Research Article GU J Sci32(3): 956-965 (2019) DOI: 10.35378/gujs.441422

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Hyperbolic Horadam Functions

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Highlights

- We define hyperbolic Horadam functions.
- We present their hyperbolic and recursive properties.
- Our findings generalize the previous studies.

Article Info

Received: 06/07/2018
Accepted: 18/03/2019

Keywords

Horadam numbers
Hyperbolic functions

Abstract

This article introduce hyperbolic functions connected to Horadam sequence. That is, we define hyperbolic Horadam functions and present their hyperbolic and recursive properties. We give some geometrical properties of hyperbolic Horadam functions.

1. INTRODUCTION

In [1], the authors appreciated the importance of hyperbolic functions and their analogs in the following words:

"Discovering that the world around us is hyperbolic is probably one of the major achievements of science. In 1827, Lobachevsky offered a new geometric system based on hyperbolic functions. In 1993, Stakhov and Tkachenko developed a new approach to the hyperbolic geometry. Using Binet formulas, they developed a new class of hyperbolic functions called the Hyperbolic Fibonacci and Lucas functions. In 2005, Stakhov and Rozin, developed a new surface of the second degree, called the golden shofar. The hyperbolic Fibonacci and Lucas functions and the surface of the golden shofar are the representatives of the golden mathematical models used for modeling hyperbolic space."

The Fibonacci and Lucas numbers are generated by the rules: $F_{n+1}=F_n+F_{n-1}$ $(n\geq 1)$ with $F_0=0$, $F_1=1$ and $L_{n+1}=L_n+L_{n-1}$ $(n\geq 1)$ with $L_0=2$ and $L_1=1$, respectively. The Fibonacci numbers have many continuous forms and generalizations [1-11]. Horadam number sequence $W_n(a,b:p,q)$, or briefly W_n, is generated by the rule:

$$W_{n+1}=pW_n+qW_{n-1}, \tag{1}$$

where $n\geq 1$, $W_0=a$, $W_1=b$, p and q are nonzero real numbers [3]. The characteristic equation of W_n is

$$t^2-pt-q=0. \tag{2}$$

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The roots of Eq. (2) are:

$$\alpha=\frac{p+\sqrt{p^2+4q}}{2} \text{ and } \beta=\frac{p-\sqrt{p^2+4q}}{2}. \tag{3}$$

Also, the Binet formula for W_n is

$$W_n=\frac{A\alpha^n-B\beta^n}{\alpha-\beta}=\frac{A\alpha^n-B\beta^n}{\sqrt{p^2+4q}}, \tag{4}$$

where

$$A=b-a\beta \text{ and } B=b-a\alpha. \tag{5}$$

Since $\alpha-\beta=\sqrt{p^2+4q}$ and $\alpha\beta=-q$, the Eq.(4) is rewritten as

$$W_n=\frac{A\alpha^n-(-1)^n Bq^n\alpha^{-n}}{\sqrt{p^2+4q}}=\begin{cases}\dfrac{A\alpha^n+Bq^n\alpha^{-n}}{\sqrt{p^2+4q}}, & n \text{ is odd}\\ \dfrac{A\alpha^n-Bq^n\alpha^{-n}}{\sqrt{p^2+4q}}, & n \text{ is even.}\end{cases} \tag{6}$$

The Horadam number W_n is reduced to some famous number sequences as:

$$\text{the Fibonacci number } F_n=W_n(0,1:1,1)=\begin{cases}\dfrac{\alpha^n+\alpha^{-n}}{\sqrt{5}}, & n \text{ is odd}\\ \dfrac{\alpha^n-\alpha^{-n}}{\sqrt{5}}, & n \text{ is even,}\end{cases} \tag{7}$$

$$\text{the Lucas number } L_n=W_n(2,1:1,1)=\begin{cases}\alpha^n-\alpha^{-n}, & n \text{ is odd}\\ \alpha^n+\alpha^{-n}, & n \text{ is even,}\end{cases} \tag{8}$$

$$U_n=W_n(0,1:p,q)=\begin{cases}\dfrac{\alpha^n+q^n\alpha^{-n}}{\sqrt{p^2+4q}}, & n \text{ is odd}\\ \dfrac{\alpha^n-q^n\alpha^{-n}}{\sqrt{p^2+4q}}, & n \text{ is even}\end{cases} \tag{9}$$

and

$$V_n=W_n(2,p:p,q)=\begin{cases}\alpha^n-q^n\alpha^{-n}, & n \text{ is odd}\\ \alpha^n+q^n\alpha^{-n}, & n \text{ is even.}\end{cases} \tag{10}$$

