, now see Theorem 2.3(6). □

If N is ϕ-primary submodule, $am \in \phi(N)$, $a \notin \sqrt{(N :_R M)}$ and $m \notin N$, then (a, m) is called a ϕ-primary twin-zero of N.

Theorem 2.20. *Let N be a ϕ-primary submodule of M and suppose that (a, m) is a ϕ-primary twin-zero of N for some $a \in R$, $m \in M$. Then*

(1) $aN \subseteq \phi(N)$.
(2) $(N :_R M)m \subseteq \phi(N)$.
(3) $(N :_R M)N \subseteq \phi(N)$.
(4) $\sqrt{(N :_R M)} = \sqrt{(\phi(N) :_R M)}$.

Proof. (1) Assume that $aN \nsubseteq \phi(N)$. Then there exists $n \in N$ with $an \notin \phi(N)$. Hence $a(m + n) \notin \phi(N)$. Since $a(m + n) \in N$ and $a \notin \sqrt{(N :_R M)}$, we deduce that $m + n \in N$ as N is a ϕ-primary submodule of M. So $m \in N$, which contradicts our hypothesis. Thus $aN \subseteq \phi(N)$.
(2) Let $xm \notin \phi(N)$ for some $x \in (N :_R M)$. Then $(a + x)m \notin \phi(N)$ as $am \in \phi(N)$. Since $xm \in N$, we get $(a + x)m \in N$. Since $m \notin N$, we have that $a + x \in \sqrt{(N :_R M)}$. Hence $a \in \sqrt{(N :_R M)}$ which contradicts the assumption that (a, m) is ϕ-primary twin-zero.
(3) Assume that $(N :_R M)N \nsubseteq \phi(N)$. Hence there are $x \in (N :_R M)$ and $n \in N$ such that $xn \notin \phi(N)$. By parts (1) and (2), $(a+x)(m+n) \in N\backslash\phi(N)$. So either $a+x \in \sqrt{(N :_R M)}$ or $m + n \in N$. Thus we have either $a \in \sqrt{(N :_R M)}$ or $m \in N$, a contradiction.
(4) By part (3) we have

$$(N :_R M)(N :_R M) \subseteq ((N :_R M)N :_R M) \subseteq (\phi(N) :_R M).$$

Therefore $\sqrt{(N :_R M)} = \sqrt{(\phi(N) :_R M)}$. □

Corollary 2.21. *Let M be a multiplication R-module. If N is a ϕ-primary submodule of M that is not primary, then the following statements hold:*

(1) $N^2 \subseteq \phi(N)$.
(2) M-rad(N)=M-rad$(\phi(N))$.

Proof. (1) It is a direct consequence of Theorem 2.20(3).
(2) By Theorem 2.20(4), we have that

$$M\text{-rad}(N) = \sqrt{(N :_R M)}M = \sqrt{(\phi(N) :_R M)}M = M\text{-rad}(\phi(N)).$$

□

A submodule N of an R-module M is called a nilpotent submodule if $(N :_R M)^k N = 0$ for some positive integer k (see [1]), and we say that $m \in M$ is nilpotent if Rm is a nilpotent submodule of M.

Corollary 2.22. *Let N be a weakly primary submodule of an R-module M that is not primary. Then*

(1) *N is nilpotent.*
(2) $\sqrt{(N :_R M)} = \sqrt{Ann_R(M)}$.

Assume that Nil(M) is the set of nilpotent elements of M. If M is faithful, then Nil(M) is a submodule of M and if M is faithful multiplication, then Nil$(M) = $ Nil$(R)M = \bigcap Q$, where the intersection runs over all prime submodules of M, [1, Theorem 6].

Corollary 2.23. *Let N be a weakly primary submodule of a multiplication R-module M. If N is not primary, then then the following statements hold:*

(1) $N^2 = 0$.
(2) M-rad(N)=M-rad$(\{0\})$. *If in addition M is faithful, then* M-rad$(N) = $ Nil(M).

Theorem 2.24. *Let N be a ϕ-2-absorbing (resp. 2-absorbing primary) submodule of M and suppose that (a,b,m) is a ϕ-triple-zero (resp. ϕ-primary triple-zero) of N for some $a,b\in R$, $m\in M$. Then*

(1) $abN\subseteq\phi(N)$.
(2) $a(N:_R M)m\subseteq\phi(N)$.
(3) $b(N:_R M)m\subseteq\phi(N)$.
(4) $(N:_R M)^2m\subseteq\phi(N)$.

Proof. (1) Suppose that N is a ϕ-2-absorbing (resp. 2-absorbing primary) submodule of M and $abN\nsubseteq\phi(N)$. Then there exists $n\in N$ with $abn\notin\phi(N)$. Hence $ab(m+n)\notin\phi(N)$. Since $ab(m+n)=abm+abn\in N$ and $ab\notin(N:_R M)$, we conclude that $a(m+n)\in N$ or $b(m+n)\in N$ (resp. $a(m+n)\in M\text{-}rad(N)$ or $b(m+n)\in M\text{-}rad(N)$). So $am\in N$ or $bm\in N$ (resp. (resp. $am\in M\text{-}rad(N)$ or $bm\in M\text{-}rad(N)$), which contradicts our hypothesis. Thus $abN\subseteq\phi(N)$.

