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Research Article

Some special differential subordinations

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Dedicated to Dato' Rosihan M. Ali, on the occasion of his sixtieth birthday

Abstract

For an analytic function p satisfying $p(0) = 1$, we obtain sharp estimates on β such that the first order differential subordination $p(z) + \beta zp'(z) \prec \mathcal{P}(z)$ or $1 + \beta zp'(z)/p^j(z) \prec \mathcal{P}(z)$, $(j = 0, 1, 2)$ implies $p(z) \prec \mathcal{Q}(z)$ where $\mathcal{P}$ and $\mathcal{Q}$ are Carathéodory functions. The key tools in the proof of main results are the theory of differential subordination and some properties of hypergeometric functions. Further, these subordination results immediately give sufficient conditions for an analytic function f to be in various well-known subclasses of starlike functions.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ be the class of univalent functions in $\mathcal{A}$. The function $f \in \mathcal{A}$ is said to be subordinate to the function $g \in \mathcal{A}$, written as $f(z) \prec g(z)$, if there is an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. Ma and Minda[17] studied distortion, growth, covering and coefficient estimates for starlike and convex functions for which either of the quantity $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a univalent superordinate function. For this purpose, they considered an analytic function φ with positive real part in the unit disk $\mathbb{D}$ and normalized by $\varphi(0) = 1$ and $\varphi'(0) > 0$. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $zf'(z)/f(z) \prec \varphi(z)$ and is denoted by $\mathcal{S}^*(\varphi)$. The convolution properties of functions in a general class $\mathcal{S}^*_g(\varphi)$ of all $f \in \mathcal{A}$ satisfying $z(f(z) * g(z))'/(f(z) * g(z)) \prec \varphi(z)$, where φ is a convex function, g is a fixed function in $\mathcal{A}$, and $*$ is the Hadamard product, was studied by Shanmugam[27]. On taking $g(z) = z/(1 - z)$, the subclass $\mathcal{S}^*_g(\varphi)$ reduces to class $\mathcal{S}^*(\varphi)$. For special choices of φ, the class $\mathcal{S}^*(\varphi)$ reduces to well known subclasses of starlike functions. For example $\mathcal{S}^*_L := \mathcal{S}^*(\sqrt{1+z})$ is the subclass of $\mathcal{S}^*$ introduced by Sokól and Stankiewicz [29] consisting

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of functions $f \in \mathcal{A}$ such that for each $z \in \mathbb{D}$, $w = zf'(z)/f(z)$ lies in the region bounded by right half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. On taking $\varphi(z) := (1 + Az)/(1 + Bz), (-1 \leq B < A \leq 1)$, the class $\mathcal{S}^*(\varphi)$ reduces to $\mathcal{S}^*[A, B]$ introduced by Janowski [13] and for $\varphi(z) = \varphi_c(z) := 1 + 4z/3 + 2z^2/3$, $\mathcal{S}^*(\varphi)$ reduces to the class $\mathcal{S}^*_c$ associated with cardiod, introduced and studied by Sharma *et al.*[28]. The class $\mathcal{S}^*_e = \mathcal{S}^*(e^z)$ introduced by Mendiratta *et al.* [20] consists of $f \in A$ satisfying the condition $|\log zf'(z)/f(z)| < 1$. Recently, Kumar and Ravichandran [15], introduced and studied the geometric properties of the class $\mathcal{S}^*_R = \mathcal{S}^*(\varphi_0)$ associated with the rational function

$$\varphi_0(z) := 1 + \frac{z}{k}\left(\frac{k+z}{k-z}\right), \quad (k = \sqrt{2} + 1). \tag{1.1}$$

Further Cho *et al.*[8] discussed the various radius and coefficient estimates of the function f in the class $\mathcal{S}^*_s = \mathcal{S}^*(\varphi_s)$ where $\varphi_s(z) := 1 + \sin z$. More details regarding these classes can be found in [4, 6, 9, 11, 23, 25].

In [24], authors consider certain subclasses of starlike and convex functions of complex order, giving necessary and sufficient conditions for functions to belong to these classes. By making use of subordination, Tuneski [30] introduced an interesting criteria for analytic functions to be in the class of Janowski starlike functions and studied a linear combination of starlike functions. A Carathéodory function $p : \mathbb{D} \to \mathbb{C}$ is of the form $p(z) = 1 + c_1 z + c_2 z + \cdots$ with $\operatorname{Re}(p(z)) > 0$. These functions are analytic in $\mathbb{D}$ and maps $\mathbb{D}$ into the right half plane. The function $p(z) = (1+z)/(1-z)$ is a leading example of function with positive real part and maps $\mathbb{D}$ onto the right-half plane. Ali *et al.*[3] determined conditions on β for which $1+\beta zp'(z)/p^j(z) \prec (1+Dz)/(1+Ez)$ $(j = 0, 1, 2)$ implies $p(z) \prec (1+Az)/(1+Bz)$, where $A, B, C, D, E, F \in [-1, 1]$. Ali *et al.*[2] determined conditions on β for $p(z) \prec \sqrt{1+z}$ when $1 + \beta zp'(z)/p^j(z) \prec \sqrt{1+z}$. In 2013, authors [14] obtained the bound on β with $-1 < E < 1$ and $|D| \leq 1$ such that $1 + \beta zp'(z)/p^j(z) \prec (1 + Dz)/(1 + Ez)$ $(j = 0, 1, 2)$ implies $p(z) \prec \sqrt{1+z}$. These results are not sharp. Recently Kumar and Ravichandran [16] obtained sharp estimates on β for which the subordination $1 + \beta zp'(z)/p^j(z)$ $(j = 0, 1, 2) \prec \varphi_0(z), \sqrt{1+z}, (1 + Az)/(1 + Bz), \varphi_s(z)$ implies $p(z) \prec e^z, (1 + Az)/(1 + Bz)$. They further used these results to obtain sufficient conditions for $f \in \mathcal{A}$ to be in certain subclasses of starlike functions. For more details, see [1, 7, 10, 18, 22].

Motivated by above work, we consider the subordination inclusions in which we determine the sharp bound on parameter β so that a given differential subordination implication holds. In the second section, we obtain estimates on β for which the differential subordination $p(z) + \beta zp'(z) \prec e^z, \sqrt{1+z}$ or $1 + z$ implies $p(z) \prec e^z, \sqrt{1+z}$, $\varphi_0(z)$ and $\varphi_s(z)$ by using the theory of hypergeometric functions. Third section provides conditions on β so that the subordination $1 + \beta zp'(z)/p^j(z) \prec \varphi_0(z)$ $(j = 0, 1, 2)$ implies $p(z) \prec \varphi_0(z)$, $\varphi_s(z)$, $\sqrt{1+z}$. In the next section, we obtain estimates on β so that $p(z) \prec e^z, (1 + Az)/(1 + Bz), \varphi_0(z), \varphi_s(z)$, $\sqrt{1+z}$ when the subordination $1 + \beta zp'(z)/p^j(z) \prec \varphi_c(z)$ $(j = 0, 1, 2)$ holds. In this sequel, we also obtain sharp estimates on β so that the subordination $1 + \beta zp'(z)/p^j(z) \prec \varphi_s(z)$ $(j = 0, 1, 2)$ implies $p(z) \prec \varphi_c(z), \varphi_s(z)$. In the last section, conditions on β are obtained so that the subordination $1+\beta zp'(z)/p^j(z) \prec e^z$ $(j = 0, 1, 2)$ implies $p(z) \prec e^z, \varphi_0(z), \varphi_s(z), \varphi_c(z)$. The estimates obtained on β are sharp and improved upon the earlier some known estimates. Further, several sufficient conditions are obtained for $f \in \mathcal{A}$ to be in certain well known subclasses of starlike functions as an application of these subordination results.

2. Subordination and hypergeometric functions

In the first result of this section, we consider the differential subordination $p(z) + \beta zp'(z) \prec e^z$ and obtain estimates on β so that $p(z) \prec e^z, \sqrt{1+z}, \varphi_s(z) = 1 + \sin z, \varphi_0(z)$ where $\varphi_0(z)$ is given by (1.1). Before stating the theorem, we recall definition and few properties of confluent hypergeometric function that are used in the proof of the next theorem. The confluent (or Kummer) hypergeometric function $\Phi(a,c;z)$ is given by the convergent power series

$$\Phi(a,c;z) = {}_1F_1(a,c;z) := 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \cdots, \tag{2.1}$$

where a and c are complex numbers with $c \neq 0, -1, -2, \ldots$. The function Φ is analytic in $\mathbb{C}$ and satisfies the Kummer's differential equation

$$zw''(z) + (c-z)w'(z) - aw(z) = 0.$$

Let $(d)_k$ denotes the Pochhammer symbol given by $(d)_k = \Gamma(d+k)/\Gamma(d) = d(d+1)\cdots(d+k-1)$ and $(d)_0 = 1$, then (2.1) can be written in the form

$$\Phi(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)}\frac{z^k}{k!}.$$

Also $c\Phi'(a,c;z) = a\phi(a+1,c+1;z)$ and the following integral representation of Φ [19, p. 5] given by

$$\Phi(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tz}dt = \int_0^1 e^{tz}d\mu(t), \tag{2.2}$$

is well known, where $d\mu(t) = \dfrac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}t^{a-1}(1-t)^{c-a-1}dt$ is a probability measure on $[0,1]$ and $\operatorname{Re} c > \operatorname{Re} a > 0$. For latest details, see [5, 21].

Theorem 2.1. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Let*

$$p(z) + \beta zp'(z) \prec e^z, \quad \beta > 0.$$

Then the following are true:

(a) $p(z) \prec e^z$.