Let us consider the positive root of Eq.(2)

$$\alpha = \frac{p+\sqrt{p^2+4q}}{2}. \tag{11}$$

The proportion (11) is called as the generalized golden (p,q)-proportion (or metallic means) [4,5]. For the case $p=q=1$, we have the golden ratio, $\phi = \frac{1+\sqrt{5}}{2}$. Also, the formula (11) is reduced to silver ratio ($p=2,\ q=1$), bronze ratio ($p=3,\ q=1$) and generalized golden *m*- proportions ($p=m,\ q=1$).
In this paper, we introduce hyperbolic functions connected to Horadam sequence. We first define hyperbolic Horadam functions and present their hyperbolic and recursive properties. Then, we give some geometrical properties of hyperbolic Horadam functions.

2. HYPERBOLIC HORADAM FUNCTIONS

The famous classical hyperbolic functions are defined as

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}. \tag{12}$$

The hyperbolic functions have many generalizations, analogs and inequalities [1,2,4,6-10,12-18]. Pandir and Ulusoy [16] defined generalized hyperbolic functions as follows:

$$\sinh_a(\xi) = \frac{pa^{k\xi} - qa^{-k\xi}}{2} \quad \text{and} \quad \cosh_a(\xi) = \frac{pa^{k\xi} + qa^{-k\xi}}{2}, \tag{13}$$

where ξ is a variable; p,q and k are constants. Stakhov and Tkachenko [7] defined the hyperbolic Fibonacci and Lucas functions replacing the discrete variable n in formulas (7) and (8) with the real variable x. Stakhov and Rozin [8] defined symmetrical forms of hyperbolic Fibonacci and Lucas functions. Koçer and et al. [4] first defined two hyperbolic functions by using the formulas (9), (10) as follows:

$$sU(x) = \frac{\alpha^{2x} - q^{2x}\alpha^{-2x}}{\sqrt{p^2+4q}} \quad \text{and} \quad cU(x) = \frac{\alpha^{2x+1} + q^{2x+1}\alpha^{-2x-1}}{\sqrt{p^2+4q}}. \tag{14}$$

Then, they introduced symmetrical representations of $sU(x)$ and $cU(x)$ using Stakhov and Rozin's approach as follows:

$$sUs(x) = \frac{\alpha^{x} - q^{x}\alpha^{-x}}{\sqrt{p^2+4q}} \quad \text{and} \quad cUs(x) = \frac{\alpha^{x} + q^{x}\alpha^{-x}}{\sqrt{p^2+4q}}. \tag{15}$$

If we compare Binet formula (6) to the generalized hyperbolic functions (13), we notice a similarity. Generalizing the hyperbolic functions (14), we have hyperbolic Horadam functions as in next definition.

Definition 1. Let α be as in (11). Then, the functions

$$sW(x) = \frac{A\alpha^{2x} - Bq^{2x}\alpha^{-2x}}{\sqrt{p^2+4q}} \tag{16}$$

and

$$cW(x)=\frac{A\alpha^{2x+1}+Bq^{2x+1}\alpha^{-2x-1}}{\sqrt{p^2+4q}} \tag{17}$$

are called hyperbolic Horadam sine and cosine functions, respectively, where p and q are nonzero real numbers, such that $p^2+4q\neq 0.$

The equations $sW(k)=W_{2k}$ and $cW(k)=W_{2k+1}$, where $k=0,\mp 1,\mp 2,\ldots$ denote the relationships between the Horadam numbers and the hyperbolic Horadam functions.

The hyperbolic Horadam functions (16) and (17) are not symmetrical with respect to the origin. So, we use Stakhov and Rozin's approach [8] to define symmetrical representations of hyperbolic Horadam functions. That is, the symmetrical hyperbolic Horadam sine function is

$$sWs(x)=\frac{A\alpha^{x}-Bq^{x}\alpha^{-x}}{\sqrt{p^2+4q}} \tag{18}$$

and

$$cWs(x)=\frac{A\alpha^{x}+Bq^{x}\alpha^{-x}}{\sqrt{p^2+4q}}. \tag{19}$$

The relationship between the Horadam numbers and the symmetrical hyperbolic Horadam functions is:

$$W_n=\begin{cases} cWs(n), & n \text{ is odd} \\ sWs(n), & n \text{ is even.}\end{cases} \tag{20}$$

The formulas (18) and (19) generalize the formulas (15) and symmetrical hyperbolic Fibonacci and Lucas *m*- functions in [6].

The functions $cWs(x)$ and $sWs(x)$ have similar properties that of hyperbolic functions in (15), with second order recurrence sequences introduced by Koçer and et al. [4]. In the sequel we present some of them.