(2) Let $axm\notin\phi(N)$ for some $x\in(N:_R M)$. Then $a(b+x)m\notin\phi(N)$ as $abm\in\phi(N)$. Since $xm\in N$, we obtain $a(b+x)m\in N$. Then $am\in N$ or $(b+x)m\in N$ or $a(b+x)\in(N:_R M)$ (resp. $am\in M\text{-rad}(N)$ or $(b+x)m\in M\text{-rad}(N)$ or $a(b+x)\in(N:_R M)$). Hence $am\in N$ or $bm\in N$ or $ab\in(N:_R M)$ (resp. $am\in M\text{-rad}(N)$ or $bm\in M\text{-rad}(N)$ or $ab\in(N:_R M)$) which contradicts the assumption that (a,b,m) is ϕ-triple-zero (resp. ϕ-primary triple-zero).

(3) The proof is similar to part (2).

(4) Assume that $x_1x_2m\notin\phi(N)$ for some $x_1,x_2\in(N:_R M)$. Then by parts (2) and (3), $(a+x_1)(b+x_2)m\notin\phi(N)$. Clearly $(a+x_1)(b+x_2)m\in N$. Then $(a+x_1)m\in N$ or $(b+x_2)m\in N$ or $(a+x_1)(b+x_2)\in(N:_R M)$ (resp. $(a+x_1)m\in M\text{-rad}(N)$ or $(b+x_2)m\in M\text{-rad}(N)$ or $(a+x_1)(b+x_2)\in(N:_R M)$). Therefore $am\in N$ or $bm\in N$ or $ab\in(N:_R M)$ (resp. $am\in M\text{-rad}(N)$ or $bm\in M\text{-rad}(N)$ or $ab\in(N:_R M)$) which is a contradiction. Consequently, $(N:_R M)^2m\subseteq\phi(N)$. □

Theorem 2.25. *If N is a ϕ-2-absorbing primary submodule of M that is not 2-absorbing primary, then $(N:_R M)^2N\subseteq\phi(N)$.*

Proof. Suppose that N is a ϕ-2-absorbing primary submodule of M that is not 2-absorbing primary. Then there exists a ϕ-primary triple-zero (a,b,m) of N for some $a,b\in R$, $m\in M$. Assume that $(N:_R M)^2N\nsubseteq\phi(N)$. Hence there are $x_1,x_2\in(N:_R M)$ and $n\in N$ such that $x_1x_2n\notin\phi(N)$. By Theorem 2.24, we get $(a+x_1)(b+x_2)(m+n)\in N\backslash\phi(N)$. So $(a+x_1)(m+n)\in M\text{-rad}(N)$ or $(b+x_1)(m+n)\in M\text{-rad}(N)$ or $(a+x_1)(b+x_2)\in(N:_R M)$. Therefore $am\in M\text{-rad}(N)$ or $bm\in M\text{-rad}(N)$ or $ab\in(N:_R M)$, a contradiction. □

Corollary 2.26. *If N is a ϕ-2-absorbing primary submodule of a multiplication R-module M that is not 2-absorbing primary, then $N^3\subseteq\phi(N)$.*

Corollary 2.27. *Let M be a multiplication R-module. If N is a ϕ-2-absorbing primary submodule of M where $\phi\leq\phi_3$, then N is a ω-2-absorbing (ω-2-absorbing primary) submodule of M.*

Corollary 2.28. *Let N be a weakly 2-absorbing primary submodule of M that is not 2-absorbing primary. Then*

(1) *N is nilpotent.*
(2) *If M is a multiplication module, then $N^3=0$.*

Theorem 2.29. *Let N be a ϕ-2-absorbing primary submodule of M. If N is not 2-absorbing primary, then*

(1) $\sqrt{(N:_R M)}=\sqrt{(\phi(N):_R M)}$.
(2) *If M is multiplication, then* $M\text{-rad}(N)=M\text{-rad}(\phi(N))$.

Proof. (1) Assume that N is not 2-absorbing primary. By Theorem 2.25, we known$(N :_R M)^2N \subseteq \phi(N)$. Then

$$\begin{aligned}(N :_R M)^3 &= (N :_R M)^2(N :_R M)\\ &\subseteq ((N :_R M)^2N :_R M)\\ &\subseteq (\phi(N) :_R M),\end{aligned}$$

and so $(N :_R M) \subseteq \sqrt{(\phi(N) :_R M)}$. Hence, we have $\sqrt{(N :_R M)} = \sqrt{(\phi(N) :_R M)}$.
(2) It is clera from part (1). □

Corollary 2.30. *Let N be a weakly 2-absorbing primary submodule of M. If N is not 2-absorbing primary, then*