(b) *If $\beta \geq \beta_L \simeq 2.35$, then $p(z) \prec \sqrt{1+z}$ where β_L is the unique root of*

$$\sqrt{2} - \sum_{k=0}^{\infty} \frac{1}{k!(1+\beta k)} = 0 \quad \text{in } (0,\infty).$$

(c) *If $\beta \geq \beta_s \simeq 0.73$, then $p(z) \prec \varphi_s(z)$ where β_s is the unique root of*

$$\sin 1 - \sum_{k=1}^{\infty} \frac{1}{k!(1+\beta k)} = 0 \quad \text{in } (0,\infty).$$

(d) *If $\beta \geq \beta_0 \simeq 3.51$, then $p(z) \prec \varphi_0(z)$ where β_0 is unique root of*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(1+\beta k)} - 2(\sqrt{2}-1) = 0 \quad \text{in } (0,\infty).$$

The bounds on β are sharp.

In order to prove this result, we need the following lemma due to Miller and Mocanu:

Lemma 2.2 ([19, Theorem 3.4h., p. 132])**.** *Let q be analytic in $\mathbb{D}$ and let θ and φ be analytic in domain U containing $q(\mathbb{D})$, with $\varphi(w) \neq 0$, when $w \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\varphi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$ and suppose that either h is convex or Q is starlike. In addition, assume that $\operatorname{Re} zh'(z)/Q(z) > 0$. If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subset \mathbb{D}$ and*

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) = h(z)$$

then $p \prec q$, and q is the best dominant.

Proof of Theorem 2.1. Consider the following first order linear differential equation

$$q(z) + \beta z q'(z) = e^z.$$

Its solution $q = q_\beta$ is given by

$$q_\beta(z) = \frac{1}{\beta}\int_0^1 e^{zt} t^{\frac{1}{\beta}-1} dt.$$

Using the integral representation given in (2.2), we see that $q_\beta(z)$ reduces to the confluent hypergeometric function given by $\Phi(\frac{1}{\beta}, \frac{1}{\beta}+1; z)$. Clearly, q_β is analytic in $\mathbb{D}$. Define $\varphi(w) := \beta$ and $\theta(w) := w$, $w \in \mathbb{C}$. Let

$$Q(z) = zq'_\beta(z)\varphi(q_\beta(z)) = \beta z q'_\beta(z) = \frac{\beta}{\beta+1} z\Phi(\frac{1}{\beta}+1, \frac{1}{\beta}+2; z).$$

To see that Q is starlike, we use [19, Corollary 4.5c.1.] which says that $z\Phi(a, c; z)$ is starlike if $c - 1 \geq N(a-1)$ where $N(a-1)$ is given by

$$N(a-1) = \begin{cases} |a-1| + 1/2 & \text{if} \quad |a-1| \geq 1/3, \\ 3(a-1)^2/2 + 2/3 & \text{if} \quad |a-1| \leq 1/3. \end{cases}$$

Here $a = 1/\beta + 1$ and $c = 1/\beta + 2$. For $\beta \geq 1/3$, $1/\beta + 1 > 1/\beta + 1/2$ and for $1/\beta \leq 1/3$, we have $3(a-1)^2/2 + 2/3 = 3/2\beta^2 + 2/3 \leq 5/6 < 1 + 1/\beta$. Thus $(c-1) \geq N(a-1)$.

Since Q is starlike and β is positive, the function $h : \mathbb{D} \to \mathbb{C}$ defined by $h(z) = \theta(q_\beta(z)) + Q(z) = q_\beta(z) + Q(z)$ satisfies $\operatorname{Re}(zh'(z)/Q(z)) = \operatorname{Re}(1/\beta + zQ'(z)/Q(z)) > 0$. Hence by making use of Lemma 2.2, we see that the subordination $p(z) + \beta z p'(z) \prec q_\beta(z) + \beta z q'_\beta(z)$ implies $p(z) \prec q_\beta(z)$. The required subordination $p(z) \prec \mathcal{P}(z)$ for different choices of $\mathcal{P}$ holds if $q_\beta(z) \prec \mathcal{P}(z)$ and the subordination $q_\beta(z) \prec \mathcal{P}(z)$ holds, then

$$\mathcal{P}(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \mathcal{P}(1).$$

Fortunately, this condition becomes sufficient also for appropriate values of β as can be seen by plotting the graphs of respective functions.

(a) For $\mathcal{P}(z) = e^z$, the inequalities $e^{-1} \leq q_\beta(-1)$ and $q_\beta(1) \leq e$ reduces to

$$f(\beta) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(1+\beta k)} - \frac{1}{e} \geq 0 \quad \text{and} \quad g(\beta) := e - \sum_{k=0}^{\infty} \frac{1}{k!(1+\beta k)} \geq 0.$$

We note that $\lim_{\beta \searrow 0} f(\beta) = \lim_{\beta \searrow 0} g(\beta) = 0$ and both $f'(\beta)$ and $g'(\beta)$ are strictly positive for every $\beta \in (0, \infty)$. Hence both f and g are increasing functions. Thus both inequalities are true for any $\beta > 0$.

(b) For $\mathcal{P}(z) = \sqrt{1+z}$, the inequalities $0 \leq q_\beta(-1)$ and $q_\beta(1) \leq \sqrt{2}$ reduces to

$$f(\beta) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(1+\beta k)} \geq 0 \quad \text{and} \quad g(\beta) := \sqrt{2} - \sum_{k=0}^{\infty} \frac{1}{k!(1+\beta k)} \geq 0.$$

Here $f(\beta) > 0$ for all $\beta \in (0, \infty)$. We note that $\lim_{\beta \searrow 0} g(\beta) = \sqrt{2} - e \simeq -1.3041 < 0$ and $\lim_{\beta \nearrow \infty} g(\beta) = \sqrt{2} - 1 > 0$. Also $g'(\beta) > 0$ for all $\beta \in (0, \infty)$. Hence g is strictly increasing in $(0, \infty)$. Let β_L denotes the unique zero of g in $(0, \infty)$. Then $g(\beta) \geq 0$ for every $\beta \geq \beta_0 \simeq 2.35$. Thus $p(z) \prec \sqrt{1+z}$ for all $\beta \geq \beta_L \simeq 2.35$.

(c) For $\mathcal{P}(z) = \varphi_s(z)$, the inequalities $\varphi_s(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \varphi_s(1)$ reduces to

$$f(\beta) := \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(1+\beta k)} + \sin 1 \geq 0 \text{ and } g(\beta) := \sin 1 - \sum_{k=1}^{\infty} \frac{1}{k!(1+\beta k)} \geq 0.$$

Again, $f(\beta) > 0$ for all $\beta \in (0, \infty)$ and $\lim_{\beta \searrow 0} g(\beta) = \sin 1 - e + 1 \simeq -0.876811 < 0$, while $\lim_{\beta \nearrow \infty} g(\beta) = \sin 1 > 0$. Also $g'(\beta) > 0$ for all $\beta \in (0, \infty)$. Hence g is strictly increasing in $(0, \infty)$. Let β_s denotes the unique zero of g in $(0, \infty)$. Then $g(\beta) \geq 0$ for every $\beta \geq \beta_s \simeq 0.73$. Thus $p(z) \prec 1 + \sin z$ for all $\beta \geq \beta_s \simeq 0.73$.

(d) Let $\mathcal{P}(z) = \varphi_0(z)$. Then the inequalities $\varphi_0(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \varphi_0(1)$ reduces to

$$f(\beta) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(1+\beta k)} - 2(\sqrt{2}-1) \geq 0 \text{ and } g(\beta) := 2 - \sum_{k=0}^{\infty} \frac{1}{k!(1+\beta k)} \geq 0.$$

We note that $\lim_{\beta \searrow 0} f(\beta) \simeq -0.460548 < 0$, $\lim_{\beta \nearrow \infty} f(\beta) = 0.171573 > 0$ and $f'(\beta) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!(1+\beta k)^2} > 0$ for all $\beta \in (0,\infty)$. Hence the function f is strictly increasing. Let $\beta_0 \simeq 3.51$ be the unique root of $f(\beta) = 0$ in $(0,\infty)$. Similarly $\lim_{\beta \searrow 0} g(\beta) \simeq -0.718282 < 0$, $\lim_{\beta \nearrow \infty} g(\beta) = 1 > 0$ and $g'(\beta) = \sum_{k=1}^{\infty} \frac{1}{(k-1)!(1+\beta k)^2} > 0$ for all $\beta \in (0,\infty)$. Hence the function g is also strictly increasing. Let $\beta_1 \simeq 0.54$ be the unique root of $g(\beta) = 0$ in $(0,\infty)$. Then $p(z) \prec \varphi_0(z)$ if $\beta \geq \max\{\beta_0, \beta_1\} = \beta_0 \simeq 3.51$. □

Remark 2.3. As a consequence of the previous theorem, let the function $f \in \mathcal{A}$ satisfies the following subordination

$$\frac{zf'(z)}{f(z)}\left(1+\beta\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right)\right) \prec e^z.$$

Then $f \in \mathcal{S}_e^*$ for $\beta \geq 0$, $f \in \mathcal{S}_L^*$ for $\beta \geq 2.35$, $f \in \mathcal{S}_s^*$ for $\beta \geq 0.73$, and $f \in \mathcal{S}_R^*$ for $\beta \geq 3.51$.

The next theorem uses the properties of Gaussian hypergeometric function. We recall that the Gaussian hypergeometric function ${}_2F_1(a,b,c;z)$ is given by the convergent power series

$$F(a,b,c;z) = {}_2F_1(a,b,c;z) := 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots,$$

where a, b and c are complex numbers with $c \neq 0, -1, -2, \ldots$. The function F is analytic in $\mathbb{C}$ and satisfies the hypergeometric differential equation

$$z(1-z)w''(z) + (c-(a+b+1)z)w'(z) - abw(z) = 0.$$

Using the Pochhammer symbol, we can rewrite F as

$$F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k!}.$$

Also, it is well known that $cF'(a,b,c;z) = abF(a+1,b+1,c+1;z)$ and if $\operatorname{Re} c > \operatorname{Re} b > 0$ then

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt. \tag{2.3}$$

Following result of Hästö *et al.* [12] will be used in proving the next theorem.