Proposition 1. ***(Recursive relation).***

$$sWs(x+2)=pcWs(x+1)+qsWs(x) \quad \text{and} \quad cWs(x+2)=psWs(x+1)+qcWs(x). \tag{21}$$

Proof.

$$pcWs(x+1)+qsWs(x)=p\left(\frac{A\alpha^{x+1}+Bq^{x+1}\alpha^{-x-1}}{\sqrt{p^2+4q}}\right)+q\left(\frac{A\alpha^{x}-Bq^{x}\alpha^{-x}}{\sqrt{p^2+4q}}\right)$$

$$=\frac{A\alpha^{x}(p\alpha+q)-Bq^{x+1}\alpha^{-x}(1-\frac{p}{\alpha})}{\sqrt{p^2+4q}}$$

$$=\frac{A\alpha^{x+2}-Bq^{x+2}\alpha^{-x-2}}{\sqrt{p^2+4q}}$$

$$=sWs(x+2), \tag{22}$$

where $p\alpha+q=\alpha^2$, $1-\frac{p}{\alpha}=\frac{q}{\alpha^2}$.

Proposition 2. ***(Cassini's identity).***

$$\left[sWs(x)\right]^2-cWs(x+1)cWs(x-1)=-ABq^{x-1} \text{ and } \left[cWs(x)\right]^2-sWs(x+1)sWs(x-1)=ABq^{x-1}. \tag{23}$$

Proof.

$$(LHS)=\frac{\left(A\alpha^{x}-Bq^{x}\alpha^{-x}\right)^2-\left(A\alpha^{x+1}+Bq^{x+1}\alpha^{-x-1}\right)\left(A\alpha^{x-1}+Bq^{x-1}\alpha^{-x+1}\right)}{\left(\sqrt{p^2+4q}\right)^2}$$

$$=\frac{-ABq^{x-1}\left(2q+\alpha^2+q^2\alpha^{-2}\right)}{p^2+4q}$$

$$=-ABq^{x-1}, \tag{24}$$

where (LHS) is the left hand side of the first identity.

This completes the proof.

The proofs of the next three propositions are similar to the proofs of the previous propositions. So, we give them without proof.

Proposition 3. ***(Pythagorean Theorem).***

$$\left[cWs(x)\right]^2-\left[sWs(x)\right]^2=4ABq^{x}(p^2+4q)^{-1}. \tag{25}$$

Proposition 4. ***(nth derivatives).***

$$\left[cWs(x)\right]^{(n)}=\begin{cases}(\ln\alpha)^n sWs(x)+\dfrac{\left(\ln\dfrac{q}{\alpha}\right)^n+\left(\ln\alpha\right)^n}{\sqrt{p^2+4q}}Bq^x\alpha^{-x}, & n \text{ is odd}\\ (\ln\alpha)^n cWs(x)+\dfrac{\left(\ln\dfrac{q}{\alpha}\right)^n-\left(\ln\alpha\right)^n}{\sqrt{p^2+4q}}Bq^x\alpha^{-x}, & n \text{ is even}\end{cases} \tag{26}$$

and

$$\left[sWs(x)\right]^{(n)}=\begin{cases}(\ln\alpha)^n cWs(x)-\dfrac{\left(\ln\dfrac{q}{\alpha}\right)^n+\left(\ln\alpha\right)^n}{\sqrt{p^2+4q}}Bq^x\alpha^{-x}, & n \text{ is odd}\\ (\ln\alpha)^n sWs(x)-\dfrac{\left(\ln\dfrac{q}{\alpha}\right)^n-\left(\ln\alpha\right)^n}{\sqrt{p^2+4q}}Bq^x\alpha^{-x}, & n \text{ is even.}\end{cases} \tag{27}$$

Proposition 5. ***(Moivre's equation).***

$$i)\ \left[cWs(x)+sWs(x)\right]^n=\left[2A\left(\sqrt{p^2+4q}\right)^{-1}\right]^{n-1}\left[cWs(nx)+sWs(nx)\right], \tag{28}$$

$$ii)\ \left[cWs(x)-sWs(x)\right]^n=\left[2B\left(\sqrt{p^2+4q}\right)^{-1}\right]^{n-1}\left[cWs(nx)-sWs(nx)\right]. \tag{29}$$

3. THE QUASI-SINE HORADAM FUNCTION

If we take into account that $\cos(n\pi)=(-1)^n$ for integer values of *n*, the formula (6) is rewritten as

$$W_n=\frac{A\alpha^n-\cos(n\pi)Bq^n\alpha^{-n}}{\sqrt{p^2+4q}}. \tag{30}$$

Then we have next definition.