(1) $\sqrt{(N :_R M)} = \sqrt{Ann_R(M)}$.
(2) *If M is multiplication, then M-rad(N)=M-rad$(\{0\})$. Furthermore, if M is faithful, then M-rad$(N) = \text{Nil}(M)$.*

Corollary 2.31. *Let M be a finitely generated multiplication R-module and suppose that M-rad$(\phi(N))$ is a 2-absorbing submodule of M. If N is a ϕ-2-absorbing primary submodule of M, then M-rad(N) is a 2-absorbing submodule of M.*

Proof. Assume that N is a ϕ-2-absorbing primary submodule of M. If N is a 2-absorbing primary submodule of M, then M-rad(N) is a 2-absorbing submodule of M, by [23, Theorem 2.6]. If N is not a 2-absorbing primary submodule of M, then by Theorem 2.29(2) and by our hypothesis, M-rad$(N) = M$-rad$(\phi(N))$ which is a 2-absorbing submodule. □

Theorem 2.32. *Let M be a multiplication R-module. Suppose that N is a ϕ-primary submodule of M that is not primary, and K is a submodule of M such that $K \subseteq N$ with $\phi(N) \subseteq \phi(K)$. Then K is a ϕ-2-absorbing primary submodule of M.*

Proof. Since N is a ϕ-primary submodule that is not primary we have M-rad$(N) = M$-rad$(\phi(N))$ by Corollary 2.21(2). Hence M-rad$(K) = M$-rad$(N) = M$-rad$(\phi(N))$ since $\phi(N) \subseteq \phi(K)$. Let $abm \in K\backslash\phi(K)$ for some $a, b \in R$ and $m \in M$ such that $ab \notin (K :_R M)$. Since $K \subseteq N$ and $\phi(N) \subseteq \phi(K)$, we have $abm \in N\backslash\phi(N)$. Consider two cases.
Case 1. Assume that $bm \notin N$. Since N is ϕ-primary, then $a \in \sqrt{(N :_R M)}$. Hence $am \in \sqrt{(N :_R M)}M = M$-rad$(N) = M$-rad$(K)$.
Case 2. Assume that $bm \in N$. Since $abm \notin \phi(N)$, we have that $bm \in N\backslash\phi(N)$. On the other hand N is a ϕ-primary submodule, so either $m \in N$ or $b \in \sqrt{(N :_R M)}$. By any of these two possibilities we have $bm \in M$-rad$(N) = M$-rad(K). Consequently, N is a ϕ-2-absorbing primary submodule of M. □

Theorem 2.33. *Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a family of submodules of M such that for every $\lambda, \lambda' \in \Lambda$, M-rad$(\phi(N_\lambda)) = M$-rad$(\phi(N_{\lambda'}))$ and $\phi(N_\lambda) \subseteq \phi(N)$. If for every $\lambda \in \Lambda$, N_λ is a ϕ-2-absorbing primary submodule of M that is not 2-absorbing primary, then $N = \bigcap_{\lambda\in\Lambda} N_\lambda$ is a ϕ-2-absorbing primary submodule of M.*

Proof. Since N_λ's are ϕ-2-absorbing primary but are not 2-absorbing primary, then for every $\lambda \in \Lambda$, M-rad$(N_\lambda) = M$-rad$(\phi(N_\lambda))$, by Theorem 2.29(2). On the other hand $\phi(N_\lambda) \subseteq \phi(N)$ for every $\lambda \in \Lambda$, and so M-rad$(\phi(N_\lambda)) \subseteq M$-rad$(N)$. Hence M-rad$(N) = M$-rad$(N_\lambda) = M$-rad$(\phi(N_\lambda))$ for every $\lambda \in \Lambda$. Let $abm \in N\backslash\phi(N)$ for some $a, b \in R$, $m \in M$, and let $ab \notin (N :_R M)$. Therefore there is a $\lambda \in \Lambda$ such that $ab \notin (N_\lambda :_R M)$. Since N_λ is ϕ-2-absorbing primary and $abm \in N_\lambda\backslash\phi(N_\lambda)$, then $am \in M$-rad$(N_\lambda) = M$-rad(N) or $bm \in M$-rad$(N_\lambda) = M$-rad(N). Consequently, N is a ϕ-2-absorbing primary submodule of M. □

Proposition 2.34. *Let N be a submodule of M and $\phi(N)$ be a 2-absorbing primary submodule of M. If N is a ϕ-2-absorbing primary submodule of M, then N is a 2-absorbing primary submodule of M.*

Proof. Let N be a ϕ-2-absorbing primary submodule of M. Assume that $abm \in N$ for some elements $a, b \in R$ and $m \in M$. If $abm \in \phi(N)$, then we conclude that $am \in M$-rad$(\phi(N)) \subseteq M$-rad(N) or $bm \in M$-rad$(\phi(N)) \subseteq M$-rad(N) or $ab \in (\phi(N) :_R M) \subseteq (N :_R M)$ since $\phi(N)$ is 2-absorbing primary, and so we are done. If $abm \notin \phi(N)$, then clearly the result follows. □

Definition 2.35. Let N be a ϕ-2-absorbing primary submodule of M and suppose that $IJK \subseteq N$ for some ideals I, J of R and any submodule K of M. We call N as a *free ϕ-triple-zero with respect to IJK* if (a, b, k) is not a ϕ-triple-zero of N for every $a \in I, b \in J$ and $k \in K$.

