Konuralp Journal of Mathematics, 9 (1) (2021) 10-18

Konuralp Journal of Mathematics

Research Paper
https://dergipark.org.tr/en/pub/konuralpjournalmath
e-ISSN: 2147-625X

Composition of Three Entire Functions with Finite Iterated Order

Dibyendu Banerjee[1*] **and Mithun Adhikary**[1]

[1]*Department of Mathematics, Visva-Bharati, Santiniketan-731235, India*
[*]*Corresponding author*

Abstract

The purpose of this paper is to investigate the growth of three composite entire functions of finite iterated order by extending some results of Jin Tu et.al [11].

Keywords: *Order, Iterated i-order, Entire function, Composition.*
2010 Mathematics Subject Classification: *30D35.*

1. Introduction and Definitions

For two transcendental entire functions $f(z)$ and $g(z)$ it is well known by a result of Clunie [3] that $\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,f)}=\infty$ and $\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}=\infty$. Many authors $[5,6,7,10,12]$ made close investigation on composition of two entire functions with finite order and obtained many interesting results. In [11], Jin Tu et.al investigated the composition of entire functions with finite iterated order and proved various results on comparative growths of $\log^{[p+q]}T(r,f\circ g)\ \ (p,q)\in\mathbb{N}$ with $\log^{[p]}T(r,f)$ and $\log^{[q]}T(r,g)$. The aim of this paper is to investigate the composition of three entire functions with finite iterated order and extend some results of Jin Tu et.al [11] for composition of three entire functions. We first introduce the notions of iterated order [5].

Definition 1.1. *The iterated i order $\rho_i(f)$ of an entire function f is defined by*

$$\rho_i(f)=\limsup_{r\to\infty}\frac{log^{[i+1]}M(r,f)}{logr}=\limsup_{r\to\infty}\frac{log^{[i]}T(r,f)}{logr},(i\in\mathbb{N}).$$

Similarly, the iterated i lower order $\mu_i(f)$ of an entire function f is defined by

$$\mu_i(f)=\liminf_{r\to\infty}\frac{log^{[i+1]}M(r,f)}{logr}=\liminf_{r\to\infty}\frac{log^{[i]}T(r,f)}{logr},\ (i\in\mathbb{N}),$$

where

$$\log^{[1]}(r)=\log(r),\ \ \log^{[i+1]}(r)=\log(\log^{[i]}(r))\ i\in\mathbb{N},\ \textit{for all sufficiently large } r.$$

Definition 1.2. *The finiteness degree of the order of an entire function $f(z)$ is defined by*

$$i(f)=\begin{cases}0 & \textit{for f polynomial,}\\ min\{j\in\mathbb{N}:\rho_j(f)<\infty\} & \textit{for f transcendental for which some } j\in\mathbb{N} \textit{ with } \rho_j(f)<\infty \textit{ exists,}\\ \infty & \textit{for f with } \rho_j(f)=\infty \textit{ for all } j\in\mathbb{N}.\end{cases}$$

Throughout we assume f, g, h etc. are non-constant entire functions of finite iterated order and c_1,c_2,c_3 etc. are suitable constants.

Email addresses: dibyendu192@rediffmail.com (Dibyendu Banerjee), adhikary.421.mithun@gmail.com (Mithun Adhikary)

2. Known lemmas

In this section we present three lemmas which will be needed in the sequel.

Lemma 2.1. *[8] Let $f(z)$ and $g(z)$ be two entire functions. If $M(r,g) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ for any $\varepsilon > 0$, then*

$$T(r, f \circ g) < (1+\varepsilon)T(M(r,g), f).$$

In particular if $g(0) = 0$, then

$$T(r, f \circ g) \le T(M(r,g), f)$$

for all $r > 0$.

Lemma 2.2. *[3] Let $f(z)$ and $g(z)$ be two entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Then for $r > 0$*

$$M(M(r,g), f) \ge M(r, f \circ g) \ge M(c(\alpha)M(\alpha r, g), f).$$

Furthermore if $\alpha = \frac{1}{2}$, for sufficiently large r

$$M(r, f \circ g) \ge M(\frac{1}{8}M(\frac{1}{2}r, g), f).$$

Lemma 2.3. *[9] Let $f(z)$ and $g(z)$ be two entire functions. Then for all large values of r*

$$T(r, f \circ g) \ge \frac{1}{3}\log M(\frac{1}{9}M(\frac{r}{4}, g), f).$$

3. Main theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let f, g, h be three entire functions of finite iterated order with $i(f) = p$, $i(g) = q$, $i(h) = s$ and if $\mu_p(f) > 0$, $\mu_q(g) > 0$ then $i(f \circ g \circ h) = p+q+s$ and $\rho_{[p+q+s]}(f \circ g \circ h) = \rho_s(h)$.*

Proof. We have for sufficiently large r and for any given $\varepsilon > 0$

$T(r, f) \le exp^{[p-1]}\left\{r^{\rho_p(f)+\varepsilon}\right\}, \quad M(r,g) \le exp^{[q]}\left\{r^{\rho_q(g)+\varepsilon}\right\}$ *and* $M(r,h) \le exp^{[s]}\left\{r^{\rho_s(h)+\varepsilon}\right\}$.

