Research Article GU J Sci 32(3): 991-1006 (2019) DOI: 10.35378/gujs.434646

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

The Modified Modal Operators over the Generalized Interval Valued Intuitionistic Fuzzy Sets

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Highlights

- The paper focuses on define newly modal operators over GIVIFSs.
- Various properties of these operators have also been investigated in details.
- Some applications of operators are one motivation for study.
- The validity of the operators is tested based on proofs and numerical examples.

Article Info

Received:19/06/2018
Accepted:07/12/2018

Keywords

Generalized interval valued intuitionistic fuzzy sets
Intuitionistic fuzzy sets
Modified modal operators

Abstract

Interval valued intuitionistic fuzzy set (IVFS) as an extension of intuitionistic fuzzy sets is described by two parameters, namely membership degree and non-membership degree which are expressed in terms of intervals rather than crisp numbers. IVFS can be used to handle uncertainty and vagueness in real world decision making problems and operators of IVFSs have a key role in this filed. Thus, in this work we define newly defined modal operators over generalized interval valued intuitionistic fuzzy sets by modifying the existing operators. The new proposed operators are the integrity and comprehensive. Then, we describe the desirable properties of the proposed operators and discuss the special cases of them in details. Furthermore, the relationship between operators is examined. Finally, an illustrative example is provided for comparison.

1. INTRODUCTION

Atanassov [1] introduced the concept of intuitionistic fuzzy sets (IFSs), which is a generalization of Zadeh's fuzzy sets [2] and defined new operations on IFSs. In IFSs, each element is assigned by membership and non-membership degrees, where the sum of the two degrees is between zero and one. However, in reality, it may not always be true that the degree of membership and degree of non-membership of an element in IFS be real numbers. Therefore, a generalization of IFS was introduced by Atanassov and Gargov [3] as interval valued intuitionistic fuzzy sets (IVIFSs) which its fundamental characteristic is that the values of its membership and non-membership degree are intervals rather than exact numbers. After the introduction of IVIFSs, many researchers have shown interest in the IVIFSs theory and applied it to the various field. Interval valued intuitionistic fuzzy sets is used to model uncertainty, imprecise, incomplete and vague information. Atanassov [4] introduced operators over IVIFSs. Xu [5], Xu and Jian [6] and Wei and Wang [7] developed some arithmetic aggregation operators and some geometric aggregation operators of IVIFS for decision making. Wang and Liu [8] considered the interval valued intuitionistic fuzzy hybrid weighted averaging operator based on Einstein operation and its application to decision making. They defined generalized interval valued intuitionistic fuzzy relation with some results. Bhowmik and Pal [9,10] defined generalized interval valued intuitionistic fuzzy sets (GIVIFSs). Bhowmik and Pal [11] defined two composite relations, four types of reflexivity and irreflexivity of GIVIFSs with some of their properties. Also they define two operators C and I with some properties over GIVIFSs.

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Li [12-14], Yue [15], Chen et al. [16], Bai [17] and Wang and Chen [18] presented methods for handling multi-criteria fuzzy decision making based on IVIFS. Mondal and Samanta [19] studied the topological properties and the category of topological spaces of IVIFSs. Zhang et al. [20] introduced a generalized interval valued intuitionistic fuzzy sets. Sudharsan and Ezhilmaran [21] defined two new operators over interval valued intuitionistic fuzzy sets. A novel way introduced to fuse several images using interval valued intuitionistic fuzzy sets by Ananthi and Balasubramaniam [22]. They prove that IVIFSs are more suitable for fusion of such uncertain images. Meng et al. [23] analyzed a method to multi-attribute decision making with interval valued intuitionistic fuzzy information problems using prospect theory based on the interval valued intuitionistic hybrid weight averaging operator. Reiser and Bedregal [24] studies the conjugate functions related to main connectives of the interval valued intuitionistic fuzzy logic.

One motivation for our study has been the significant performance achieved by the use of operators over IVIFSs implications in some applications. For example, some applications of operators have been: medical diagnosis (Ahn et al. [25], Ezhilmaran and Sudharsan [26]); decision making problem (Bhowmik and Pal [11]); exploitation investment evaluation (Qi et al. [27]); mathematical programming (Wang et al. [28]); evaluation about the performance of e-government (Zhang et al. [29]); multiple attribute group decision making (Tan et al. [30]); supplier selection with multi criteria group decision making (Makui et al. [31]); medical diagnosis using logical operators (Pathinathan et al. [32]); enterprise e-marketing performance evaluation (Zhou, [33]); etc.