Definition 2. The function

$$WW(x)=\frac{A\alpha^x-\cos(\pi x)Bq^x\alpha^{-x}}{\sqrt{p^2+4q}} \tag{31}$$

is called the quasi-sine Horadam function, where *x* is any real number.

Notice that $WW(n)=W_n$ for all integer *n*. The quasi-sine Horadam function generalizes some quasi-sine functions such as quasi-sine Fibonacci and Lucas function. The graphs and more information of the special cases of the quasi-sine Horadam function are given in [4,9].

Now we give two properties of quasi-sine Horadam function without proof.

Proposition 6. ***(Recursive relation).***

$$WW(x+2)=pWW(x+1)+qWW(x). \tag{32}$$

Proposition 7. ***(Cassini's identity).***

$$\left[WW(x)\right]^2-WW(x+1)WW(x-1)=-ABq^{x-1}\cos(\pi x). \tag{33}$$

3.1. The Metallic Shofars

We introduce in this subsection some curves and surfaces connected to the Horadam number. Koçer and et al. [4] defined three-dimensional spiral for U_n the sequence in (9) as:

$$CUU(x)=\frac{\alpha^x-\cos(\pi x)q^x\alpha^{-x}}{\sqrt{p^2+4q}}+i\frac{\sin(\pi x)q^x\alpha^{-x}}{\sqrt{p^2+4q}}. \tag{34}$$

As for us, we define three-dimensional Horadam spiral by generalizing formula (34).

Definition 3. The function

$$CWW(x)=\frac{A\alpha^x-\cos(\pi x)Bq^x\alpha^{-x}}{\sqrt{p^2+4q}}+i\frac{\sin(\pi x)Bq^x\alpha^{-x}}{\sqrt{p^2+4q}}=\frac{A\alpha^x+ie^{i\pi(\frac{1}{2}-x)}Bq^x\alpha^{-x}}{\sqrt{p^2+4q}} \tag{35}$$

is called the three-dimensional Horadam spiral, where $i^2=-1$.

Note that the real part of $CWW(x)$ equals to $WW(x)$.

Proposition 8. ***(Recursive relation).***

$$CWW(x+2)=pCWW(x+1)+qCWW(x). \tag{36}$$

Proof.

$$pCWW(x+1)+qCWW(x)=p\left(\frac{A\alpha^{x+1}+ie^{i\pi(\frac{1}{2}-x-1)}Bq^{x+1}\alpha^{-x-1}}{\sqrt{p^2+4q}}\right)+q\left(\frac{A\alpha^{x}+ie^{i\pi(\frac{1}{2}-x)}Bq^{x}\alpha^{-x}}{\sqrt{p^2+4q}}\right)$$

$$=\frac{A\alpha^{x}(p\alpha+q)+ie^{-i\pi x}Bq^{x+1}\alpha^{-x-1}(pe^{-i\frac{\pi}{2}}+e^{i\frac{\pi}{2}}\alpha)}{\sqrt{p^2+4q}}$$

$$=\frac{A\alpha^{x+2}+ie^{-i\pi x}Bq^{x+1}\alpha^{-x-1}(\alpha-p)}{\sqrt{p^2+4q}}$$

$$=\frac{A\alpha^{x+2}+ie^{-i\pi(x+2)}Bq^{x+2}\alpha^{-x-2}}{\sqrt{p^2+4q}}$$

$$=CWW(x+2), \qquad (37)$$

where $p\alpha+q=\alpha^2$ and $\alpha-p=\dfrac{q}{\alpha}$. This completes the proof.

Let *Y* and *Z* axises be real and imaginary axises, respectively. Then, we have the system of equations depend on three-dimensional Horadam spiral:

$$y(x)-\frac{A\alpha^{x}}{\sqrt{p^2+4q}}=\frac{-\cos(\pi x)Bq^{x}\alpha^{-x}}{\sqrt{p^2+4q}}, \qquad (38)$$

$$z(x)=\frac{\sin(\pi x)Bq^{x}\alpha^{-x}}{\sqrt{p^2+4q}}. \qquad (39)$$

By summing up the squares of equations of the above system we have

$$\left(y-\frac{A\alpha^{x}}{\sqrt{p^2+4q}}\right)^2+z^2=\left(\frac{Bq^{x}\alpha^{-x}}{\sqrt{p^2+4q}}\right)^2. \qquad (40)$$