Using Lemma 2.1, we have for sufficiently large r

$$\begin{aligned} T(r, f \circ g \circ h) &\le (1+o(1))T(M(r,h), f \circ g) \\ &\le (1+o(1))T(M(M(r,h), g), f) \\ &\le (1+o(1))exp^{[p-1]}\left\{[M(M(r,h),g)]^{\rho_p(f)+\varepsilon}\right\} \\ &\le (1+o(1))exp^{[p]}\left\{c_1 exp^{[q-1]}\left\{[M(r,h)]^{\rho_q(g)+\varepsilon}\right\}\right\} \\ &\le (1+o(1))exp^{[p]}\left\{c_1 exp^{[q]}\left\{c_2 exp^{[s-1]}\left\{r^{\rho_s(h)+\varepsilon}\right\}\right\}\right\} \\ &\le exp^{[p+q+s-1]}\left\{r^{\rho_s(h)+2\varepsilon}\right\}. \end{aligned} \tag{3.1}$$

Now by (3.1) we have

$$\limsup_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f \circ g \circ h)}{\log r} \le \rho_s(h). \tag{3.2}$$

Again $i(h) = s$, so we have

$$\limsup_{r\to\infty} \frac{\log^{[s+1]} M(r,h)}{\log r} = \rho_s(h).$$

If $\rho_s(h) > 0$, there exists a sequence $\{r_n\} \to \infty$ such that for any ε $(0 < \varepsilon < \rho_s(h))$ and for sufficiently large r_n, we have

$$M(r_n, h) \ge exp^{[s]}\left\{r_n^{\rho_s(h)-\varepsilon}\right\}. \tag{3.3}$$

We denote $\{r_n\}$ a sequence tending to infinity not necessarily the same at each occurrence. Since $\mu_p(f) > 0$, $\mu_q(g) > 0$, then from Lemma 2.2 and Lemma 2.3 we have

$$\begin{aligned} T(r_n, f \circ g \circ h) &\ge \frac{1}{3}\log M(\frac{1}{9}M(\frac{1}{8}M(\frac{r_n}{8}, h), g)f) \\ &\ge \frac{1}{3}exp^{[p-1]}\left\{[\frac{1}{9}M(\frac{1}{8}M(\frac{r_n}{8}, h), g)]^{\mu_p(f)-\varepsilon}\right\} \\ &\ge \frac{1}{3}exp^{[p]}\left\{c_3 exp^{[q-1]}\left\{[M(\frac{r_n}{8}, h)]^{\mu_q(g)-\varepsilon}\right\}\right\}. \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\ge \frac{1}{3}exp^{[p]}\left\{c_3 exp^{[q]}\left\{c_4 exp^{[s-1]}\left\{r_n^{\rho_s(h)-\varepsilon}\right\}\right\}\right\} \quad \textit{using } (3.3) \\ &\ge exp^{[p+q+s-1]}\left\{r_n^{\rho_s(h)-2\varepsilon}\right\}. \end{aligned} \tag{3.5}$$

So,

$$\limsup_{r\to\infty}\frac{\log^{[p+q+s]}T(r,f\circ g\circ h)}{\log r}\geq\rho_s(h). \tag{3.6}$$

Therefore from (3.2) and (3.6) we have

$$\limsup_{r\to\infty}\frac{\log^{[p+q+s]}T(r,f\circ g\circ h)}{\log r}=\rho_s(h).$$

Thus $i(f\circ g\circ h)=p+q+s$ and $\rho_{[p+q+s]}(f\circ g\circ h)=\rho_s(h)$ for $\rho_s(h)>0$.
If $\rho_s(h)=0$, then by definition we have

$$\limsup_{r\to\infty}\frac{\log^{[s]}M(r,h)}{\log r}=\infty.$$

Hence there exists a sequence $\{r_n\}\to\infty$ such that for any arbitrary $A>0$, we have

$$\frac{\log^{[s]}M(r_n,h)}{\log r_n}\geq A \quad i.e., M(r_n,h)\geq exp^{[s-1]}\left\{r_n^A\right\}. \tag{3.7}$$

So from (3.4) and (3.7) we have

$$\begin{aligned} T(r_n,f\circ g\circ h) &\geq \frac{1}{3}exp^{[p]}\left\{c_5 exp^{[q]}\left\{c_6\log M(\frac{r_n}{4},h)\right\}\right\}\\ &\geq \frac{1}{3}exp^{[p]}\left\{c_5 exp^{[q]}\left\{c_6 exp^{[s-2]}\left\{(\frac{r_n}{4})^A\right\}\right\}\right\}\\ &\geq \frac{1}{3}exp^{[p+q+s-2]}\left\{(\frac{r_n}{4})^{A-\varepsilon}\right\}. \end{aligned}$$

So,

$$\frac{\log^{[p+q+s-1]}T(r_n,f\circ g\circ h)}{\log r_n}\geq(A-\varepsilon).$$

Hence $\lim_{r\to\infty}\frac{\log^{[p+q+s-1]}T(r,f\circ g\circ h)}{\log r}\geq A$. Since A is arbitrary large, then we get

$$\limsup_{r\to\infty}\frac{\log^{[p+q+s-1]}T(r,f\circ g\circ h)}{\log r}=\infty.$$

Therefore $i(f\circ g\circ h)=p+q+s$ and $\rho_{[p+q+s]}(f\circ g\circ h)=\rho_s(h)$.