Lemma 2.36. *Let N be a ϕ-2-absorbing primary submodule of M and suppose that $abK \subseteq N$, for some $a, b \in R$ and any submodule K of M. Suppose that (a, b, k) is not a ϕ-triple-zero of N for every $k \in K$. If $ab \notin (N :_R M)$, then $aK \subseteq M$-rad(N) or $bK \subseteq M$-rad(N).*

Proof. Suppose that $ab \notin (N :_R M)$. Assume that $aK \nsubseteq M\text{-}rad(N)$ and $bK \nsubseteq M\text{-}rad(N)$. Then there are $k_1, k_2 \in K$ such that $ak_1 \notin M\text{-}rad(N)$ and $bk_2 \notin M\text{-}rad(N)$. If $abk_1 \notin \phi(N)$, then we have $bk_1 \in M\text{-}rad(N)$ as $ab \notin (N :_R M)$ and N is a ϕ-2-absorbing primary submodule of M. If $abk_1 \in \phi(N)$, then since $abk_1 \in N$, $ab \notin (N :_R M)$, $ak_1 \notin M\text{-}rad(N)$ and (a, b, k_1) is not a ϕ-triple-zero of N, we conclude again $bk_1 \in M\text{-}rad(N)$. By the similar argument, if $abk_2 \notin \phi(N)$, then we get $ak_2 \in M\text{-}rad(N)$ as N is a ϕ-2-absorbing primary submodule of M. Also if $abk_2 \in \phi(N)$, since $abk_2 \in N$, $ab \notin (N :_R M)$, $bk_2 \notin M\text{-}rad(N)$ and (a, b, k_2) is not a ϕ-triple-zero of N, we have $ak_2 \in M\text{-}rad(N)$. From our hypothesis, $(a, b, k_1 + k_2)$ is not a ϕ-triple-zero of N and $ab(k_1 + k_2) \in N$ and $ab \notin (N :_R M)$. Hence we have either $a(k_1 + k_2) \in M\text{-}rad(N)$ or $b(k_1 + k_2) \in M\text{-}rad(N)$. If $a(k_1 + k_2) = ak_1 + ak_2 \in M\text{-}rad(N)$, then since $ak_2 \in M\text{-}rad(N)$, we have $ak_1 \in M\text{-}rad(N)$, a contradiction. If $b(k_1 + k_2) = bk_1 + bk_2 \in M\text{-}rad(N)$, then since $bk_1 \in M\text{-}rad(N)$, we have $bk_2 \in M\text{-}rad(N)$, a contradiction again. Thus $aK \subseteq M\text{-}rad(N)$ or $bK \subseteq M\text{-}rad(N)$. □

Remark 2.37. Let N be a ϕ-2-absorbing primary submodule of M and suppose that $IJK \subseteq N$ for some ideals I, J of R and any submodule K of M such that N is a free ϕ-triple-zero with respect to IJK. Then if $a \in I, b \in J$ and $k \in K$, then $ab \in (N :_R M)$ or $ak \in M\text{-}rad(N)$ or $bk \in M\text{-}rad(N)$.

Theorem 2.38. *Let N be a ϕ-2-absorbing primary submodule of M and suppose that $IJK \subseteq N$, $IJK \nsubseteq \phi(N)$ for some ideals I, J of R, any submodule K of M such that N is a free ϕ-triple-zero with respect to IJK. Then $IJ \subseteq (N :_R M)$ or $IK \subseteq M$-rad(N) or $JK \subseteq M$-rad(N).*

Proof. Suppose that N is a ϕ-2-absorbing primary submodule of M and $IJK \subseteq N$, $IJK \nsubseteq \phi(N)$ for some ideals I, J of R and any submodule K of M such that N is a free ϕ-triple-zero with respect to IJK. Suppose that $IJ \nsubseteq (N :_R M)$. We show that $IK \subseteq M\text{-}rad(N)$ or $JK \subseteq M\text{-}rad(N)$.

On the contrary, assume that $IK \nsubseteq M\text{-}rad(N)$ and $JK \nsubseteq M\text{-}rad(N)$. Then there are $a_1 \in I$ and $b_1 \in J$ with $a_1K \nsubseteq M\text{-}rad(N)$ and $b_1K \nsubseteq M\text{-}rad(N)$. Since $a_1b_1K \subseteq N$, $a_1K \nsubseteq M\text{-}rad(N)$ and $b_1K \nsubseteq M\text{-}rad(N)$, we have $a_1b_1 \in (N :_R M)$ by Lemma 2.36. Recall that our assumption is $IJ \nsubseteq (N :_R M)$. Hence there are $a \in I$, $b \in J$ such that $ab \notin (N :_R M)$. Since $abK \subseteq N$ and $ab \notin (N :_R M)$, we have $aK \subseteq M\text{-}rad(N)$ or $bK \subseteq M\text{-}rad(N)$ by Lemma 2.36. In here there are three cases.