Theorem 2.4. *Let a, b and c be nonzero real numbers such that $F(a,b,c;z)$ has no zeros in $\mathbb{D}$. Then $zF(a,b,c;z)$ is starlike if*

(1) $c \geq \max\{1+a+b-ab, 2+2ab-(a+b)\}$ *and*
(2) $(c-1)(c-2) \geq a^2+b^2-ab-a-b$.

Theorem 2.5. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Let the subordination*

$$p(z) + \beta zp'(z) \prec \sqrt{1+z}, \quad \beta > 0$$

holds. Let $\chi(\beta)$ denotes the infinite series given by

$$\chi(\beta) = \frac{1}{\Gamma(-\frac{1}{2})} \sum_{k=0}^{\infty} \frac{\Gamma(-\frac{1}{2}+k)}{k!(1+\beta k)}.$$

Then the following are true:

(a) *If $\beta \geq \beta_e \simeq 0.198099$, then $p(z) \prec e^z$ where β_e is the unique root of $\chi(\beta) - 1/e = 0$ in $(0,\infty)$.*

(b) *If* $\beta \geq \beta_s \simeq 0.0327862$, *then* $p(z) \prec \varphi_s(z)$ *where* β_s *is the unique root of* $\chi(\beta) - 1 + \sin 1 = 0$ *in* $(0, \infty)$.

(c) *If* $\beta \geq \beta_0 \simeq 2.71181$, *then* $p(z) \prec \varphi_0(z)$ *where* β_0 *is the unique root of* $\chi(\beta) - 2(\sqrt{2} - 1) = 0$ *in* $(0, \infty)$.

(d) *If* $\beta \geq \beta_c \simeq 0.158374$, *then* $p(z) \prec \varphi_c(z) = 1 + 4z/3 + 2z^2/3$ *where* β_c *is the unique root of* $\chi(\beta) - 1/3 = 0$ *in* $(0, \infty)$.

Estimates on β *are sharp.*

Proof. Consider the linear differential equation $q(z) + \beta z q'(z) = \sqrt{1+z}$. Its solution $q = q_\beta$ is given by the function

$$q_\beta(z) = \frac{1}{\beta} \int_0^1 t^{\frac{1}{\beta}-1} (1 + zt)^{\frac{1}{2}} dt.$$

Using integral representation of Gauss hypergeometric function given in equation (2.3), we see that $q_\beta(z) = F(-1/2, 1/\beta, 1/\beta + 1; -z)$. Clearly $q_\beta(z)$, is analytic in $\mathbb{D}$. As defined in previous theorem, let $\varphi(w) := \beta$ and $\theta(w) := w$, $w \in \mathbb{C}$. Then

$$Q(z) = z q_\beta'(z) \varphi(q_\beta(z)) = \beta z q_\beta'(z) = \frac{\beta}{2(\beta+1)} z F\left(\frac{1}{2}, \frac{1}{\beta} + 1, \frac{1}{\beta} + 2; -z\right).$$

To see that the function Q is starlike in $\mathbb{D}$, we use Theorem 2.4. Here $a = 1/2, b = 1/\beta + 1$, and $c = 1/\beta + 2$. Since $0 < a < b < c$, by [12, Lemma 1.11], $F(a, b, c; z)$ has no zeros in $\mathbb{D}$. Now

(1) $1 + a + b - ab = 2 + \frac{1}{2\beta} < 2 + \frac{1}{\beta} = c$ and $2 + 2ab - (a + b) = \frac{3}{2} < \frac{1}{\beta} + 2 = c$.

(2) $a^2 + b^2 - ab - a - b = \frac{1}{\beta}\left(\frac{1}{\beta} + \frac{1}{2}\right) - \frac{3}{4} < \frac{1}{\beta}\left(\frac{1}{\beta} + 1\right) = (c - 1)(c - 2)$.

Hence, by Theorem 2.4, Q is starlike in $\mathbb{D}$. Also the function $h : \mathbb{D} \to \mathbb{C}$ defined by $h(z) = \theta(q_\beta(z)) + Q(z) = q_\beta(z) + Q(z)$ satisfies $\operatorname{Re} zh'(z)/Q(z) = \operatorname{Re}(1/\beta + zQ'(z)/Q(z)) > 0$ as Q is starlike and β is positive. Thus, by making use of Lemma 2.2, the subordination $p(z) + \beta z p'(z) \prec q_\beta(z) + \beta z q_\beta'(z)$ implies $p(z) \prec q_\beta(z)$. The required subordination $p(z) \prec \mathcal{P}(z)$ for different choices of $\mathcal{P}$ holds if $q_\beta(z) \prec \mathcal{P}(z)$, and the subordination $q_\beta(z) \prec \mathcal{P}(z)$ holds, then

$$\mathcal{P}(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \mathcal{P}(1).$$

Fortunately, this condition becomes sufficient also for appropriate values of β. Before finding the values of β, we note that

$$q_\beta(-1) = F\left(\frac{-1}{2}, \frac{1}{\beta}, \frac{1}{\beta} + 1; 1\right) = \frac{1}{\Gamma(-\frac{1}{2})} \sum_{k=0}^{\infty} \frac{\Gamma(-\frac{1}{2} + k)}{k!(1 + \beta k)}$$

and

$$q_\beta(1) = F\left(-\frac{1}{2}, \frac{1}{\beta}, \frac{1}{\beta} + 1; -1\right) = \frac{1}{\Gamma(-\frac{1}{2})} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(-\frac{1}{2} + k)}{k!(1 + \beta k)}.$$

(a) Let $\mathcal{P}(z) = e^z$. Then the inequalities $e^{-1} \leq q_\beta(-1)$ and $q_\beta(1) \leq e$ reduce to

$$f(\beta) := \frac{1}{\Gamma(-\frac{1}{2})} \sum_{k=0}^{\infty} \frac{\Gamma(-\frac{1}{2} + k)}{k!(1 + \beta k)} - \frac{1}{e} \geq 0$$

$$\text{and } g(\beta) := e - \frac{1}{\Gamma(-\frac{1}{2})} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(-\frac{1}{2} + k)}{k!(1 + \beta k)} \geq 0.$$

We note that $g(\beta) > 0$ for all $\beta \in (0, \infty)$ but $\lim_{\beta \searrow 0} f(\beta) = -1/e < 0$ and $\lim_{\beta \nearrow \infty} f(\beta) = (e - 1)/e > 0$. Also $f'(\beta) > 0$ for every $\beta \in (0, \infty)$. Let β_e denotes the unique zero of $f(\beta)$ in $(0, \infty)$. Then $p(z) \prec e^z$ if $\beta \geq \beta_e \simeq 0.198099$.

(b) For $\mathcal{P}(z) = \varphi_s(z) = 1 + \sin z$, the inequality $1 - \sin 1 \leq q_\beta(-1)$ reduces to

$$f(\beta) := \frac{1}{\Gamma(-\frac{1}{2})} \sum_{k=1}^{\infty} \frac{\Gamma(-\frac{1}{2}+k)}{k!(1+\beta k)} + \sin 1 \geq 0.$$

Note that $\lim_{\beta \searrow 0} f(\beta) = -0.158529 < 0$ and $\lim_{\beta \nearrow \infty} f(\beta) = \sin 1 > 0$. The other inequality $q_\beta(1) \leq 1 + \sin 1$ gives

$$g(\beta) := \sin 1 - \frac{1}{\Gamma(-\frac{1}{2})} \sum_{k=1}^{\infty} \frac{(-1)^k \Gamma(-\frac{1}{2}+k)}{k!(1+\beta k)} \geq 0$$

which is true for all $\beta > 0$. Hence $p(z) \prec \varphi_s(z)$ for $\beta \geq \beta_s \simeq 0.0327862$ where β_s is unique root of $f(\beta) = 0$ in $(0, \infty)$.

(c) Let $\mathcal{P}(z) = \varphi_0(z)$. Then $\varphi_0(-1) = 2(\sqrt{2} - 1)$ and $\varphi_0(1) = 2$. Proceeding as in previous parts, the inequalities $\varphi_0(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \varphi_0(1)$ are true for $\beta \geq \beta_0$ where β_0 is unique root of

$$\frac{1}{\Gamma(-\frac{1}{2})} \sum_{k=0}^{\infty} \frac{\Gamma(-\frac{1}{2}+k)}{k!(1+\beta k)} - 2(\sqrt{2}-1) = 0 \quad \text{in} \quad (0, \infty).$$

Hence $p(z) \prec \varphi_0(z)$ if $\beta \geq \beta_0 \simeq 2.71181$.

(d) For $\mathcal{P}(z) = \varphi_c(z)$, the inequalities $\varphi_c(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \varphi_c(1)$ are true for $\beta \geq \beta_c \simeq 0.156$, where β_0 is unique root of

$$\frac{1}{\Gamma(-\frac{1}{2})} \sum_{k=0}^{\infty} \frac{\Gamma(-\frac{1}{2}+k)}{k!(1+\beta k)} - \frac{1}{3} = 0 \quad \text{in} \quad (0, \infty).$$

Thus, $p(z) \prec \varphi_c(z)$ for $\beta \geq \beta_c$. □

Remark 2.6. Let the function $f \in \mathcal{A}$ satisfies the following subordination

$$\frac{zf'(z)}{f(z)} \left(1 + \beta \left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right)\right) \prec \sqrt{1+z}.$$

Then $f \in \mathcal{S}^*_e$ for $\beta \geq 0.198099$, $f \in \mathcal{S}^*_s$ for $\beta \geq 0.0327862$, $f \in \mathcal{S}^*_R$ for $\beta \geq 2.71131$ and $f \in \mathcal{S}^*_c$ for $\beta \geq 0.158374$.