Theorem 3.2. *Let f, g, h be three entire functions of finite iterated order such that $0<\rho_p(f)<\infty$, $0<\mu_q(g)\leq\rho_q(g)<\infty$ and $0<\mu_s(h)\leq\rho_s(h)<\infty$, then $i(f\circ g\circ h)=p+q+s$ and $\mu_s(h)\leq\rho_{[p+q+s]}(f\circ g\circ h)\leq\rho_s(h)$.*

Proof. Since $\rho_p(f)>0$, there exists a sequence $\{R_n\}$ tending to infinity such that for any ε $(0<\varepsilon<\rho_p(f))$ and sufficiently large R_n, we have

$$M(R_n,f)\geq exp^{[p]}\left\{{R_n}^{\rho_p(f)-\varepsilon}\right\}. \tag{3.8}$$

Since $M(r,h)$ is an increasing continuous function, then there exists a sequence $\{r_n\}$ tending to infinity satisfying $R_n=\frac{1}{8}M(\frac{1}{16}M(\frac{r_n}{2},h),g)$ such that for sufficiently large r_n and by Lemma 2.2, we have

$$\begin{aligned} M(r_n,f\circ g\circ h) &\geq M(\frac{1}{8}M(\frac{1}{16}M(\frac{r_n}{2},h),g),f)\\ &\geq exp^{[p]}\left\{{R_n}^{\rho_p(f)-\varepsilon}\right\} \text{ using (3.8)}\\ &\geq exp^{[p]}[\left\{\frac{1}{8}M(\frac{1}{16}M(\frac{r_n}{2},h),g)\right\}^{\rho_p(f)-\varepsilon}]\\ &\geq exp^{[p+1]}\left\{c_1 exp^{[q-1]}\left\{\frac{1}{16}M(\frac{r_n}{2},h)\right\}^{\mu_q(g)-\varepsilon}\right\}\\ &\geq exp^{[p+1]}\left\{c_1 exp^{[q]}\left\{c_2 exp^{[s-1]}(\frac{r_n}{2})^{\mu_s(h)-\varepsilon}\right\}\right\}\\ &\geq exp^{[p+q+s]}\left\{{r_n}^{\mu_s(h)-2\varepsilon}\right\}. \end{aligned} \tag{3.9}$$

So,

$$\frac{\log^{[p+q+s+1]}M(r_n,f\circ g\circ h)}{\log r_n}\geq\mu_s(h)-2\varepsilon$$

i.e.,

$$\rho_{[p+q+s]}(f\circ g\circ h)\geq\mu_s(h). \tag{3.10}$$

Again for sufficiently large r, we have from Lemma 2.2

$$\begin{aligned} M(r, f\circ g\circ h) &\leq M(M(M(r,h),g),f) \\ &\leq exp^{[p]}[\{M(M(r,h),g)\}^{\rho_p(f)+\varepsilon}] \\ &\leq exp^{[p+1]}[c_3 exp^{[q-1]}\{M(r,h)\}^{\rho_q(g)+\varepsilon}] \\ &\leq exp^{[p+1]}[c_3 exp^{[q]}\left\{c_4 exp^{[s-1]}\left\{r^{\rho_s(h)+\varepsilon}\right\}\right\}] \\ &\leq exp^{[p+q+s]}\left\{r^{\rho_s(h)+2\varepsilon}\right\}. \end{aligned} \tag{3.11}$$

So,

$$\frac{\log^{[p+q+s+1]} M(r, f\circ g\circ h)}{\log r} \leq \rho_s(h)+2\varepsilon$$

i.e.,

$$\rho_{[p+q+s]}(f\circ g\circ h) \leq \rho_s(h). \tag{3.12}$$

Therefore from (3.10) and (3.12) we get

$$\mu_s(h) \leq \rho_{[p+q+s]}(f\circ g\circ h) \leq \rho_s(h).$$

This completes the proof of the Theorem 3.2.

Theorem 3.3. *Let f, g, h be three entire functions of iterated order with $i(f)=p$, $i(g)=q$, $i(h)=s$ and $\rho_s(h)<\mu_p(f)\leq\rho_p(f)$, then* $\lim_{r\to\infty}\frac{\log^{[q+s]}T(r,f\circ g\circ h)}{T(r,f)}=0$ *and* $\lim_{r\to\infty}\frac{\log^{[q+s+1]}M(r,f\circ g\circ h)}{\log M(r,f)}=0$.

Proof. By definition, for sufficiently large r, we have

$$exp^{[p-1]}\left\{r^{\mu_p(f)-\varepsilon}\right\} \leq T(r,f) \leq exp^{[p-1]}\left\{r^{\rho_p(f)+\varepsilon}\right\},\ \ M(r,h) \leq exp^{[s]}\left\{r^{\rho_s(h)+\varepsilon}\right\}. \tag{3.13}$$

By (3.1), we have

$$T(r, f\circ g\circ h) \leq exp^{[p+q+s-1]}\left\{r^{\rho_s(h)+2\varepsilon}\right\}.$$

Hence for sufficiently large r and for any given ε, we have from (3.13)

$$\frac{\log^{[q+s]} T(r, f\circ g\circ h)}{T(r,f)} \leq \frac{exp^{[p-1]}\left\{r^{\rho_s(h)+2\varepsilon}\right\}}{exp^{[p-1]}\left\{r^{\mu_p(f)-\varepsilon}\right\}} \to 0$$

i.e.,

$$\lim_{r\to\infty}\frac{\log^{[q+s]} T(r, f\circ g\circ h)}{T(r,f)} = 0.$$

Similarly for sufficiently large r, we have

$$exp^{[p-1]}\left\{r^{\mu_p(f)-\varepsilon}\right\} \leq \log M(r,f) \leq exp^{[p-1]}\left\{r^{\rho_p(f)+\varepsilon}\right\},\ \ M(r,h) \leq exp^{[s]}\left\{r^{\rho_s(h)+\varepsilon}\right\}. \tag{3.14}$$

Again by (3.11), we have

$$M(r, f\circ g\circ h) \leq exp^{[p+q+s]}\left\{r^{\rho_s(h)+2\varepsilon}\right\}.$$

Hence from (3.14), we get

$$\frac{\log^{[q+s+1]} M(r, f\circ g\circ h)}{\log M(r,f)} \leq \frac{exp^{[p-1]}\left\{r^{\rho_s(h)+2\varepsilon}\right\}}{exp^{[p-1]}\left\{r^{\mu_p(f)-\varepsilon}\right\}} \to 0$$

i.e.,

$$\lim_{r\to\infty}\frac{\log^{[q+s+1]} M(r, f\circ g\circ h)}{\log M(r,f)} = 0.$$