Case 1. Suppose that $aK \subseteq M\text{-}rad(N)$, but $bK \nsubseteq M\text{-}rad(N)$. Since $a_1bK \subseteq N$, $bK \nsubseteq M\text{-}rad(N)$ and $a_1K \nsubseteq M\text{-}rad(N)$, then $a_1b \in (N :_R M)$ by Lemma 2.36. Since $(a + a_1)bK \subseteq N$ and $aK \subseteq M\text{-}rad(N)$, but $a_1K \nsubseteq M\text{-}rad(N)$, we get $(a + a_1)K \nsubseteq M\text{-}rad(N)$. Since $bK \nsubseteq M\text{-}rad(N)$ and $(a+a_1)K \nsubseteq M\text{-}rad(N)$, we have $(a+a_1)b \in (N :_R M)$ by Lemma 2.36. Since $(a + a_1)b = ab + a_1b \in (N :_R M)$ and $a_1b \in (N :_R M)$, we conclude that $ab \in (N :_R M)$, a contradiction.

Case 2. Suppose that $bK \subseteq M\text{-}rad(N)$, but $aK \nsubseteq M\text{-}rad(N)$. Since $ab_1K \subseteq N$, $aK \nsubseteq M\text{-}rad(N)$ and $b_1K \nsubseteq M\text{-}rad(N)$, we deduce that $ab_1 \in (N :_R M)$. Since $a(b+b_1)K \subseteq N$ and $bK \subseteq M\text{-}rad(N)$, but $b_1K \nsubseteq M\text{-}rad(N)$, we have $(b+b_1)K \nsubseteq M\text{-}rad(N)$. Since $aK \nsubseteq M\text{-}rad(N)$ and $(b+b_1)K \nsubseteq M\text{-}rad(N)$, we get $a(b+b_1) \in (N :_R M)$ by Lemma 2.36. Since $a(b+b_1) = ab + ab_1 \in (N :_R M)$ and $ab_1 \in (N :_R M)$, we get $ab \in (N :_R M)$, a contradiction.

Case 3. Suppose that $aK \subseteq M\text{-}rad(N)$ and $bK \subseteq M\text{-}rad(N)$. Hence $(b+b_1)K \nsubseteq M\text{-}rad(N)$ as $bK \subseteq M\text{-}rad(N)$ and $b_1K \nsubseteq M\text{-}rad(N)$. Since $a_1(b+b_1)K \subseteq N$ and neither $a_1K \subseteq M\text{-}rad(N)$ nor $(b+b_1)K \subseteq M\text{-}rad(N)$, we obtain that $a_1(b+b_1) = a_1b + a_1b_1 \in (N :_R M)$ by Lemma 2.36. Since $a_1b_1 \in (N :_R M)$ and $a_1b + a_1b_1 \in (N :_R M)$, we have $ba_1 \in (N :_R M)$. Since $aK \subseteq M\text{-}rad(N)$ and $a_1K \nsubseteq M\text{-}rad(N)$, we deduce that $(a+a_1)K \nsubseteq M\text{-}rad(N)$. Since $(a+a_1)b_1K \subseteq N$, $b_1K \nsubseteq M\text{-}rad(N)$, $(a+a_1)K \nsubseteq M\text{-}rad(N)$, we get $(a+a_1)b_1 = ab_1 + a_1b_1 \in (N :_R M)$ by Lemma 2.36. Since $a_1b_1 \in (N :_R M)$ and $ab_1 + a_1b_1 \in (N :_R M)$, we conclude that $ab_1 \in (N :_R M)$. Now, since $(a+a_1)(b+b_1)K \subseteq N$ and neither $(a+a_1)K \subseteq M\text{-}rad(N)$ nor $(b+b_1)K \subseteq M\text{-}rad(N)$, it follows $(a+a_1)(b+b_1) = ab + ab_1 + ba_1 + a_1b_1 \in (N :_R M)$ by Lemma 2.36. Since $ab_1, ba_1, a_1b_1 \in (N :_R M)$, we get $ab \in (N :_R M)$, a contradiction. Thus $IK \subseteq M\text{-}rad(N)$ or $JK \subseteq M\text{-}rad(N)$. □

Theorem 2.39. *Let N be a submodule of M with $\phi(M\text{-}rad(N)) \subseteq \phi(N)$. If $M\text{-}rad(N)$ is a ϕ-prime submodule of M, then N is a ϕ-2-absorbing primary submodule of M.*

Proof. Suppose that M-rad(N) is a ϕ-prime submodule of M. Let $a, b \in R$ and $m \in M$ be such that $abm \in N \backslash \phi(N)$, $am \notin M$-rad(N). Since M-rad(N) is ϕ-prime submodule and $abm \in M$-rad$(N) \backslash \phi(M\text{-}rad(N))$, then $b \in (M\text{-rad}(N) :_R M)$. So $bm \in M$-rad(N). Consequently, N is a ϕ-2-absorbing primary submodule of M. □

In [24], Quartararo et al. said that a commutative ring R is a u-ring provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a um-ring is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them.