Theorem 2.7. *If the analytic function $p : \mathbb{D} \to \mathbb{C}$ satisfies $p(0) = 1$ and the subordination*

$$p(z) + \beta z p'(z) \prec 1 + z, \quad \beta > 0,$$

then the following are true:

(a) *If $\beta \geq 1/(e-1)$, then $p(z) \prec e^z$.*
(b) *If $\beta \geq (1 - \sin 1)/\sin 1$, then $p(z) \prec \varphi_s(z) = 1 + \sin z$.*
(c) *If $\beta \geq 2(1+\sqrt{2})$, then $p(z) \prec \varphi_0(z)$, where $\varphi_0(z)$ is given by (1.1).*
(d) *If $\beta \geq 1/2$, then $p(z) \prec \varphi_c(z) = 1 + 4z/3 + 2z^2/3$.*

Estimates on β are sharp.

Proof. The function

$$q(z) = q_\beta(z) := \frac{1}{\beta} \int_0^1 t^{\frac{1}{\beta}-1}(1+zt)dt$$

is the solution of linear differential equation $q(z) + \beta z q'(z) = 1 + z$. Using integral representation of Gauss hypergeometric function given in (2.3), we see that $q_\beta(z) = F(-1, \frac{1}{\beta}, \frac{1}{\beta}+1; -z)$. Thus $q_\beta(z)$ is clearly analytic in $\mathbb{D}$. Let $\varphi(w) := \beta$ and $\theta(w) := w$, $w \in \mathbb{C}$. Then

$$\begin{aligned} Q(z) &= zq'_\beta(z)\varphi(q_\beta(z)) = \beta z q'_\beta(z) \\ &= \frac{\beta}{\beta+1} z F\left(0, \frac{1}{\beta}+1, \frac{1}{\beta}+2; -z\right) = \frac{z\beta}{\beta+1}. \end{aligned}$$

Clearly Q is starlike in $\mathbb{D}$. Also the function $h : \mathbb{D} \to \mathbb{C}$ defined as $h(z) = \theta(q_\beta(z)) + Q(z) = q_\beta(z) + Q(z)$ satisfies $\operatorname{Re} zh'(z)/Q(z) = \operatorname{Re}(1/\beta + zQ'(z)/Q(z)) > 0$ as Q is starlike and β is positive. Hence by making use of Lemma 2.2, the subordination $p(z) + \beta zp'(z) \prec q_\beta(z) + \beta z q'_\beta(z)$ implies $p(z) \prec q_\beta(z)$. The required subordination $p(z) \prec \mathcal{P}(z)$ for different choices of $\mathcal{P}$ holds if $q_\beta(z) \prec \mathcal{P}(z)$, and if the subordination $q_\beta(z) \prec \mathcal{P}(z)$ holds , then

$$\mathcal{P}(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \mathcal{P}(1).$$

This condition becomes sufficient also for appropriate values of β as can be seen by plotting the graphs of respective functions. We note that

$$q_\beta(-1) = F\left(-1, \frac{1}{\beta}, \frac{1}{\beta} + 1; 1\right) = 1 - \frac{1}{1+\beta}$$

and

$$q_\beta(1) = F\left(-1, \frac{1}{\beta}, \frac{1}{\beta} + 1; -1\right) = 1 + \frac{1}{1+\beta}.$$

(a) Let $\mathcal{P}(z) = e^z$. Then the inequalities $e^{-1} \leq q_\beta(-1)$ and $q_\beta(1) \leq e$ give $\beta \geq 1/(e-1)$ and $\beta \geq (2-e)/(e-1)$ respectively. Hence $p(z) \prec e^z$ if

$$\beta \geq \max\left\{\frac{1}{e-1}, \frac{2-e}{e-1}\right\} = \frac{1}{e-1}.$$

(b) For $\mathcal{P}(z) = \varphi_s(z) = 1 + \sin z$, both the inequalities $1 - \sin 1 \leq q_\beta(-1)$ and $q_\beta(1) \leq 1 + \sin 1$ give $\beta \geq (1 - \sin 1)/\sin 1$.

(c) Let $\mathcal{P}(z) = \varphi_0(z)$. Then $\varphi_0(-1) = 2(\sqrt{2} - 1)$ and $\varphi_0(1) = 2$. The inequalities $\varphi_0(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \varphi_0(1)$ give $\beta \geq 2(1 + \sqrt{2})$ and $\beta \geq 0$ respectively. Hence $p(z) \prec \varphi_0(z)$ if $\beta \geq 2(1 + \sqrt{2})$.

(d) For $\mathcal{P}(z) = \varphi_c(z)$, the inequalities $\varphi_c(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \varphi_c(1)$ give $\beta \geq 1/2$ and $\beta \geq -1/2$. Hence $p(z) \prec \varphi_c(z)$ if $\beta \geq 1/2$. $\square$

Remark 2.8. Replacing the function p by zf'/f in Theorem 2.7, sufficient conditions can be derived for $f \in \mathcal{A}$ to be in the subclasses of starlike functions $\mathcal{S}^*_e$, $\mathcal{S}^*_s$, $\mathcal{S}^*_R$ and $\mathcal{S}^*_c$.

3. Subordination associated with rational function

In this section, we consider the subordination

$$1 + \frac{\beta z p'(z)}{p^j(z)} \prec \varphi_0(z) \quad j = 0, 1, 2, \quad (k = \sqrt{2} + 1)$$

and obtain sharp estimates on β for which $p(z) \prec \varphi_0(z)$, $\varphi_s(z) = 1 + \sin z$, $\sqrt{1+z}$, where φ_0 is given by (1.1).

Theorem 3.1. *Let p be the analytic function in $\mathbb{D}$ with $p(0) = 1$ and satisfies the subordination*

$$1 + \beta z p'(z) \prec \varphi_0(z).$$

Then the following are true:

(a) *If $\beta \geq (1 + \sqrt{2})\left(-1 + 2k \log\left(1 + \frac{1}{k}\right)\right) \simeq 1.62574$, then $p(z) \prec \varphi_0(z)$.*

(b) *If $\beta \geq \frac{-1}{k \sin 1}\left(1 + 2k \log\left(1 - \frac{1}{k}\right)\right) \simeq 0.778858$, then $p(z) \prec \varphi_s(z)$.*

(c) *If $\beta \geq -\left(1 + 2k \log\left(1 - \frac{1}{k}\right)\right) \simeq 1.58224$, then $p(z) \prec \sqrt{1+z}$.*

The bounds on β are sharp.

Proof. The function

$$q_\beta(z) := 1 - \frac{1}{\beta k}\left(z + 2k \log\left(1 - \frac{z}{k}\right)\right), \quad z \in \overline{\mathbb{D}},$$

is a solution of differential equation $1 + \beta z q'(z) = \varphi_0(z)$. Clearly, the function q_β is analytic in $\mathbb{D}$. Define $\varphi(w) := \beta$ and $\theta(w) := 1$, $w \in \mathbb{C}$. We define a function Q on $\overline{\mathbb{D}}$ as $Q(z) = zq'_\beta(z)\varphi(q_\beta(z)) = \beta z q'_\beta(z) = z(k+z)/k(k-z)$. Then Q is starlike in $\mathbb{D}$. We also define a function h on $\overline{\mathbb{D}}$ as $h(z) = \theta(q_\beta(z)) + Q(z) = 1 + z(k+z)/k(k-z)$. We note that $zh'(z)/Q(z) = zQ'(z)/Q(z)$ is a function with positive real part in $\mathbb{D}$. Hence, by making use of Lemma 2.2, the subordination $1 + \beta z p'(z) \prec 1 + \beta z q'_\beta(z)$ implies $p(z) \prec q_\beta(z)$. The required subordination $p(z) \prec \mathcal{P}(z)$ for different choices of $\mathcal{P}$ holds if $q_\beta(z) \prec \mathcal{P}(z)$. And the subordination $q_\beta(z) \prec \mathcal{P}(z)$ holds if and only if $\mathcal{P}(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \mathcal{P}(1)$ as it is clear from the graph of respective functions.

(a) For $\mathcal{P}(z) = \varphi_0(z)$, the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ reduce to $\beta \geq (k+1)(-1+2k\log(1+1/k))/(k-1) \simeq 1.62574 = \beta_1$ and $\beta \geq (1-k)(1+2k\log(1-1/k))/(k+1) \simeq 0.655386 = \beta_2$ respectively. Hence the required subordination $p(z) \prec \varphi_0(z)$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(b) For $\mathcal{P}(z) = \varphi_s(z)$, the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ reduce to $1 - \sin 1 \leq q_\beta(-1)$ and $q_\beta(1) \leq 1 + \sin 1$. These inequalities give $\beta \geq (-1 + 2k\log(1 + 1/k))/k\sin 1 \simeq 0.331483 = \beta_1$ and $\beta \geq -(1+2k\log(1-1/k))/k\sin 1 \simeq 0.778858 = \beta_2$. Hence the required subordination $p(z) \prec 1 + \sin z$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$.

(c) For $\mathcal{P}(z) = \sqrt{1+z}$, the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ reduce to $q_\beta(-1) \geq 0$ and $q_\beta(1) \leq \sqrt{2}$ respectively, which on further calculations give $\beta \geq (-1 + 2k\log(1+1/k))/k \simeq 0.278934 = \beta_1$ and $\beta \geq -(1+2k\log(1-1/k))/k(\sqrt{2}-1) \simeq 1.58224 = \beta_2$. Hence the required subordination $p(z) \prec \sqrt{1+z}$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$. □

Remark 3.2. We note that $|(k+e^{i\theta})|/|k(k-e^{i\theta})| \geq (k-1)/k(k+1) = 3 - 2\sqrt{2}$. Hence the inequality $|w(z)| \leq 3 - 2\sqrt{2}$ implies that $w(z) \prec z(k+z)/k(k-z)$. Using this fact and Theorem 3.1, we obtain the following sufficient conditions for $f \in \mathcal{A}$ to be in various subclasses of $\mathcal{S}^*$.

(a) If $\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| \leq \frac{3-2\sqrt{2}}{(1+\sqrt{2})\left(-1+2k\log\left(1+\frac{1}{k}\right)\right)} \simeq 0.105535$, then $f \in \mathcal{S}^*_R$.