Example 3.1. The condition $\rho_s(h)<\mu_p(f)$ in Theorem 3.3 is necessary. To see this we consider the following example.
Let $f(z)=exp(z)$, $g(z)=exp^{[2]}(z)$, $h(z)=exp^{[3]}(z)$ and $p=1$, $q=2$, $s=3$. Then we have
$\rho_3(h)$=$\limsup_{r\to\infty}\frac{\log^{[4]}M(r,h)}{\log r}=\lim_{r\to\infty}\frac{\log r}{\log r}=1$ and $\mu_1(f)=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r}=1$.
But $\lim_{r\to\infty}\frac{\log^{[6]}M(r,f\circ g\circ h)}{\log M(r,f)}=\lim_{r\to\infty}\frac{r}{r}=1\neq 0$.

Theorem 3.4. *Let f, g, h be three entire functions of finite iterated order with $i(f)=p$, $i(g)=q$, $i(h)=s$ and $\mu_s(h)<\mu_p(f)\leq\rho_p(f)$ then* $\liminf_{r\to\infty}\frac{\log^{[q+s]}T(r,f\circ g\circ h)}{T(r,f)}=0$ *and* $\liminf_{r\to\infty}\frac{\log^{[q+s+1]}M(r,f\circ g\circ h)}{\log M(r,f)}=0$.

Proof. By definition for sufficiently large r we have

$$exp^{[p-1]}\left\{r^{\mu_p(f)-\varepsilon}\right\} \leq T(r,f) \leq exp^{[p-1]}\left\{r^{\rho_p(f)+\varepsilon}\right\}, \;\; M(r,h) \leq exp^{[s]}\left\{r^{\rho_s(h)+\varepsilon}\right\}. \tag{3.15}$$

By (3.15) and using Lemma 2.1, we get

$$\begin{aligned} T(r,f\circ g\circ h) &\leq 2T(M(M(r,h),g),f) \\ &\leq 2exp^{[p-1]}[\{M(M(r,h),g)\}^{\rho_p(f)+\varepsilon}] \\ &\leq 2exp^{[p]}[c_1 exp^{[q-1]}\left\{\{M(r,h)\}^{\rho_q(g)+\varepsilon}\right\}]. \end{aligned}$$

Hence for a sequence $\{r_n\} \to \infty$ we can get from above

$$T(r_n,f\circ g\circ h) \leq 2exp^{[p]}[c_1 exp^{[q]}\left\{c_2 exp^{[s-1]}(r_n^{\mu_s(h)+\varepsilon})\right\}].$$

So,

$$T(r_n,f\circ g\circ h) \leq 2exp^{[p+q+s-1]}\left\{r_n^{\mu_s(h)+2\varepsilon}\right\}. \tag{3.16}$$

From (3.15) and (3.16) for a sequence of values of $\{r_n\}$ tending to infinity, we get

$$\frac{\log^{[q+s]} T(r_n,f\circ g\circ h)}{T(r_n,f)} \leq \frac{exp^{[p-1]}\left\{r_n^{\mu_s(h)+2\varepsilon}\right\}}{exp^{[p-1]}\left\{r_n^{\mu_p(f)-\varepsilon}\right\}} \to 0.$$

Therefore

$$\liminf_{r\to\infty} \frac{\log^{[q+s]} T(r,f\circ g\circ h)}{T(r,f)} = 0.$$

Again for sufficiently large r, we have

$$exp^{[p-1]}\left\{r^{\mu_p(f)-\varepsilon}\right\} \leq \log M(r,f) \leq exp^{[p-1]}\left\{r^{\rho_p(f)+\varepsilon}\right\} \tag{3.17}$$

and

$$M(r,h) \leq exp^{[s]}\left\{r^{\rho_s(h)+\varepsilon}\right\}.$$

So for all large values of r, using Lemma 2.2

$$\begin{aligned} M(r,f\circ g\circ h) &\leq M(M(M(r,h),g),f) \\ &\leq exp^{[p]}[\{M(M(r,h),g)\}^{\rho_p(f)+\varepsilon} \\ &\leq exp^{[p+1]}[c_3 exp^{[q-1]}\left\{(M(r,h))^{\rho_q(g)+\varepsilon}\right\}]. \end{aligned}$$

Hence for a sequence $\{r_n\} \to \infty$ we get

$$\begin{aligned} M(r_n,f\circ g\circ h) &\leq exp^{[p+1]}[c_3 exp^{[q]}\left\{\left\{c_4 exp^{[s-1]}(r_n)^{\mu_s(h)+\varepsilon}\right\}\right\}] \\ &\leq exp^{[p+q+s]}\left\{(r_n)^{\mu_s(h)+2\varepsilon}\right\}. \end{aligned} \tag{3.18}$$

From (3.17) and (3.18) we get for a sequence $\{r_n\} \to \infty$

$$\frac{\log^{[q+s+1]} M(r_n,f\circ g\circ h)}{\log M(r,f)} \leq \frac{exp^{[p-1]}\left\{r_n^{\mu_s(h)-2\varepsilon}\right\}}{exp^{[p-1]}\left\{r^{\mu_p(f)-\varepsilon}\right\}} \to 0$$

i.e.,

$$\liminf_{r\to\infty} \frac{\log^{[q+s+1]} M(r,f\circ g\circ h)}{\log M(r,f)} = 0.$$