By considering the formulas (18) and (19), the formula (40) is of the form

$$z^2=\left[cWs(x)-y\right]\left[y-sWs(x)\right]. \qquad (41)$$

The Eq. (40) corresponds to Metallic Shofar [5]. Let us consider the definition of $W_n\left(a,b:p,q\right)$ given in (1). In the case $a=0,\ b=1,\ p=m$ and $q=1$ the Metallic Shofar is expressed with the equation

$$\left(y-\frac{\alpha^{x}}{\sqrt{m^2+4}}\right)^2+z^2=\left(\frac{\alpha^{-x}}{\sqrt{m^2+4}}\right)^2,$$

(42)

where α is the golden (*m*,1) proportion [4]. For the cases *m=1,2,3*, the Eq. (40) corresponds to Golden Shofar [4,9], Silver Shofar [4,5] and Bronze Shofar [4,5], respectively. That is, we have the equations

$$\left(y-\frac{\phi^{x}}{\sqrt{5}}\right)^{2}+z^{2}=\left(\frac{\phi^{-x}}{\sqrt{5}}\right)^{2},\ \left(y-\frac{\psi^{x}}{\sqrt{8}}\right)^{2}+z^{2}=\left(\frac{\psi^{-x}}{\sqrt{8}}\right)^{2}\ \text{and}\ \left(y-\frac{\tau^{x}}{\sqrt{13}}\right)^{2}+z^{2}=\left(\frac{\tau^{-x}}{\sqrt{13}}\right)^{2}, \quad (43)$$

where ϕ, ψ and τ are golden, silver and bronze proportion, respectively.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Stakhov, A. P., Rozin, B., "The "golden" hyperbolic models of Universe", Chaos, Solitons & Fractals, 34(2): 159-171, (2007).

[2] Falcón, S., Plaza, Á., "The k-Fibonacci hyperbolic functions", Chaos, Solitons and Fractals, 38: 409-420, (2008).

[3] Horadam, A.F., "Basic properties of a certain generalized sequence of numbers", The Fibonacci Quarterly, 3: 161-176, (1965).

[4] Koçer, E.G., Tuğlu, N., Stakhov, A., "Hyperbolic Functions with Second Order Recurrence Sequences", Ars Combinatoria, 88: 65-81, (2008).

[5] De Spinadel, V.W., "From the Golden Mean to Chaos", Nueva Libreria, (1998) (second edition, Nobuko, 2004).

[6] Stakhov, A. P., "Gazale Formulas, a New Class of the Hyperbolic Fibonacci and Lucas Functions, and the Improved Method of the `Golden' Cryptography", Academy of Trinitarism, 77-6567: 1-32, (2006).

[7] Stakhov, A. P., Tkachenko, I.S., "Hyperbolic Fibonacci trigonometry", Reports of Ukraine Academy of Science, 7: 9-14 (1993).

[8] Stakhov, A. P., Rozin, B., "On a new class of hyperbolic functions", Chaos, Solitons & Fractals, 23(2): 379-389, (2005).

[9] Stakhov, A. P., Rozin, B., "The Golden Shofar", Chaos, Solitons & Fractals, 26(3): 677-684, (2005).

[10] Stakhov, A. P., Rozin, B., "The continuous functions for the Fibonacci and Lucas p-numbers", Chaos, Solitons & Fractals, 28(4): 1014-1025, (2006).

[11] Vajda, S., "Fibonacci & Lucas Numbers, and the Golden Section. Theory and Applications", Ellis Horwood Limited, (1989).

[12] Baricza Á., Bhayo, B.A., Pogány, T.K., "Functional inequalities for generalized inverse trigonometric and hyperbolic functions", Journal of Mathematical Analysis and Applications, 417: 244-259, (2014).

[13] Cieśliński, J.L., "New definitions of exponential, hyperbolic and trigonometric functions on time scales", Journal of Mathematical Analysis and Applications, 388: 8-22, (2012).

[14] Klén, R., Vuorinen, M., Zhang, X.H., "Inequalities for the generalized trigonometric and hyperbolic functions", Journal of Mathematical Analysis and Applications, 409: 521-529, (2014).

[15] Lv, Y., Wang, G., Chu, Y., "A note on Jordan type inequalities for hyperbolic functions", Applied Mathematics Letters, 25: 505-508, (2012).

[16] Pandir, Y., Ulusoy, H., "New Generalized Hyperbolic Functions to Find New Exact Solutions of the Nonlinear Partial Differential Equations", Journal of Mathematics, 2013: Article ID 201276, (2013).

[17] Taşçı, D., Azman, H., "The k-Lucas Hyperbolic Functions", Communications in Mathematics and Applications, 5(1): 11-21, (2014).

[18] Yang, C.-Y., "Inequalities on generalized trigonometric and hyperbolic functions", Journal of Mathematical Analysis and Applications, 419: 775-782, (2014).