Baloui Jamkhaneh and Nadarajah [34] considered a generalized intuitionistic fuzzy sets ($\mathrm{GIFS_B}s$) and introduced some operators over $\mathrm{GIFS_B}$. Afterwards, level operators, modal-like operators, modal operators and some operations were introduced on $\mathrm{GIFS_B}s$ in Baloui Jamkhaneh [35], Baloui Jamkhaneh and Nadi Ghara [36], Baloui Jamkhaneh and Nadarajah [37], Baloui Jamkhaneh and Garg [38]. Baloui Jamkhaneh [39] considered generalized interval valued intuitionistic fuzzy sets ($\mathrm{GIVIFS_B}s$), dealing with uncertainty and vagueness. Afterwards, some operations were introduced on $\mathrm{GIVIFS_B}s$ in Baloui Jamkhaneh [40]. Recently Baloui Jamkhaneh [41] and Baloui Jamkhaneh and Amirzadi [42] defined some operators over $\mathrm{GIVIFS_B}s$ due to Baloui Jamkhaneh [39]. According to the definition in Baloui Jamkhaneh [39], degree of membership and degree of non-membership of $\mathrm{GIVIFS_B}$ are subintervals of the interval [0,1]. In order to establish this condition for operators due to Baloui Jamkhaneh [41], the α and β parameters must be in the specific subset of [0,1]. This means that the values of the parameters must be limited. In this case, this reduces the integrity and comprehensiveness of the operator. For this purpose, in this paper, modified operators are defined in which parameters are not limited.

In this paper we shall introduce the some of the modified modal operators (as $\mathrm{D}_\alpha(\mathrm{A}), \mathrm{F}_{\alpha,\beta}(\mathrm{A})$, $\mathrm{J}_{\alpha,\beta}(\mathrm{A})$, $\mathrm{d}_\alpha(\mathrm{A})$, $\mathrm{f}_{\alpha,\beta}(\mathrm{A})$, $\mathrm{j}_{\alpha,\beta}(\mathrm{A})$, $\mathrm{H}_{\alpha,\beta}(\mathrm{A})$, $\mathrm{h}_{\alpha,\beta}(\mathrm{A})$) over $\mathrm{GIVIFS_B}$ and we will discuss their properties. Some of these properties are the following: i) All these operators are $\mathrm{GIVIFS_B}$ ii) All these operators are increasing relative to α iii) All these operators are decreasing relative to β iv) $\mathrm{D}_0(\mathrm{A}) = \mathrm{F}_{0,1}(\mathrm{A}) = \overline{\mathrm{d}_1(\mathrm{A})} = \overline{\mathrm{f}_{1,0}(\mathrm{A})} = \mathrm{H}_{1,1}(\mathrm{A}) = \overline{\mathrm{j}_{1,1}(\mathrm{A})} = \square\mathrm{A}$ v) $\mathrm{D}_1(\mathrm{A}) = \mathrm{F}_{1,0}(\mathrm{A}) = \mathrm{J}_{1,1}(\mathrm{A}) = \overline{\mathrm{d}_0(\mathrm{A})} = \overline{\mathrm{f}_{0,1}(\mathrm{A})} = \overline{\mathrm{h}_{1,1}(\mathrm{A})} = \Diamond\mathrm{A}$ vi) $\mathrm{A} = \mathrm{F}_{0,0}(\mathrm{A}) = \mathrm{J}_{0,1}(\mathrm{A}) = \mathrm{H}_{1,0}(\mathrm{A})$ vii) $\overline{\mathrm{A}} = \mathrm{f}_{0,0}(\mathrm{A}) = \mathrm{j}_{0,1}(\mathrm{A}) = \mathrm{h}_{1,0}(\mathrm{A})$ viii) $\mathrm{A} \subset \mathrm{J}_{\alpha,\beta}(\mathrm{A}), \overline{\mathrm{A}} \subset \mathrm{j}_{\alpha,\beta}(\mathrm{A})$ ix) $\mathrm{D}_\alpha(\square\mathrm{A}) = \mathrm{F}_{\alpha,\beta}(\square\mathrm{A}) = \overline{\mathrm{d}_\alpha(\square\mathrm{A})} = \overline{\mathrm{f}_{\alpha,\beta}(\square\mathrm{A})} = \square\mathrm{A}$ x) $\mathrm{D}_\alpha(\Diamond\mathrm{A}) = \mathrm{F}_{\alpha,\beta}(\Diamond\mathrm{A}) = \overline{\mathrm{d}_\alpha(\Diamond\mathrm{A})} = \overline{\mathrm{f}_{\alpha,\beta}(\Diamond\mathrm{A})} = \Diamond\mathrm{A}$, etc. The remainder of the paper is organized as follows. In Section 2, we briefly introduce IFS and its generalizations. In Section 3 define modified operators over generalized interval valued intuitionistic fuzzy sets. The paper is concluded in Section 4.