(b) If $\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| \leq \frac{(2\sqrt{2}-3)k\sin 1}{\left(1+2k\log\left(1-\frac{1}{k}\right)\right)} \simeq 0.220288$, then $f \in \mathcal{S}^*_s$.

(c) If $\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| \leq \frac{(2\sqrt{2}-3)}{\left(1+2k\log\left(1-\frac{1}{k}\right)\right)} \simeq 0.108437$, then $f \in \mathcal{S}^*_L$.

Theorem 3.3. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Let*

$$1 + \frac{\beta z p'(z)}{p(z)} \prec \varphi_0(z).$$

(a) *If $\beta \geq -\left(-1 + 2k\log\left(1 + \frac{1}{k}\right)\right)/k\log(2(\sqrt{2}+1)) \simeq 1.4819$, then $p(z) \prec \varphi_0(z)$.*

(b) *If $\beta \geq \frac{-1}{k\log(1+\sin 1)}\left(1 + 2k\log\left(1 - \frac{1}{k}\right)\right) \simeq 1.07341$, then $p(z) \prec \varphi_s(z)$.*

(c) *If $\beta \geq \frac{-1}{k\log\sqrt{2}}\left(1 + 2k\log\left(1 - \frac{1}{k}\right)\right) \simeq 1.89105$, then $p(z) \prec \sqrt{1+z}$.*

The bounds on β are sharp.

Proof. The function

$$q_\beta(z) := \exp\left(-\frac{1}{\beta k}\left(z + 2k\log\left(1 - \frac{z}{k}\right)\right)\right), \quad z \in \overline{\mathbb{D}},$$

is a solution of differential equation $1 + \beta z q'(z)/q(z) = \varphi_0(z)$. Clearly q_β is analytic in $\mathbb{D}$. Define $\varphi(w) := \beta/w$ and $\theta(w) := 1$, $w \in \mathbb{C}$. The function Q defined on $\overline{\mathbb{D}}$ as $Q(z) = zq'_\beta(z)\varphi(q_\beta(z)) = \beta z q'_\beta(z)/q_\beta(z) = z(k+z)/k(k-z)$ is starlike in $\mathbb{D}$. And the function $h(z) = \theta(q_\beta(z)) + Q(z) = 1 + z(k + z)/k(k - z)$ satisfies $\operatorname{Re} zh'(z)/Q(z) > 0$ in $\mathbb{D}$. Hence, by making use of Lemma 2.2, the subordination $1 + \beta zp'(z)/p(z) \prec 1 + \beta zq'_\beta(z)/q_\beta(z)$ implies $p(z) \prec q_\beta(z)$. The required subordination $p(z) \prec \mathcal{P}(z)$ for different choices of $\mathcal{P}$ holds if $q_\beta(z) \prec \mathcal{P}(z)$, and the subordination $q_\beta(z) \prec \mathcal{P}(z)$ holds if and only if $\mathcal{P}(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \mathcal{P}(1)$ as it is clear from the graph of respective functions.

(a) For $\mathcal{P}(z) = \varphi_0(z)$, the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ reduce to $\beta \geq -(-1 + 2k \log(1 + 1/k))/k \log((k^2 + 1)/(k^2 + k)) \simeq 1.4819 = \beta_1$ and $\beta \geq -(1 + 2k \log(1 - 1/k))/k \log\left((k^2 + 1)/(k^2 - k)\right) \simeq 0.945523 = \beta_2$ respectively. Hence the required subordination $p(z) \prec \varphi_0(z)$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(b) For $\mathcal{P}(z) = \varphi_s(z)$, the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ reduce to $1 - \sin 1 \leq q_\beta(-1)$ and $q_\beta(1) \leq 1 + \sin 1$. These inequalities give $\beta \geq -(-1 + 2k \log(1+1/k))/k(\log(1-\sin 1) \simeq 0.151445 = \beta_1$ and $\beta \geq -(1+2k \log(1-1/k))/k \log(1+\sin 1) \simeq 1.07341 = \beta_2$. Hence the required subordination $p(z) \prec 1 + \sin z$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$.

(c) For $\mathcal{P}(z) = \sqrt{1 + z}$, the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ reduce to $q_\beta(-1) \geq 0$ and $q_\beta(1) \leq \sqrt{2}$. The inequality $q_\beta(-1) \geq 0$ is true for all β and the inequality $q_\beta(1) \leq \sqrt{2}$ gives $\beta \geq -(1 + 2k \log(1 - 1/k))/k \log \sqrt{2} \simeq 1.89105$. Hence the required subordination $p(z) \prec \sqrt{1 + z}$ holds if $\beta \geq 1.89105$. $\square$

Remark 3.4. As done in Remark 3.2, we have the following sufficient conditions for $f \in \mathcal{A}$ to be in various subclasses of starlike functions.

(a) If $\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| \leq \frac{(2\sqrt{2}-3)k \log(2(\sqrt{2}+1))}{\left(-1+2k \log\left(1+\frac{1}{k}\right)\right)} \simeq 0.115779$, then $f \in \mathcal{S}^*_R$.

(b) If $\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| \leq \frac{(2\sqrt{2}-3)k \log(1+\sin 1)}{\left(1+2k \log\left(1-\frac{1}{k}\right)\right)} \simeq 0.159839$, then $f \in \mathcal{S}^*_s$.

(c) If $\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| \leq \frac{(2\sqrt{2}-3)k \log \sqrt{2}}{\left(1+2k \log\left(1-\frac{1}{k}\right)\right)} \simeq 0.0907289$, then $f \in \mathcal{S}^*_L$.

Theorem 3.5. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Let*

$$1 + \frac{\beta zp'(z)}{p^2(z)} \prec \varphi_0(z).$$

(a) *If $\beta \geq 2\left(-1 + 2k \log\left(1 + \frac{1}{k}\right)\right) \simeq 1.34681$, then $p(z) \prec \varphi_0(z)$.*

(b) *If $\beta \geq \frac{-(1+\sin 1)}{k \sin 1}\left(1 + 2k \log\left(1 - \frac{1}{k}\right)\right) \simeq 1.43424$, then $p(z) \prec \varphi_s(z)$.*

(c) *If $\beta \geq -\sqrt{2}\left(1 + 2k \log\left(1 - \frac{1}{k}\right)\right) \simeq 2.23763$, then $p(z) \prec \sqrt{1 + z}$.*

The bounds on β are sharp.

Proof. The function

$$q_\beta(z) := \left(1 + \frac{1}{\beta k}\left(z + 2k \log\left(1 - \frac{z}{k}\right)\right)\right)^{-1}, \quad z \in \overline{\mathbb{D}},$$

is a solution of differential equation

$$1 + \beta \frac{zq'(z)}{q^2(z)} = \varphi_0(z).$$

Clearly q_β is analytic in $\mathbb{D}$. Define $\varphi(w) := \beta/w^2$ and $\theta(w) := 1$, $w \in \mathbb{C}$. The function Q defined on $\overline{\mathbb{D}}$ as $Q(z) = zq'_\beta(z)\varphi(q_\beta(z)) = \beta zq'_\beta(z)/q^2_\beta(z) = z(k+z)/k(k-z)$ is starlike in $\mathbb{D}$. Also the function $h(z) = \theta(q_\beta(z))+Q(z) = 1+z(k+z)/k(k-z)$ satisfies $\operatorname{Re} zh'(z)/Q(z) > 0$ in $\mathbb{D}$. Hence, by making use of Lemma 2.2, the subordination $1 + \beta zp'(z)/p^2(z) \prec 1 + \beta zq'_\beta(z)/q^2_\beta(z)$ implies $p(z) \prec q_\beta(z)$. The required subordination $p(z) \prec \mathcal{P}(z)$ for different choices of $\mathcal{P}$ holds if $q_\beta(z) \prec \mathcal{P}(z)$, and the subordination $q_\beta(z) \prec \mathcal{P}(z)$ holds if and only if $\mathcal{P}(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \mathcal{P}(1)$ as it is clear from the graph of respective functions.

(a) For $\mathcal{P}(z) = \varphi_0(z)$, the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ reduce to $\beta \geq (-1 + 2k \log(1 + 1/k))(k^2 + 1)/(k^2 - k) \simeq 1.34681 = \beta_1$ and $\beta \geq -(k^2 + 1)(1 + 2k \log(1 - 1/k))/(k^2 + k) \simeq 1.31077 = \beta_2$ respectively. Hence the required subordination $p(z) \prec \varphi_0(z)$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(b) For $\mathcal{P}(z) = \varphi_s(z)$, the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ reduce to $1 - \sin 1 \leq q_\beta(-1)$ and $q_\beta(1) \leq 1 + \sin 1$. These inequalities give $\beta \geq (1 - \sin 1)(-1 + 2k \log(1+1/k))/k \sin 1 \simeq 0.0525497 = \beta_1$ and $\beta \geq -(1+\sin 1)(1+2k \log(1-1/k))/k \sin 1 \simeq 1.43424 = \beta_2$. Hence the required subordination $p(z) \prec 1+\sin z$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$.

(c) For $\mathcal{P}(z) = \sqrt{1+z}$, the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ reduce to $q_\beta(-1) \geq 0$ and $q_\beta(1) \leq \sqrt{2}$ respectively, which on further calculations give $\beta \geq -(-1+2k \log(1+1/k))/k \simeq -0.278934 = \beta_1$ and $\beta \geq -\sqrt{2}(1+2k \log(1-1/k))/k(\sqrt{2}-1) \simeq 2.23763 = \beta_2$. Hence the required subordination $p(z) \prec \sqrt{1+z}$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$. □

Remark 3.6. Replacing the function p by zf'/f in Theorem 3.5, sufficient conditions can be derived for $f \in \mathcal{A}$ to be in various subclasses of starlike functions as done in Remark 3.2.