Theorem 3.5. *Let f, g, h be three entire functions of finite iterated order with $i(f)=p$, $i(g)=q$, $i(h)=s$, $\mu_q(g)>0$ and $0<\mu_p(f)\leq \rho_p(f)<\rho_s(h)<\infty$ then $\liminf_{r\to\infty}\frac{\log^{[q+s]} T(r,f\circ g\circ h)}{T(r,f)}=\infty$ and $\liminf_{r\to\infty}\frac{\log^{[q+s+1]} M(r,f\circ g\circ h)}{\log M(r,f)}=\infty$.*

Proof. By definition, there exists a sequence $\{r_n\} \to \infty$ and for any given $\varepsilon(>0)$, we have

$$M(r_n,h) \geq exp^{[s]}\left\{r_n^{\rho_s(h)-\varepsilon}\right\}, \quad T(r_n,f) \leq exp^{[p-1]}\left\{r_n^{\rho_p(f)+\varepsilon}\right\}. \tag{3.19}$$

Also from (3.5)

$$\log^{[q+s]} T(r_n, f\circ g\circ h) \geq exp^{[p-1]}\left\{r_n^{\rho_s(h)-2\varepsilon}\right\}.$$

Hence from (3.19)

$$\frac{\log^{[q+s]} T(r_n, f\circ g\circ h)}{T(r_n,f)} \geq \frac{exp^{[p-1]}\left\{r_n^{\rho_s(h)-2\varepsilon}\right\}}{exp^{[p-1]}\left\{r_n^{\rho_p(f)+\varepsilon}\right\}}.$$

Since $\rho_s(h) > \rho_p(f)$, so

$$\liminf_{r\to\infty} \frac{\log^{[q+s]} T(r, f\circ g\circ h)}{T(r,f)} = \infty.$$

Similarly we also have

$$\liminf_{r\to\infty} \frac{\log^{[q+s+1]} M(r, f\circ g\circ h)}{\log M(r,f)} = \infty.$$

Theorem 3.6. *Let f, g, h be three entire functions of finite iterated order such that $0 < \mu_p(f) < \mu_s(h) < \infty$, then $\limsup_{r\to\infty} \frac{\log^{[q+s]} T(r,f\circ g\circ h)}{T(r,f)} = \infty$ and $\limsup_{r\to\infty} \frac{\log^{[q+s+1]} M(r,f\circ g\circ h)}{\log M(r,f)} = \infty$.*

Proof. For all large r we get using Lemma 2.2 and 2.3

$$\begin{aligned} T(r, f\circ g\circ h) &\geq \frac{1}{3}\log M\left(\frac{1}{9}M\left(\frac{1}{8}M\left(\frac{r}{8},h\right),g\right)f\right) \\ &\geq \frac{1}{3}exp^{[p-1]}\left\{\left[\frac{1}{9}M\left(\frac{1}{8}M\left(\frac{r}{8},h\right),g\right)\right]^{\mu_p(f)-\varepsilon}\right\} \\ &\geq \frac{1}{3}exp^{[p]}\left\{c_1 exp^{[q-1]}\left\{\left[M\left(\frac{r}{8},h\right)\right]^{\mu_q(g)-\varepsilon}\right\}\right\} \\ &\geq \frac{1}{3}exp^{[p]}\left\{c_1 exp^{[q]}\left\{c_2 exp^{[s-1]}\left\{r^{\mu_s(h)-\varepsilon}\right\}\right\}\right\} \\ &\geq exp^{[p+q+s-1]}\left\{r^{\mu_s(h)-2\varepsilon}\right\}. \end{aligned} \tag{3.20}$$

By definition there exists a sequence $\{r_n\} \to \infty$ such that for any given $\varepsilon(>0)$, we have

$$T(r_n,f) \leq exp^{[p-1]}\left\{r_n^{\mu_p(f)+\varepsilon}\right\}. \tag{3.21}$$

From (3.20) and (3.21) we get for a sequence $\{r_n\}$ of values of r tending to infinity

$$\frac{\log^{[q+s]} T(r_n, f\circ g\circ h)}{T(r_n,f)} \geq \frac{exp^{[p-1]}\left\{r_n^{\mu_s(h)-2\varepsilon}\right\}}{exp^{[p-1]}\left\{r_n^{\mu_p(f)+\varepsilon}\right\}} \to \infty$$

i.e., $\limsup_{r\to\infty} \frac{\log^{[q+s]} T(r,f\circ g\circ h)}{T(r,f)} = \infty$.
Similarly we can prove that

$$\limsup_{r\to\infty} \frac{\log^{[q+s+1]} M(r, f\circ g\circ h)}{\log M(r,f)} = \infty.$$

Example 3.2. The condition $\mu_p(f) < \mu_s(h)$ in Theorem 3.6 is necessary. To see this we consider the following example.
Let $f(z) = exp^{[2]}(z)$, $g(z) = exp^{[3]}(z)$ $h(z) = exp(z)$, and $p = 2$, $q = 3$, $s = 1$. Then we have
$\mu_2(f)=\liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} = \lim_{r\to\infty} \frac{\log r}{\log r} = 1$ and $\mu_1(h) = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,h)}{\log r} = 1$.
But $\lim_{r\to\infty} \frac{\log^{[5]} M(r,f\circ g\circ h)}{\log M(r,f)} = \lim_{r\to\infty} \frac{\exp r}{\exp r} = 1 \neq \infty$.