2. REMARKS ON THE $\mathrm{GIVIFS_B}$

In this section, we give some basic definition. Let X be a non-empty universal set.

Definition 2.1. [1] An IFS A in X is defined as an object of the form $\mathrm{A} = \{\langle \mathrm{x}, \mu_\mathrm{A}(\mathrm{x}), \nu_\mathrm{A}(\mathrm{x})\rangle : \mathrm{x} \in \mathrm{X}\}$ where the functions $\mu_\mathrm{A}: \mathrm{X} \to [0,1]$ and $\nu_\mathrm{A}: \mathrm{X} \to [0,1]$ denote the degree of membership and degree of non-membership of the element x in A respectively, satisfying $0 \le \mu_\mathrm{A}(\mathrm{x}) + \nu_\mathrm{A}(\mathrm{x}) \le 1$ for each $\mathrm{x} \in \mathrm{X}$.

Definition 2.2. Let [I] be the set of all closed subintervals of the interval $[0,1]$ and $M_A(x) = [M_{AL}(x), M_{AU}(x)] \in [I]$ and $N_A(x) = [N_{AL}(x), N_{AU}(x)] \in [I]$ then $N_A(x) \le M_A(x)$ if and only if $N_{AL}(x) \le M_{AL}(x)$ and $N_{AU}(x) \le M_{AU}(x)$.

Definition 2.3. [3] Interval valued intuitionistic fuzzy set (IVIFS) A in X, is defined as an object of the form $A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$ where the functions $M_A(x): X \to [\,I\,]$ and $N_A(x): X \to [\,I\,]$, denote the degree of membership and degree of non-membership of the element x in A respectively, where $M_A(x) = [M_{AL}(x), M_{AU}(x)]$, $N_A(x) = [N_{AL}(x), N_{AU}(x)]$, and $0 \le M_{AU}(x) + N_{AU}(x) \le 1$ for each $x \in X$.

Definition 2.4. [34] Let X be a non-empty set. Generalized intuitionistic fuzzy set A in X, is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\nu_A: X \to [0,1]$, denote the degree of membership and degree of non-membership of the element x in A respectively, and $0 \le \mu_A(x)^\delta + \nu_A(x)^\delta \le 1$ for each $x \in X$, and $\delta = n$ or $\frac{1}{n}, n = 1,2, \dots, N$.

Definition 2.5. [39] Generalized interval valued intuitionistic fuzzy set ($GIVIFS_B$) A in X, is defined as an object of the form $A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$ where the functions $M_A(x): X \to [\,I\,]$ and $N_A(x): X \to [\,I\,]$, denote the degree of membership and degree of non-membership of the element x in A respectively, and $M_A(x) = [M_{AL}(x), M_{AU}(x)]$, $N_A(x) = [N_{AL}(x), N_{AU}(x)]$, where $0 \le M_{AU}(x)^\delta + N_{AU}(x)^\delta \le 1$, for each $x \in X$ and $\delta = n$ or $\frac{1}{n}, n = 1,2, \dots, N$. The collection of all $GIVIFS_B(\delta)$ is denoted by $GIVIFS_B(\delta, X)$.

Definition 2.6. The degree of non-determinacy (uncertainty) of an element $x \in X$ to the $GIVIFS_B$ A is defined by

$$\pi_A(x) = [\pi_{AL}(x), \pi_{AU}(x)] = [\,(1 - M_{AU}(x)^\delta - N_{AU}(x)^\delta)^{\frac{1}{\delta}}, (1 - M_{AL}(x)^\delta - N_{AL}(x)^\delta)^{\frac{1}{\delta}}].$$

Definition 2.7. [39] Let A and B be two $GIVIFS_B$s such that

$A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$, $B = \{\langle x, M_B(x), N_B(x)\rangle : x \in X\}$,
$M_A(x) = [M_{AL}(x), M_{AU}(x)]$, $N_A(x) = [N_{AL}(x), N_{AU}(x)]$,
$M_B(x) = [M_{BL}(x), M_{BU}(x)]$, $N_B(x) = [N_{BL}(x), N_{BU}(x)]$.