4. Subordination associated with cardioid

In this section, we consider the subordination

$$1 + \frac{\beta z p'(z)}{p^j(z)} \prec \varphi_c(z) = 1 + \frac{4z}{3} + \frac{2z^2}{3}, \quad j = 0, 1, 2$$

and obtain sharp estimates on β for which $p(z) \prec e^z$, $\sqrt{1+z}$, $\frac{1+Az}{1+Bz}$, $\varphi_s(z) = 1+\sin z$, and $\varphi_0(z)$, where φ_0 is given by (1.1).

Theorem 4.1. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Let*

$$1 + \beta z p'(z) \prec \varphi_c(z).$$

Then

(a) *If $\beta \geq e/(e-1)$, then $p(z) \prec e^z$.*
(b) *If $\beta \geq 3 + 2\sqrt{2}$, then $p(z) \prec \varphi_0(z)$.*
(c) *If $\beta \geq 5/(3 \sin 1)$, then $p(z) \prec \varphi_s(z)$.*
(d) *If $\beta \geq 5/(3\sqrt{2} - 3)$, then $p(z) \prec \sqrt{1+z}$.*
(e) *If $\beta \geq \max\{(1-B)/(A-B), 5(1+B)/3(A-B)\}$, then $p(z) \prec (1+Az)/(1+Bz)$*

The results are sharp.

Proof. Consider the function $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ given by

$$q_\beta(z) := 1 + \frac{1}{3\beta}(4z + z^2).$$

It can be easily verified that $q_\beta(z)$ is a solution of differential equation $1+\beta z q'(z) = \varphi_c(z)$. Clearly q_β is analytic in $\mathbb{D}$. Let $\varphi(w) := \beta$ and $\theta(w) := 1$, $w \in \mathbb{C}$. Let the function Q be defined on $\mathbb{D}$ as $Q(z) = z q_\beta'(z) \varphi(q_\beta(z)) = \beta z q_\beta'(z) = 2z(z+2)/3$. Then Q is starlike as $\operatorname{Re} zQ'(z)/Q(z) = 2 \operatorname{Re}(z+1)/(z+2) > 0$ for $z \in \mathbb{D}$. Then the function h defined on $\mathbb{D}$ as

$$h(z) = \theta(q_\beta(z)) + Q(z) = 1 + \frac{2}{3}z(z+2),$$

satisfies $\operatorname{Re} zh'(z)/Q(z) = \operatorname{Re} zQ'(z)/Q(z) > 0$. Hence, by making use of Lemma 2.2, the subordination $1 + \beta z p'(z) \prec 1 + \beta z q_\beta'(z)$ implies $p \prec q_\beta$. The required subordination $p \prec \mathcal{P}$ holds for different $\mathcal{P}$ if $q_\beta \prec \mathcal{P}$, and the subordination $q_\beta \prec \mathcal{P}$ holds if and only if $\mathcal{P}(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \mathcal{P}(1)$ as it is clear from the graph of respective functions.

(a) For $\mathcal{P}(z) = e^z$, the inequalities $q_\beta(-1) \geq 1/e$ and $q_\beta(1) \leq e$ reduce to $\beta \geq e/(e-1)$ and $\beta \geq 5/3(e-1)$ respectively. Therefore the subordination $q_\beta(z) \prec e^z$ holds if $\beta \geq \max\{e/(e-1), 5/3(e-1)\} = e/(e-1)$.

(b) For $\mathcal{P}(z) = \varphi_0(z)$, the inequalities $\varphi_0(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \varphi_0(1)$ give $\beta \geq (k^2+k)/(k-1) = 3+2\sqrt{2}$ and $\beta \geq 5k(k-1)/3(k+1) = 5/3$. Hence $p(z) \prec \varphi_0(z)$ if $\beta \geq \max\{3+2\sqrt{2}, 5/3\} = 3+2\sqrt{2}$.

(c) For $\mathcal{P}(z) = \varphi_s(z)$, the inequalities $1-\sin 1 \leq q_\beta(-1)$ and $q_\beta(1) \leq 1+\sin 1$ give $\beta \geq 1/\sin 1 \simeq 1.1884$ and $\beta \geq 5/3\sin 1 \simeq 1.98066$. Hence $p(z) \prec \varphi_s(z)$ if $\beta \geq 5/3\sin 1$.

(d) For $\mathcal{P}(z) = \sqrt{1+z}$, the inequalities $0 \leq q_\beta(-1)$ and $q_\beta(1) \leq \sqrt{2}$ give $\beta \geq 1$ and $\beta \geq 5/3(\sqrt{2}-1)$. Since $5/3(\sqrt{2}-1) > 1$, $p(z) \prec \sqrt{1+z}$ if $\beta \geq 5/3(\sqrt{2}-1)$.

(e) For $\mathcal{P}(z) = (1+Az)/(1+Bz)$, the inequalities $(1-A)/(1-B) \leq q_\beta(-1)$ and $q_\beta(1) \leq (1+A)/(1+B)$ yield $\beta \geq (1-B)/(A-B) = \beta_1$ and $\beta \geq 5(1+B)/3(A-B) = \beta_2$. Hence $p(z) \prec (1+Az)/(1+Bz)$ if $\beta \geq \max\{\beta_1, \beta_2\}$. □

Remark 4.2. Let the function $f \in \mathcal{A}$ satisfies the following subordination

$$1+\beta\frac{zf'(z)}{f(z)}\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right) \prec \varphi_c(z).$$

Then $f \in \mathcal{S}^*_e$ if $\beta \geq e/(e-1)$, $f \in \mathcal{S}^*_R$ if $\beta \geq 3+2\sqrt{2}$, $f \in \mathcal{S}^*_s$ if $\beta \geq \frac{5}{3\sin 1}$, $f \in \mathcal{S}^*_L$ if $\beta \geq \frac{5}{3(\sqrt{2}-1)}$, and $f \in \mathcal{S}^*[A,B]$ if $\beta \geq \max\{(1-B)/(A-B), 5(1+B)/3(A-B)\}$.

Theorem 4.3. *Let p be an analytic function in $\mathbb{D}$ with $p(0)=1$. Let*

$$1+\frac{\beta zp'(z)}{p(z)} \prec \varphi_c(z).$$

Then

(a) *If $\beta \geq 5/3$, then $p(z) \prec e^z$.*
(b) *If $\beta \geq (\log(1/2(\sqrt{2}-1)))^{-1} \simeq 5.31275$, then $p(z) \prec \varphi_0(z)$.*
(c) *If $\beta \geq 5/(3\log(1+\sin 1))$, then $p(z) \prec \varphi_s(z)$.*
(d) *If $\beta \geq 5/3\log\sqrt{2}$, then $p(z) \prec \sqrt{1+z}$.*
(e) *If $\beta \geq \max\{5(3\log((1+A)/(1+B)))^{-1}, (\log((1-B)/(1-A)))^{-1}\}$, then $p(z) \prec (1+Az)/(1+Bz)$.*

The results are sharp.

Proof. Define a function $q_\beta : \mathbb{D} \to \mathbb{C}$ as

$$q_\beta(z) := \exp\left(\frac{1}{3\beta}(4z+z^2)\right).$$

Clearly q_β is analytic in $\mathbb{D}$ and is a solution of differential equation $1+\beta zq'(z)/q(z) = \varphi_c(z)$. Define $\varphi(w) := \beta/w$ and $\theta(w) := 1$, $w \in \mathbb{C}$. Then the function

$$Q(z) = zq'_\beta(z)\varphi(q_\beta(z)) = \frac{\beta zq'_\beta(z)}{q_\beta(z)} = \frac{2}{3}z(z+2)$$

is starlike in $\mathbb{D}$. Also the function $h(z) = \theta(q_\beta(z)) + Q(z) = 1+2z(z+2)/3$ satisfies $\operatorname{Re} zh'(z)/Q(z) > 0$ in $\mathbb{D}$. Hence, by making use of Lemma 2.2, the subordination $1+\beta zp'(z)/p(z) \prec 1+\beta zq'_\beta(z)/q_\beta(z)$ implies $p(z) \prec q_\beta(z)$. The required subordination $p(z) \prec \mathcal{P}(z)$ holds for different choices of $\mathcal{P}$ if $q_\beta(z) \prec \mathcal{P}(z)$. And the subordination $q_\beta(z) \prec \mathcal{P}(z)$ holds if and only if $\mathcal{P}(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \mathcal{P}(1)$ as it is clear from the graph of respective functions.

(a) For $\mathcal{P}(z) = e^z$, the inequalities $q_\beta(-1) \geq 1/e$ and $q_\beta(1) \leq e$ reduce to $\beta \geq 1$ and $\beta \geq 5/3$ respectively. Therefore the subordination $q_\beta(z) \prec e^z$ holds if $\beta \geq \max\{1, 5/3\} = 5/3$.

(b) For $\mathcal{P}(z)=\varphi_0(z)$, the inequalities $\varphi_0(-1)\leq q_\beta(-1)$ and $q_\beta(1)\leq \varphi_0(1)$ give $\beta\geq 1/\log((k^2+k)/(k^2+1))=(\log(1/2(\sqrt{2}-1)))^{-1}\simeq 5.31275$ and $\beta\geq 5/3\log((k^2+1)/(k^2-k))=5/3\log 2\simeq 2.40449$. Hence $p(z)\prec\varphi_0(z)$ if $\beta\geq 5.31275$.

(c) For $\mathcal{P}(z)=\varphi_s(z)=1+\sin z$, the inequalities $1-\sin 1\leq q_\beta(-1)$ and $q_\beta(1)\leq 1+\sin 1$ give $\beta\geq(\log(1/(1-\sin 1)))^{-1}\simeq 0.542942$ and $\beta\geq 5/3(\log(1+\sin 1))\simeq 2.72971$. Hence $p(z)\prec\varphi_s(z)$ if $\beta\geq 5/3(\log(1+\sin 1))$.