Theorem 3.7. *Let f, g, h be three entire functions of finite iterated order with $i(f) = p$, $i(g) = q$, $i(h) = s$ and $0 < \mu_p(f) \leq \rho_p(f) < \mu_s(h) < \infty$ then $\lim_{r\to\infty} \frac{\log^{[q+s]} T(r,f\circ g\circ h)}{T(r,f)} = \infty$ and $\lim_{r\to\infty} \frac{\log^{[q+s+1]} M(r,f\circ g\circ h)}{\log M(r,f)} = \infty$.*

Proof. By definition, for large r and for any given $\varepsilon(>0)$, we have

$$T(r,f) \leq exp^{[p-1]}\left\{r^{\rho_p(f)+\varepsilon}\right\}. \tag{3.22}$$

Again for all large values of r we get from (3.20)

$$T(r, f\circ g\circ h) \geq exp^{[p+q+s-1]}\left\{r^{\mu_s(h)-2\varepsilon}\right\}. \tag{3.23}$$

So from (3.22) and (3.23) we have

$$\frac{\log^{[q+s]} T(r, f\circ g\circ h)}{T(r,f)} \geq \frac{exp^{[p-1]}\left\{r^{\mu_s(h)-2\varepsilon}\right\}}{exp^{[p-1]}\left\{r^{\rho_p(f)+\varepsilon}\right\}}.$$

Since $\mu_s(h) > \rho_p(f)$ and $\varepsilon(>0)$ is arbitrary, so

$$\lim_{r\to\infty} \frac{\log^{[q+s]} T(r, f\circ g\circ h)}{T(r,f)} = \infty.$$

Now from (3.17) and (3.9) we get

$$\frac{\log^{[q+s+1]} M(r_n, f\circ g\circ h)}{\log M(r,f)} \geq \frac{exp^{[p-1]}\left\{r^{\mu_s(h)-2\varepsilon}\right\}}{exp^{[p-1]}\left\{r^{\rho_p(f)+\varepsilon}\right\}} \to \infty$$

i.e.,

$$\lim_{r\to\infty} \frac{\log^{[q+s+1]} M(r, f\circ g\circ h)}{\log M(r,f)} = \infty.$$

Theorem 3.8. *Let f, g, h be three entire functions of finite iterated order such that $0<\mu_p(f)\leq\rho_p(f)<\infty$ and $0<\mu_s(h)\leq\rho_s(h)<\infty$, then*
$\frac{\mu_s(h)}{\rho_p(f)} \leq \liminf_{r\to\infty} \frac{\log^{[p+q+s]} T(r,f\circ g\circ h)}{\log^{[p]} T(r,f)} \leq min\left\{\frac{\mu_s(h)}{\mu_p(f)}, \frac{\rho_s(h)}{\rho_p(f)}\right\} \leq max\left\{\frac{\mu_s(h)}{\mu_p(f)}, \frac{\rho_s(h)}{\rho_p(f)}\right\}$
$\leq \limsup_{r\to\infty} \frac{\log^{[p+q+s]} T(r,f\circ g\circ h)}{\log^{[p]} T(r,f)} \leq \frac{\rho_s(h)}{\mu_p(f)}.$

Proof. By definition for sufficiently large r and for any $\varepsilon(>0)$ we have

$$(\mu_p(f)-\varepsilon)\log r \leq \log^{[p]} T(r,f) \leq (\rho_p(f)+\varepsilon)\log r. \tag{3.24}$$

From (3.5) we can easily say that

$$T(r_n, f\circ g\circ h) \geq \frac{1}{3} exp^{[p+q+s-1]}\left\{r_n^{\mu_s(h)-2\varepsilon}\right\}.$$

So from above and for all large r and any $\varepsilon(>0)$ we have from (3.2)

$$(\mu_s(h)-2\varepsilon)\log r \leq \log^{[p+q+s]} T(r, f\circ g\circ h) \leq (\rho_s(h)+\varepsilon)\log r. \tag{3.25}$$

From (3.24) and (3.25) we get for sufficiently large values of r

$$\frac{\rho_s(h)+\varepsilon}{\mu_p(f)-\varepsilon} \geq \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[p]} T(r,f)} \geq \frac{\mu_s(h)-2\varepsilon}{\rho_p(f)+\varepsilon}. \tag{3.26}$$

Since $\varepsilon>0$, is arbitrary we get from (3.26)

$$\liminf_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[p]} T(r,f)} \geq \frac{\mu_s(h)}{\rho_p(f)}$$

and

$$\limsup_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[p]} T(r,f)} \leq \frac{\rho_s(h)}{\mu_p(f)}.$$

Again by definition, there exist two sequences $\{r_n\}$ and $\{R_n\}$ tending to infinity such that

$$\log^{[p]} T(r_n,f) \geq (\rho_p(f)-\varepsilon)\log r_n, \quad \log^{[p]} T(R_n,f) \leq (\mu_p(f)+\varepsilon)\log R_n. \tag{3.27}$$

From (3.1)

$$T(r, f\circ g\circ h) \leq exp^{[p+q+s-1]}\left\{r^{\mu_s(h)+2\varepsilon}\right\}.$$

So from above and (3.5) there exists two sequences $\{r_n^{'}\}$ and $\{R_n^{'}\}$ tending to infinity such that

$$T(r_n^{'}, f\circ g\circ h) \leq exp^{[p+q+s-1]}\left\{{r_n^{'}}^{\mu_s(h)+2\varepsilon}\right\}$$

and

$$T(R_n^{'}, f\circ g\circ h) \geq exp^{[p+q+s-1]}\left\{{R_n^{'}}^{\rho_s(h)-2\varepsilon}\right\}.$$

Hence

$$\log^{[p+q+s]} T(r_n^{'}, f\circ g\circ h) \leq (\mu_s(h)+2\varepsilon)\log r_n^{'} \;\; and \;\; \log^{[p+q+s]} T(R_n^{'}, f\circ g\circ h) \geq (\rho_s(h)-2\varepsilon)\log R_n^{'}. \tag{3.28}$$

From (3.25) and (3.27) we get

$$\frac{\log^{[p+q+s]} T(r_n, f\circ g\circ h)}{\log^{[p]} T(r_n,f)} \leq \frac{(\rho_s(h)+\varepsilon)\log r_n}{(\rho_p(f)-\varepsilon)\log r_n}$$

i.e.,

$$\liminf_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[p]} T(r,f)} \leq \frac{\rho_s(h)}{\rho_p(f)}.$$