Lemma 2.40. *A ring R is a um-ring if and only if $M \subseteq \bigcup_{i=1}^{n} M_i$, where M_i's are some R-modules, implies that $M \subseteq M_i$ for some $1 \leq i \leq n$.*

Proof. ($\Leftarrow$) It is clear.

($\Rightarrow$) Suppose that R is a um-ring. Let $M \subseteq \bigcup_{i=1}^{n} M_i$ for some R-modules $M_1, M_2, \ldots, M_n$. Then $M = \bigcup_{i=1}^{n} (M_i \cap M)$ and so $M = M_i \cap M$ for some $1 \leq i \leq n$. Therefore $M \subseteq M_i$ for some $1 \leq i \leq n$. □

Theorem 2.41. *Let R be a um-ring, M be an R-module and N be a proper submodule of M. Then the following conditions are equivalent:*

(1) N is a ϕ-2-absorbing submodule of M.
(2) If $ab \notin (N :_R M)$ for some $a, b \in R$, then
$$(N :_M ab) = (N :_M a) \cup (N :_M b) \cup (\phi(N) :_M ab).$$
(3) If $ab \notin (N :_R M)$ for some $a, b \in R$, then $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$ or $(N :_M ab) = (\phi(N) :_M ab)$.
(4) If $abK \subseteq N$ and $abK \nsubseteq \phi(N)$ for some $a, b \in R$ and any submodule K of M, then either $aK \subseteq N$ or $bK \subseteq N$ or $ab \in (N :_R M)$.

(5) If $aK \nsubseteq N$ for some $a \in R$ and some submodule K of M, then $(N :_R aK) = (N :_R K)$ or $(N :_R aK) = (N :_R aM)$ or $(N :_R aK) = (\phi(N) :_R aK)$.
(6) If $aIK \subseteq N$ and $aIK \nsubseteq \phi(N)$ for some $a \in R$, any ideal I of R and any submodule K of M, then either $aK \subseteq N$ or $IK \subseteq N$ or $aI \subseteq (N :_R M)$.
(7) If $IK \nsubseteq N$ for any ideal I of R and any submodule K of M, then $(N :_R IK) = (N :_R K)$ or $(N :_R IK) = (N :_R IM)$ or $(N :_R IK) = (\phi(N) :_R IK)$.
(8) If $IJK \subseteq N$ and $IJK \nsubseteq \phi(N)$ for some ideals I, J of R and any submodule K of M, then either $IK \subseteq N$ or $JK \subseteq N$ or $IJ \subseteq (N :_R M)$.

Proof. (1) ⇒ (2) Suppose that $a, b \in R$ such that $ab \notin (N :_R M)$. Take $m \in (N :_M ab)$. If $abm \in \phi(N)$, then $m \in (\phi(N) :_M ab)$. If $abm \notin \phi(N)$, then $am \in N$ or $bm \in N$ since N is a ϕ-2-absorbing submodule of M. Thus we have $m \in (N :_M a)$ or $m \in (N :_M b)$. Consequently, $(N :_M ab) \subseteq (N :_M a) \cup (N :_M b) \cup (\phi(N) :_M ab)$. On the other hand $(N :_R a) \subseteq (N :_M ab)$, $(N :_M b) \subseteq (N :_M ab)$ and $(\phi(N) :_M ab) \subseteq (N :_M ab)$ are always hold, so we conclude that $(N :_M ab) = (N :_M a) \cup (N :_M b) \cup (\phi(N) :_M ab)$.

(2) ⇒ (3) Assume that $ab \notin (N :_R M)$ for some $a, b \in R$. By part (2), we have $(N :_M ab) = (N :_M a) \cup (N :_M b) \cup (\phi(N) :_M ab)$. Since R is a *um*-ring, then either $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$ or $(N :_M ab) = (\phi(N) :_M ab)$.

(3) ⇒ (4) Suppose that $abK \subseteq N$ and $abK \nsubseteq \phi(N)$ for some $a, b \in R$ and any submodule K of M. Then $K \subseteq (N :_M ab)$. Assume that $ab \notin (N :_R M)$. Then by part (3), we have $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$ or $(N :_M ab) = (\phi(N) :_M ab)$. Hence $K \subseteq (N :_M a)$ or $K \subseteq (N :_M b)$ or $K \subseteq (\phi(N) :_M ab)$. In the first case, we have $aK \subseteq N$, and in the second case, we have $bK \subseteq N$. Notice that the third case can not hold as $abK \nsubseteq \phi(N)$.