(d) For $\mathcal{P}(z)=\sqrt{1+z}$, then the inequality $0\leq q_\beta(-1)$ is true for all β and the inequality $q_\beta(1)\leq\sqrt{2}$ gives $\beta\geq 5/3\log\sqrt{2}$. Hence $p(z)\prec\sqrt{1+z}$ if $\beta\geq 5/3\log\sqrt{2}$.

(e) For $\mathcal{P}(z)=(1+Az)/(1+Bz)$, the inequalities $(1-A)/(1-B)\leq q_\beta(-1)$ and $q_\beta(1)\leq(1+A)/(1+B)$ yields $\beta\geq 1/\log((1-B)/(1-A))=\beta_1$ and $\beta\geq 5/3\log((1+A)/(1+B))=\beta_2$. Hence $p(z)\prec(1+Az)/(1+Bz)$ if $\beta\geq\max\{\beta_1,\beta_2\}$. □

Remark 4.4. Let the function $f\in\mathcal{A}$ satisfies the following subordination

$$1+\beta\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right)\prec\varphi_c(z).$$

Then $f\in\mathcal{S}^*_e$ if $\beta\geq 5/3$, $f\in\mathcal{S}^*_R$ if $\beta\geq(\log(1/2(\sqrt{2}-1)))^{-1}\simeq 5.31275$, $f\in\mathcal{S}^*_s$ if $\beta\geq 5/(3\log(1+\sin 1))$, $f\in\mathcal{S}^*_L$ if $\beta\geq 5/3\log\sqrt{2}$, and $f\in\mathcal{S}^*[A,B]$ if

$$\beta\geq\max\left\{5(3\log((1+A)/(1+B)))^{-1},\ (\log((1-B)/(1-A)))^{-1}\right\}.$$

Theorem 4.5. *Let p be an analytic function in $\mathbb{D}$ with $p(0)=1$. Let*

$$1+\frac{\beta zp'(z)}{p^2(z)}\prec\varphi_c(z).$$

(a) *If $\beta\geq 5e/3(e-1)$, then $p(z)\prec e^z$.*
(b) *If $\beta\geq 2(1+\sqrt{2})$, then $p(z)\prec\varphi_0(z)$.*
(c) *If $\beta\geq 5(1+\sin 1)/3\sin 1$, then $p(z)\prec\varphi_s(z)$.*
(d) *If $\beta\geq 5\sqrt{2}/3(\sqrt{2}-1)$, then $p(z)\prec\sqrt{1+z}$.*
(e) *If $\beta\geq\max\{(1-A)/(A-B),5(1+A)/(3(A-B))\}$, then $p(z)\prec(1+Az)/(1+Bz)$.*

The results are sharp.

Proof. Define a function $q_\beta:\mathbb{D}\to\mathbb{C}$ as

$$q_\beta(z):=\left(1-\frac{1}{3\beta}(4z+z^2)\right)^{-1}.$$

Clearly q_β is analytic in $\mathbb{D}$ and is a solution of differential equation $1+\beta zq'(z)/q^2(z)=\varphi_c(z)$. Define $\varphi(w):=\beta/w^2$ and $\theta(w):=1$, $w\in\mathbb{C}$. The the function

$$Q(z)=zq'_\beta(z)\varphi(q_\beta(z))=\frac{\beta zq'_\beta(z)}{q^2_\beta(z)}=\frac{2}{3}z(z+2)$$

is starlike in $\mathbb{D}$. And the function $h(z)=\theta(q_\beta(z))+Q(z)=1+2z(z+2)/3$ satisfies $\operatorname{Re} zh'(z)/Q(z)>0$ in $\mathbb{D}$. Hence, by making use of Lemma 2.2, the subordination $1+\beta zp'(z)/p^2(z)\prec 1+\beta zq'_\beta(z)/q^2_\beta(z)$ implies $p(z)\prec q_\beta(z)$. The required subordination $p(z)\prec e^z$ holds if $q_\beta(z)\prec e^z$. The required subordination $p(z)\prec\mathcal{P}(z)$ holds for different choices of $\mathcal{P}$ if $q_\beta(z)\prec\mathcal{P}(z)$, and the subordination $q_\beta(z)\prec\mathcal{P}(z)$ holds if and only if $\mathcal{P}(-1)\leq q_\beta(-1)\leq q_\beta(1)\leq\mathcal{P}(1)$ as it is clear from the graph of respective functions.

(a) For $\mathcal{P}(z)=e^z$, the inequalities $q_\beta(-1)\geq 1/e$ and $q_\beta(1)\leq e$ reduce to $\beta\geq 1/(e-1)$ and $\beta\geq 5e/3(e-1)$ respectively. Therefore the subordination $q_\beta(z)\prec e^z$ holds if $\beta\geq\max\{1/(e-1),5e/3(e-1)\}=5e/3(e-1)$.

(b)For $\mathcal{P}(z)=\varphi_0(z)$, the inequalities $\varphi_0(-1)\leq q_\beta(-1)$ and $q_\beta(1)\leq\varphi_0(1)$ give $\beta\geq(k^2+1)/(k-1)=2(1+\sqrt{2})$ and $\beta\geq 5(k^2+1)/3(k+1)=10/3$. Hence $p(z)\prec\varphi_0(z)$ if $\beta\geq\max\{2(1+\sqrt{2}),10/3\}=2(1+\sqrt{2})$.

(c) For $\mathcal{P}(z) = \varphi_s(z)$, the inequalities $1 - \sin 1 \leq q_\beta(-1)$ and $q_\beta(1) \leq 1 + \sin 1$ give $\beta \geq (1-\sin 1)/\sin 1 \simeq 0.188395$ and $\beta \geq 5(1+\sin 1)/3\sin 1 \simeq 3.64733$. Hence $p(z) \prec \varphi_s(z)$ if $\beta \geq 5(1+\sin 1)/3\sin 1$.

(d) For $\mathcal{P}(z) = \sqrt{1+z}$, then the inequality $0 \leq q_\beta(-1)$ gives $\beta \geq -1$ and the inequality $q_\beta(1) \leq \sqrt{2}$ gives $\beta \geq 5\sqrt{2}/3(\sqrt{2}-1)$. Hence $p(z) \prec \sqrt{1+z}$ if $\beta \geq 5\sqrt{2}/3(\sqrt{2}-1)$.

(e) For $\mathcal{P}(z) = (1+Az)/(1+Bz)$, the inequalities $(1-A)/(1-B) \leq q_\beta(-1)$ and $q_\beta(1) \leq (1+A)/(1+B)$ yield $\beta \geq (1-A)/(A-B) = \beta_1$ and $\beta \geq 5(1+A)/3(A-B) = \beta_2$. Hence $p(z) \prec (1+Az)/(1+Bz)$ if $\beta \geq \max\{\beta_1, \beta_2\}$. □

Remark 4.6. We would like to point out that authors in [26] have obtained conditions on real parameter β so that the subordination $1 + \beta zp'(z)/p^j(z)$, $(j = 0, 1, 2)$ implies $p(z) \prec (2+z)/(2-z), 1+(1-\alpha)z, (1+(1-2\alpha)z)/(1-z), (0 \leq \alpha < 1), e^z$, or $\sqrt{1+z}$. But our results improve upon the estimates obtained by them.

Remark 4.7. Let the function $f \in \mathcal{A}$ satisfies the following subordination
$$1 + \beta\left(\frac{zf'(z)}{f(z)}\right)^{-1}\left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right) \prec \varphi_c(z).$$
Then $f \in \mathcal{S}^*_e$ for $\beta \geq 5e/3(e-1)$, $f \in \mathcal{S}^*_R$ for $\beta \geq 2(1+\sqrt{2})$, $f \in \mathcal{S}^*_s$ for $\beta \geq 5(1+\sin 1)/3\sin 1$, $f \in \mathcal{S}^*_L$ for $\beta \geq 5\sqrt{2}/3(\sqrt{2}-1)$, and the function $f \in \mathcal{S}^*[A, B]$ for
$$\beta \geq \max\left\{(1-A)/(A-B), 5(1+A)/(3(A-B))\right\}.$$

5. Subordination associated with sine function

In this section, we consider the subordination
$$1 + \frac{\beta zp'(z)}{p^j(z)} \prec \varphi_s(z) = 1 + \sin z \quad j = 0, 1, 2,$$
and obtain sharp estimates on β for which $p(z) \prec \varphi_s(z),\ \varphi_c(z)$.

Theorem 5.1. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Let*
$$1 + \beta zp'(z) \prec \varphi_s(z).$$

(a) *If $\beta \geq \frac{1}{\sin 1}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \simeq 1.12432$, then $p(z) \prec \varphi_s(z)$.*

(b) *If $\beta \geq \frac{3}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \simeq 1.41912$, then $p(z) \prec \varphi_c(z)$.*

Proof. The function
$$q_\beta(z) := 1 + \frac{1}{\beta}\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!}, \quad z \in \overline{\mathbb{D}},$$
is a solution of differential equation $1 + \beta zq'(z) = \varphi_s(z)$. Clearly q_β is analytic in $\mathbb{D}$. Define $\varphi(w) := \beta$ and $\theta(w) := 1$, $w \in \mathbb{C}$. Let the function Q be defined on $\overline{\mathbb{D}}$ as $Q(z) = zq'_\beta(z)\varphi(q_\beta(z)) = \beta zq'_\beta(z) = \sin z$, which is starlike in $\mathbb{D}$. Then the function h on $\overline{\mathbb{D}}$ defined as $h(z) = \theta(q_\beta(z)) + Q(z) = 1 + \sin z$ satisfies $\operatorname{Re} zh'(z)/Q(z) = \operatorname{Re} zQ'(z)/Q(z) > 0$. Hence, by making use of Lemma 2.2, the subordination $1 + \beta zp'(z) \prec 1 + \beta zq'_\beta(z)$ implies $p(z) \prec q_\beta(z)$. The required subordination $p(z) \prec \mathcal{P}(z)$ for different choices of $\mathcal{P}$ holds if $q_\beta(z) \prec \mathcal{P}(z)$. And the subordination $q_\beta(z) \prec \mathcal{P}(z)$ holds if and only if $\mathcal{P}(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \mathcal{P}(1)$.