From (3.24) and (3.28) we get

$$\frac{\log^{[p+q+s]} T(r_n', f\circ g\circ h)}{\log^{[p]} T(r_n',f)} \leq \frac{(\mu_s(h)+2\varepsilon)\log r_n'}{(\mu_p(f)-\varepsilon)\log r_n'}$$

i.e.,

$$\liminf_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[p]} T(r,f)} \leq \frac{\mu_s(h)}{\mu_p(f)}.$$

Hence

$$\liminf_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[p]} T(r,f)} \leq min\left\{\frac{\mu_s(h)}{\mu_p(f)}, \frac{\rho_s(h)}{\rho_p(f)}\right\}.$$

Again from (3.25) and (3.27) we get

$$\frac{\log^{[p+q+s]} T(R_n, f\circ g\circ h)}{\log^{[p]} T(R_n,f)} \geq \frac{(\mu_s(h)-2\varepsilon)\log R_n}{(\mu_p(f)+\varepsilon)\log R_n}$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[p]} T(r,f)} \geq \frac{\mu_s(h)}{\mu_p(f)}.$$

From (3.24) and (3.28) we get

$$\frac{\log^{[p+q+s]} T(R_n', f\circ g\circ h)}{\log^{[p]} T(R_n',f)} \geq \frac{(\rho_s(h)-2\varepsilon)\log R_n'}{(\rho_p(f)+\varepsilon)\log R_n'}$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[p]} T(r,f)} \geq \frac{\rho_s(h)}{\rho_p(f)}.$$

Hence

$$\limsup_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[p]} T(r,f)} \geq max\left\{\frac{\mu_s(h)}{\mu_p(f)}, \frac{\rho_s(h)}{\rho_p(f)}\right\}.$$

Therefore
$\frac{\mu_s(h)}{\rho_p(f)} \leq \liminf_{r\to\infty} \frac{\log^{[p+q+s]} T(r,f\circ g\circ h)}{\log^{[p]} T(r,f)} \leq min\left\{\frac{\mu_s(h)}{\mu_p(f)}, \frac{\rho_s(h)}{\rho_p(f)}\right\} \leq max\left\{\frac{\mu_s(h)}{\mu_p(f)}, \frac{\rho_s(h)}{\rho_p(f)}\right\}$
$\leq \limsup_{r\to\infty} \frac{\log^{[p+q+s]} T(r,f\circ g\circ h)}{\log^{[p]} T(r,f)} \leq \frac{\rho_s(h)}{\mu_p(f)}$.
This completes the proof.

Corollary 3.1. Let f, g, h satisfy the hypotheses of Theorem 3.8, then $\frac{\mu_s(h)}{\rho_p(f)} \leq \liminf_{r\to\infty} \frac{\log^{[p+q+s]} T(r,f\circ g\circ h)}{\log^{[p]} T(r,f^{(k)})} \leq min\left\{\frac{\mu_s(h)}{\mu_p(f)}, \frac{\rho_s(h)}{\rho_p(f)}\right\} \leq max\left\{\frac{\mu_s(h)}{\mu_p(f)}, \frac{\rho_s(h)}{\rho_p(f)}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[p+q+s]} T(r,f\circ g\circ h)}{\log^{[p]} T(r,f^{(k)})} \leq \frac{\rho_s(h)}{\mu_p(f)}$. for k=1,2,..

Corollary 3.2. We can obtain the same result when we replace $T(r, f\circ g\circ h), T(r, f)$ with $\log M(r, f\circ g\circ h), \log M(r, f)$ in Theorem 3.8.

Theorem 3.9. *Let f, g, h be three entire functions of finite iterated order such that $0<\mu_p(f)\leq\rho_p(f)<\infty$ and $0<\mu_s(h)\leq\rho_s(h)<\infty$, then*
$\frac{\mu_s(h)}{\rho_s(h)} \leq \liminf_{r\to\infty} \frac{\log^{[p+q+s]} T(r,f\circ g\circ h)}{\log^{[s]} T(r,h)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[p+q+s]} T(r,f\circ g\circ h)}{\log^{[s]} T(r,h)} \leq \frac{\rho_s(h)}{\mu_s(h)}$ *and*
$\frac{\mu_s(h)}{\rho_s(h)} \leq \liminf_{r\to\infty} \frac{\log^{[p+q+s+1]} M(r,f\circ g\circ h)}{\log^{[s+1]} M(r,h)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[p+q+s+1]} M(r,f\circ g\circ h)}{\log^{[s+1]} M(r,h)} \leq \frac{\rho_s(h)}{\mu_s(h)}$.

Proof. For sufficiently large r and for any $\varepsilon>0$, we have

$$\log^{[s]} T(r,h) \leq (\rho_s(h)+\varepsilon)\log r. \tag{3.29}$$

Again for sufficiently large r and Lemma 2.3, we have

$$\begin{aligned} T(r_n, f\circ g\circ h) &\geq \frac{1}{3}\log M(\frac{1}{9}M(\frac{1}{8}M(\frac{r_n}{8},h),g)f) \\ &\geq \frac{1}{3}exp^{[p-1]}\left\{[\frac{1}{9}M(\frac{1}{8}M(\frac{r_n}{8},h),g)]^{\mu_p(f)-\varepsilon}\right\} \\ &\geq \frac{1}{3}exp^{[p]}\left\{c_1 exp^{[q-1]}\left\{[M(\frac{r_n}{8},h)]^{\mu_q(g)-\varepsilon}\right\}\right\} \\ &\geq \frac{1}{3}exp^{[p]}\left\{c_1 exp^{[q]}\left\{c_2 exp^{[s-1]}\left\{r_n^{\mu_s(h)-\varepsilon}\right\}\right\}\right\} \\ &\geq exp^{[p+q+s-1]}\left\{r_n^{\mu_s(h)-2\varepsilon}\right\}. \end{aligned} \tag{3.30}$$