(4) ⇒ (5) Let $aK \nsubseteq N$ for some $a \in R$ and some submodule K of M. Assume that $x \in (N :_R aK)$. Then $axK \subseteq N$. If $axK \subseteq \phi(N)$, then $x \in (\phi(N) :_R aK)$. We may assume that $axK \nsubseteq \phi(N)$. Then by part (4), we conclude that either $aK \subseteq N$ or $xK \subseteq N$ or $ax \in (N :_R M)$. By assumption, the first case can not happen. Therefore $x \in (N :_R K)$ or $x \in (N :_R aM)$. So $(N :_R aK) = (N :_R K) \cup (N :_R aM) \cup (\phi(N) :_R aK)$. Now, since R is a *um*-ring, then $(N :_R aK) = (N :_R K)$ or $(N :_R aK) = (N :_R aM)$ or $(N :_R aK) = (\phi(N) :_R aK)$.

(5) ⇒ (6) Let $aIK \subseteq N$ and $aIK \nsubseteq \phi(N)$ for some $a \in R$, any ideal I of R and any submodule K of M. Then $I \subseteq (N :_R aK)$. If $aK \subseteq N$, then we are done. Let $aK \nsubseteq N$. By part (5), $(N :_R aK) = (N :_R K)$ or $(N :_R aK) = (N :_R aM)$ or $(N :_R aK) = (\phi(N) :_R aK)$. Since $aIK \subseteq N \backslash \phi(N)$, then $(N :_R aK) \neq (\phi(N) :_R aK)$. If $(N :_R aK) = (N :_R K)$, then $IK \subseteq N$. If $(N :_R aK) = (N :_R aM)$, then $aI \subseteq (N :_R M)$.

(6) ⇒ (7), (7) ⇒ (8) have similar proof to that of the previous implications.

(8) ⇒ (1) is trivial. □

Theorem 2.42. *Let R be a um-ring, N be a proper submodule of an R-module M. Then the following conditions are equivalent:*

(1) *N is a ϕ-2-absorbing primary submodule of M.*
(2) *If $ab \notin (N :_R M)$ for some $a, b \in R$, then*

$$(N :_M ab) \subseteq (M\text{-}rad(N) :_M a) \cup (M\text{-}rad(N) :_M b) \cup (\phi(N) :_M ab).$$

(3) *If $ab \notin (N :_R M)$ for some $a, b \in R$, then $(N :_M ab) \subseteq (M\text{-}rad(N) :_M a)$ or $(N :_M ab) \subseteq (M\text{-}rad(N) :_M b)$ or $(N :_M ab) = (\phi(N) :_M ab)$.*
(4) *If $abK \subseteq N$ and $abK \nsubseteq \phi(N)$ for some $a, b \in R$ and any submodule K of M, then either $aK \subseteq M\text{-}rad(N)$ or $bK \subseteq M\text{-}rad(N)$ or $ab \in (N :_R M)$.*
(5) *If $aK \nsubseteq M\text{-}rad(N)$ for some $a \in R$ and any submodule K of M, then $(N :_R aK) \subseteq (M\text{-}rad(N) :_R K)$ or $(N :_R aK) = (N :_R aM)$ or $(N :_R aK) = (\phi(N) :_R aK)$.*
(6) *If $aIK \subseteq N$ and $aIK \nsubseteq \phi(N)$ for some $a \in R$, any ideal I of R and any submodule K of M, then either $aK \subseteq M\text{-}rad(N)$ or $IK \subseteq M\text{-}rad(N)$ or $aI \subseteq (N :_R M)$.*

(7) *If $IK \nsubseteq M$-rad(N) for any ideal I of R and any submodule K of M, then $(N :_R IK) \subseteq (M$-rad$(N) :_R K)$ or $(N :_R IK) = (N :_R IM)$ or $(N :_R IK) = (\phi(N) :_R IK)$.*

(8) *If $IJK \subseteq N$ and $IJK \nsubseteq \phi(N)$ for some ideals I, J of R and any submodule K of M, then either $IK \subseteq M$-rad(N) or $JK \subseteq M$-rad(N) or $IJ \subseteq (N :_R M)$.*

Proof. (1) ⇒ (2) Suppose that $a, b \in R$ such that $ab \notin (N :_R M)$ and take $m \in (N :_M ab)$. Then $abm \in N$. If $abm \notin \phi(N)$, then $am \in M$-*rad*(N) or $bm \in M$-*rad*(N) as N is a ϕ-2-absorbing primary submodule of M. Therefore $m \in (M$-*rad*$(N) :_M a)$ or $m \in (M$-*rad*$(N) :_M b)$. If $abm \in \phi(N)$, then $m \in (\phi(N) :_M ab)$. Thus $(N :_M ab) \subseteq (M$-*rad*$(N) :_M a) \cup (M$-*rad*$(N) :_M b) \cup (\phi(N) :_M ab)$.

(2) ⇒ (3) Assume that $ab \notin (N :_R M)$ for some $a, b \in R$. Hence we have $(N :_M ab) \subseteq (M$-*rad*$(N) :_M a) \cup (M$-*rad*$(N) :_M b) \cup (\phi(N) :_M ab)$ by part (2). Since R is a *um*-ring, and $(\phi(N) :_M ab) \subseteq (N :_M ab)$ is always satisfied, we conclude that either $(N :_M ab) \subseteq (M$-*rad*$(N) :_M a)$ or $(N :_M ab) \subseteq (M$-*rad*$(N) :_M b)$ or $(N :_M ab) = (\phi(N) :_M ab)$.