(a) Let $\mathcal{P}(z) = \varphi_s(z)$. Then both the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ give
$$\beta \geq \frac{1}{\sin 1}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \simeq 1.12432.$$
Hence $p(z) \prec \varphi_s(z)$ if $\beta \geq 1.12432$.

(b) For $\mathcal{P}(z) = \varphi_c(z)$, the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ reduce to $1/3 \leq q_\beta(-1)$ and $q_\beta(1) \leq 3$. These inequalities give $\beta \geq \frac{3}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \simeq 1.41912 = \beta_1$ and $\beta \geq \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \simeq 0.473042 = \beta_2$. Hence the required subordination $p(z) \prec \varphi_c(z)$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$. $\square$

Theorem 5.2. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Let*

$$1 + \frac{\beta z p'(z)}{p(z)} \prec \varphi_s(z).$$

(a) *If $\beta \geq \frac{1}{\log(1+\sin 1)}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \simeq 1.54952$, then $p(z) \prec \varphi_s(z)$.*

(b) *If $\beta \geq \frac{1}{\log 3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \simeq 0.861162$, then $p(z) \prec \varphi_c(z)$.*

Proof. The function

$$q_\beta(z) := \exp\left(\frac{1}{\beta}\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!}\right), \quad z \in \overline{\mathbb{D}},$$

is a solution of differential equation $1 + \beta z q'(z)/q(z) = \varphi_s(z)$. Clearly q_β is analytic in $\mathbb{D}$. Define $\varphi(w) := \beta/w$ and $\theta(w) := 1$, $w \in \mathbb{C}$. Then the function $Q(z) = zq'_\beta(z)\varphi(q_\beta(z)) = \beta z q'_\beta(z)/q_\beta(z) = \sin z$, is starlike in $\mathbb{D}$. And the function $h(z) = \theta(q_\beta(z)) + Q(z) = 1 + \sin z$ satisfies $\operatorname{Re} zh'(z)/Q(z) > 0$. Hence, by making use of Lemma 2.2, the subordination $1 + \beta z p'(z)/p(z) \prec 1 + \beta z q'_\beta(z)/q_\beta(z)$ implies $p(z) \prec q_\beta$. The required subordination $p(z) \prec \mathcal{P}(z)$ for different $\mathcal{P}'s$ holds if $q_\beta(z) \prec \mathcal{P}(z)$, and the subordination $q_\beta(z) \prec \mathcal{P}(z)$ holds if and only if $\mathcal{P}(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \mathcal{P}(1)$.

(a) Let $\mathcal{P}(z) = \varphi_s(z)$. Then the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{-1}{\ln(1-\sin 1)}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$$
$$\text{and } \beta_2 = \frac{1}{\ln(1+\sin 1)}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}.$$

Hence $p(z) \prec \varphi_s(z)$ if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2 \simeq 1.54952$.

(b) For $\mathcal{P}(z) = \varphi_c(z)$, the inequalities $\mathcal{P}(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \mathcal{P}(1)$ reduces to $1/3 \leq q_\beta(-1)$ and $q_\beta(1) \leq 3$. These both inequalities give

$$\beta \geq \frac{1}{\log 3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \simeq 0.861162.$$

Hence the required subordination $p(z) \prec \varphi_c(z)$ holds if $\beta \geq 0.861162$. $\square$

Theorem 5.3. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Let*

$$1 + \frac{\beta z p'(z)}{p^2(z)} \prec \varphi_s(z).$$

(a) *If $\beta \geq \frac{1+\sin 1}{\sin 1}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \simeq 2.0704$, then $p(z) \prec \varphi_s(z)$.*

(b) *If $\beta \geq \frac{3}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \simeq 1.41912$, then $p(z) \prec \varphi_c(z)$.*

Proof. The function

$$q_\beta(z) = \left(1 - \frac{1}{\beta}\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!}\right)^{-1}, \quad z \in \overline{\mathbb{D}},$$

is a solution of differential equation $1 + \beta z q'(z)/q^2(z) = \varphi_s(z)$. Clearly q_β is analytic in $\mathbb{D}$. Define $\varphi(w) := \beta/w^2$ and $\theta(w) := 1$, $w \in \mathbb{C}$. Then the function $Q(z) = zq'_\beta(z)\varphi(q_\beta(z)) =$

$\beta zq'_\beta(z)/q^2_\beta(z) = \sin z$ is starlike in $\mathbb{D}$. Also the function $h(z) = \theta(q_\beta(z)) + Q(z) = 1 + \sin z$ satisfies $\operatorname{Re} zh'(z)/Q(z) > 0$. Hence, by making use of Lemma 2.2, the subordination $1 + \beta zp'(z)/p^2(z) \prec 1 + \beta zq'_\beta(z)/q^2_\beta(z)$ implies $p(z) \prec q_\beta$. The required subordination $p(z) \prec \mathcal{P}(z)$ for different choices of $\mathcal{P}$ holds if and only if $q_\beta(z) \prec \mathcal{P}(z)$.

Part (a) and (b) can be proved as in Theorem 5.2. We omit the details here. ☐

Remark 5.4. Taking $p(z) = zf'(z)/f(z)$, $f \in \mathcal{A}$, in Theorems 5.1, 5.2 and 5.3, and assuming that the subordination $1 + \beta zp'(z)/p^j(z) \prec \varphi_s(z)$, $j = 0, 1, 2$ holds, sufficient conditions in terms of β can be derived for f to be in classes $\mathcal{S}^*_c$ and $\mathcal{S}^*_s$.

6. Subordination associated with exponential function

In this section, we consider the subordination

$$1 + \frac{\beta zp'(z)}{p^j(z)} \prec e^z \quad j = 0, 1, 2,$$

and obtain sharp estimates on β for which $p(z) \prec e^z$, $\varphi_s(z)$, $\varphi_0(z)$, $\varphi_c(z)$, where φ_0 is given by (1.1).

Theorem 6.1. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Let*

$$1 + \beta zp'(z) \prec e^z.$$

(a) *If $\beta \geq \frac{e}{e-1}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n!n} \simeq 1.2602$, then $p(z) \prec e^z$.*
(b) *If $\beta \geq \frac{1}{\sin 1}\sum_{n=1}^\infty \frac{1}{n!n} \simeq 1.56619$, then $p(z) \prec \varphi_s(z)$.*
(c) *If $\beta \geq (2\sqrt{2}+3)\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n!n} \simeq 4.64292$, then $p(z) \prec \varphi_0(z)$.*
(d) *If $\beta \geq \frac{3}{2}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n!n} \simeq 1.1949$, then $p(z) \prec \varphi_c(z)$.*

The bounds on β are sharp.

Proof. The function

$$q_\beta(z) = 1 + \frac{1}{\beta}\sum_{n=1}^\infty \frac{z^n}{n!n}, \quad z \in \overline{\mathbb{D}},$$

is a solution of differential equation $1 + \beta zq'(z) = e^z$. Clearly q_β is analytic in $\mathbb{D}$. Define $\varphi(w) := \beta$ and $\theta(w) := 1$, $w \in \mathbb{C}$. Then the function $Q(z) = zq'_\beta(z)\varphi(q_\beta(z)) = \beta zq'_\beta(z) = e^z - 1$ is starlike in $\mathbb{D}$. And the function $h(z) = \theta(q_\beta(z)) + Q(z) = e^z$ satisfies $\operatorname{Re} zh'(z)/Q(z) > 0$. Hence, by making use of Lemma 2.2, the subordination $1 + \beta zp'(z) \prec 1 + \beta zq'_\beta(z)$ implies $p(z) \prec q_\beta$. The required subordination $p(z) \prec \mathcal{P}(z)$ for different choices of $\mathcal{P}$ holds if $q_\beta(z) \prec \mathcal{P}(z)$, and the subordination $q_\beta(z) \prec \mathcal{P}(z)$ holds if and only if $\mathcal{P}(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \mathcal{P}(1)$. Then the parts (a)-(d) can be proved by following the similar procedure as in previous theorems. ☐

Next, we state the following two theorems without proof.

Theorem 6.2. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Let $1 + \dfrac{\beta zp'(z)}{p(z)} \prec e^z$.*

(a) *If $\beta \geq \sum_{n=1}^\infty \frac{1}{n!n} \simeq 1.3179$, then $p(z) \prec e^z$.*
(b) *If $\beta \geq \frac{1}{\log(1+\sin 1)}\sum_{n=1}^\infty \frac{1}{n!n} \simeq 2.1585$, then $p(z) \prec \varphi_s(z)$.*
(c) *If $\beta \geq \frac{1}{\log 2}\sum_{n=1}^\infty \frac{1}{n!n} \simeq 0.9135$, then $p(z) \prec \varphi_0(z)$.*
(d) *If $\beta \geq \frac{1}{\log 3}\sum_{n=1}^\infty \frac{1}{n!n} \simeq 1.19961$, then $p(z) \prec \varphi_c(z)$.*

The bounds on β are sharp.

Theorem 6.3. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Let $1 + \dfrac{\beta zp'(z)}{p^2(z)} \prec e^z$.*

(a) *If $\beta \geq \frac{e}{e-1}\sum_{n=1}^\infty \frac{1}{n!n} \simeq 2.08489$, then $p(z) \prec e^z$.*

(b) *If* $\beta \geq \frac{1+\sin 1}{\sin 1}\sum_{n=1}^{\infty}\frac{1}{n!n} \simeq 2.88409$*, then* $p(z) \prec \varphi_s(z)$*.*
(c) *If* $\beta \geq 2\sum_{n=1}^{\infty}\frac{1}{n!n} \simeq 2.6358$*, then* $p(z) \prec \varphi_0(z)$*.*
(d) *If* $\beta \geq \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n!n} \simeq 1.97685$*, then* $p(z) \prec \varphi_c(z)$*.*

The bounds on β *are sharp.*

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