From (3.29) and (3.30) we get

$$\frac{\log^{[p+q+s]} T(r_n, f\circ g\circ h)}{\log^{[s]} T(r_n,h)} \geq \frac{(\mu_s(h)-2\varepsilon)\log r_n}{(\rho_s(h)+\varepsilon)\log r_n}.$$

As $\varepsilon > 0$ is any arbitrary so,

$$\liminf_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[s]} T(r,h)} \geq \frac{\mu_s(h)}{\rho_s(h)}.$$

Again by definition, there exists a sequence $\{r_n\}$ tending to infinity such that

$$\log^{[s]} T(r_n,h) \geq (\rho_s(h)-\varepsilon)\log r_n. \tag{3.31}$$

From (3.2) for any given $\varepsilon > 0$ and sufficiently large r, we have

$$\log^{[p+q+s]} T(r, f\circ g\circ h) \leq (\rho_s(h)+\varepsilon)\log r, \quad \log^{[s]} T(r,h) \leq (\rho_s(h)+\varepsilon)\log r, \qquad \log^{[s]} T(r,h) \geq (\mu_s(h)-\varepsilon)\log r. \tag{3.32}$$

From (3.31) and (3.32) we get

$$\frac{\log^{[p+q+s]} T(r_n, f\circ g\circ h)}{\log^{[s]} T(r_n,h)} \leq \frac{(\rho_s(h)+\varepsilon)\log r_n}{(\rho_s(h)-\varepsilon)\log r_n}$$

i.e.,

$$\liminf_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[s]} T(r,h)} \leq 1.$$

Again from (3.32) we get

$$\frac{\log^{[p+q+s]} T(r_n, f\circ g\circ h)}{\log^{[s]} T(r_n,h)} \leq \frac{(\rho_s(h)+\varepsilon)\log r_n}{(\mu_s(h)-\varepsilon)\log r_n}$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[s]} T(r,h)} \leq \frac{\rho_s(h)}{\mu_s(h)}.$$

Again for a sequence $\{R_n\}$ tending to infinity we have from (3.5)

$$T(R_n, f\circ g\circ h) \geq exp^{[p+q+s-1]}\left\{R_n^{\rho_s(h)-2\varepsilon}\right\}. \tag{3.33}$$

From (3.32) and (3.33), we get

$$\frac{\log^{[p+q+s]} T(R_n, f\circ g\circ h)}{\log^{[s]} T(R_n,h)} \geq \frac{(\rho_s(h)-2\varepsilon)\log R_n}{(\rho_s(h)+\varepsilon)\log R_n}.$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log^{[p+q+s]} T(r, f\circ g\circ h)}{\log^{[s]} T(r,h)} \geq 1.$$

Combining all we get
$\frac{\mu_s(h)}{\rho_s(h)} \leq \liminf_{r\to\infty} \frac{\log^{[p+q+s]} T(r,f\circ g\circ h)}{\log^{[s]} T(r,h)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[p+q+s]} T(r,f\circ g\circ h)}{\log^{[s]} T(r,h)} \leq \frac{\rho_s(h)}{\mu_s(h)}.$
Similarly as above we can show that
$\frac{\mu_s(h)}{\rho_s(h)} \leq \liminf_{r\to\infty} \frac{\log^{[p+q+s+1]} M(r,f\circ g\circ h)}{\log^{[s+1]} M(r,h)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[p+q+s+1]} M(r,f\circ g\circ h)}{\log^{[s+1]} M(r,h)} \leq \frac{\rho_s(h)}{\mu_s(h)}.$

Acknowledgement

The authors are thankful to the referees for several suggestions which considerably improve the presentation of the paper.

References

[1] W. Bergweiler , On the growth rate of composite meromorphic functions, complex Var. Elliptic Equ., 14(1990), 187-196.
[2] W. Bergweiler, Order and lower order of composite meromorphic functions, Michigam Math. J., 36(1989), 135-146.
[3] J. Clunie, The composition of entire and meromorphic functions, Mathematical essays dedicated to A.J. Macintyre, Ohio University Press (1970), 75-92.
[4] W.K.Hayman, Meromorphic functions, The clarendon Press, Oxford(1964).
[5] L. Kinnunen, Linear differential equations with solutions of finite iterated order, Southeast Asian Bull. Math. 22(4), (1998) 385-405.
[6] I.Lahiri and D.K.Sharma . Growth of composite entire and meromorphic functions,Indian journal pure appl. math,26(4)(1995),5451-58.
[7] I.Lahiri and S.K.Datta, on the growth of entire and meromorphic functions,Indian journal pure appl. math,35(4)(2004),525-43.
[8] K.Niino and N. Smita, Growth of composite function of entiure functions, Kodai Math. J., 3(1980),374-379.
[9] K.Niino and C.C. Yang, Some growth relationship on factors of two compositive entire functions, in: Factorization theory of Meromorphic functions and related Topics, Marcel Dekker Inc., New York/Basel,1982, pp. 95-99.
[10] A. P. Singh, Growth of composite entire functions, Kodai Math. J.,8(1985), 99-102.
[11] J. Tu, Z.-X.Chen and X.-M.Zheng, Composition of entire fuynctions with finite iterated order , J. Math. Anal. Appl., 353(2009), 295-304.
[12] Z.-Z.Zhou, Growth of composite entire functions, Kodai Math. J., 9(1986), 419-420.