(3) ⇒ (4) Assume that $abK \subseteq N$ and $abK \nsubseteq \phi(N)$ for some $a, b \in R$ and aany submodule K of M. Then $K \subseteq (N :_M ab)$. Suppose that $ab \notin (N :_R M)$. Then by (3) we have $(N :_M ab) \subseteq (M$-*rad*$(N) :_M a)$ or $(N :_M ab) \subseteq (M$-*rad*$(N) :_M b)$ or $(N :_M ab) = (\phi(N) :_M ab)$. Thus $K \subseteq (M$-*rad*$(N) :_M a)$ or $K \subseteq (M$-*rad*$(N) :_M b)$ or $K \subseteq (\phi(N) :_M ab)$. The first case implies that $aK \subseteq M$-*rad*(N), and in the second case, we have $bK \subseteq M$-*rad*(N). The third case can not hold, because $abK \nsubseteq \phi(N)$.

(4) ⇒ (5) Suppose that $aK \nsubseteq M$-*rad*(N) for some $a \in R$ and any submodule K of M. Assume that $b \in (N :_R aK)$. Then $abK \subseteq N$. If $abK \subseteq \phi(N)$, then $b \in (\phi(N) :_R aK)$. We may assume that $abK \nsubseteq \phi(N)$. Then by (4), either $aK \subseteq M$-*rad*(N) or $bK \subseteq M$-*rad*(N) or $ab \in (N :_R M)$. By assumption, the first case can not happen. Therefore $b \in (M$-*rad*$(N) :_R K)$ or $b \in (N :_R aM)$. So $(N :_R aK) \subseteq (M$-*rad*$(N) :_R K) \cup (N :_R aM) \cup (\phi(N) :_R aK)$. Now, since R is a *um*-ring, then we conclude that $(N :_R aK) \subseteq (M$-*rad*$(N) :_R K)$ or $(N :_R aK) = (N :_R aM)$ or $(N :_R aK) = (\phi(N) :_R aK)$.

(5) ⇒ (6) Let $aIK \subseteq N$ and $aIK \nsubseteq \phi(N)$ for some $a \in R$, any ideal I of R and any submodule K of M. Then $I \subseteq (N :_R aK)$. If $aK \subseteq M$-*rad*(N), then we are done. Let $aK \nsubseteq M$-*rad*(N). By part (5), $(N :_R aK) \subseteq (M$-*rad*$(N) :_R K)$ or $(N :_R aK) = (N :_R aM)$ or $(N :_R aK) = (\phi(N) :_R aK)$. Since $aIK \subseteq N \backslash \phi(N)$, then $(N :_R aK) \neq (\phi(N) :_R aK)$. If $(N :_R aK) \subseteq (M$-*rad*$(N) :_R K)$, then $IK \subseteq M$-*rad*(N). If $(N :_R aK) = (N :_R aM)$, then $aI \subseteq (N :_R M)$.

The proofs of (6) ⇒ (7), (7) ⇒ (8) are similar to the previous implications.

(8) ⇒ (1) is obvious. □

Theorem 2.43. *Let N be a proper submodule of an R-module M. If N is a ϕ-2-absorbing primary submodule of M, then the following statements hold:*

(1) *If $abm \notin N$ for $a, b \in R$, $m \in M$, then $(N :_R abm) = (M$-rad$(N) :_R am) \cup (M$-rad$(N) :_R bm) \cup (\phi(N) :_R abm)$.*

(2) *Let R be a u-ring. If $abm \notin N$ for $a, b \in R$, $m \in M$, then $(N :_R abm) \subseteq (M$-rad$(N) :_R am)$ or $(N :_R abm) \subseteq (M$-rad$(N) :_R bm)$ or $(N :_R abm) = (\phi(N) :_R abm)$.*

Proof. (1) Suppose that $abm \notin N$ for some $a, b \in R$, $m \in M$. Take $r \in (N :_R abm)$. Then $rabm \in N$. If $rabm \in \phi(N)$, then $r \in (\phi(N) :_R abm)$. So assume that $rabm \notin \phi(N)$. Hence we conclude either $ram \in M$-*rad*(N) or $rbm \in M$-*rad*(N) or $ab \in (N :_R M)$ as N is a ϕ-2-absorbing primary submodule of M. But $ab \notin (N :_R M)$ since $abm \notin N$. So $r \in (M$-*rad*$(N) :_R am)$ or $r \in (M$-*rad*$(N) :_R bm)$. Thus $(N :_R abm) = (M$-*rad*$(N) :_R am) \cup (M$-*rad*$(N) :_R bm) \cup (\phi(N) :_R abm)$.

(2) Suppose that R is a u-ring. Then the result is obtained from part (1). □

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