Define the following relations on A and B

i. $A \subset B$ if and only if $M_A(x) \le M_B(x)$ and $N_A(x) \ge N_B(x)$, $\forall x \epsilon X$,
ii. $A \subset_{\Box} B$ if and only if $M_A(x) \le M_B(x)$, $\forall x \epsilon X$,
iii. $A \subset_{\Diamond} B$ if and only if $N_A(x) \ge N_B(x)$, $\forall x \epsilon X$,
iv. $A \cup B = \{\langle x, [\max(M_{AL}(x), M_{BL}(x)), \max(M_{AU}(x), M_{BU}(x))], [\min(N_{AL}(x), N_{BL}(x)), \min(N_{AU}(x), N_{BU}(x))]\rangle : x \in X\}$,
v. $A \cap B = \{\langle x, [\min(M_{AL}(x), M_{BL}(x)), \min(M_{AU}(x), M_{BU}(x))], [\max(N_{AL}(x), N_{BL}(x)), \max(N_{AU}(x), N_{BU}(x))]\rangle : x \in X\}$,
vi. $\overline{A} = \{\langle x, N_A(x), M_A(x)\rangle : x \in X\}$.

Definition 2.8. [42] For every $GIVIFS_B$ $A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$, the modal logic operators defined as follows

The Necessity measure on A:

$$\Box A = \left\{\langle\, x, [M_{AL}(x), M_{AU}(x)], [N_{AL}(x), (1 - M_{AU}(x)^\delta)^{\frac{1}{\delta}}]\rangle : x \epsilon X\right\},$$

The Possibility measure on A:

$$\Diamond\, A=\left\{\langle x,[M_{AL}(x),\left(1-\ N_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}],[N_{AL}(x),N_{AU}(x)]\rangle : x\epsilon\, X\right\}.$$

3. THE MODIFIED MODAL OPERATORS OF $GIVIFS_B$

Here, we will introduce new operators over the $GIVIFS_B$, which modified some operators due to Baloui Jamkhaneh [41] related to $GIVIFS_B$. Let X is a non-empty finite set and $A=\{\langle x, M_A(x), N_A(x)\rangle : x\in X\}$ is a $GIVIFS_B$.

Definition 3.1. Let $A\in GIVIFS_B$ and $\alpha\in[0,1]$, we define the operator of $D_\alpha(A,\delta)$ (in summary, we will show as $D_\alpha(A)$) as follows

$$D_\alpha(A)=\left\{\langle x, M_{D_\alpha}(A)\,, N_{D_\alpha}(A)\rangle : x\in X\right\},$$
$$M_{D_\alpha}(A)=\left[M_{AL}(x),\left(M_{AU}(x)^{\delta}+\alpha\pi_{AL}(x)^{\delta}\ \right)^{\frac{1}{\delta}}\right], N_{D_\alpha}(A)=[N_{AL}(x),(N_{AU}(x)^{\delta}+(1-\alpha)\pi_{AL}(x)^{\delta}\ )^{\frac{1}{\delta}}].$$

It can be easily shown that $\pi_{D_\alpha(A)L}(x)=0$.

Theorem 3.1. For every $A\in GIVIFS_B$ and $\alpha\in[0,1]$, it holds that

i. $D_\alpha(A)\in GIVIFS_B$,
ii. $D_0(A)=\Box\, A$,
iii. $D_1(A)=\Diamond\, A$,
iv. $D_\alpha(\overline{A})=\overline{D_{1-\alpha}(A)}$,
v. $D_\alpha(D_\alpha(A))=D_\alpha(A)$.

Proof. The proof of part (i) is straightforward.

(ii) Note that

$$M_{D_0}(A)=\left[M_{AL}(x),\left(M_{AU}(x)^{\delta}+0\times\pi_{AL}(x)^{\delta}\ \right)^{\frac{1}{\delta}}\right],$$
$$=[M_{AL}(x),M_{AU}(x)],$$
$$N_{D_0}(A)=\left[N_{AL}(x),\left(N_{AU}(x)^{\delta}+(1-0)\pi_{AL}(x)^{\delta}\ \right)^{\frac{1}{\delta}}\right],$$
$$=\left[N_{AL}(x),\left(N_{AU}(x)^{\delta}+\pi_{AL}(x)^{\delta}\ \right)^{\frac{1}{\delta}}\right].$$

Since $\pi_{AL}(x)^{\delta}=1-\ M_{AU}(x)^{\delta}-N_{AU}(x)^{\delta}$, then $N_{D_0}(A)=\left[N_{AL}(x),\left(1-\ M_{AU}(x)^{\delta}\ \right)^{\frac{1}{\delta}}\right]$, finally we have

$$D_0(A)=\left\{\langle x,[M_{AL}(x),M_{AU}(x)]\,,\left[N_{AL}(x),\left(1-\ M_{AU}(x)^{\delta}\ \right)^{\frac{1}{\delta}}\right]\rangle : x\in X\right\}=\Box\, A.$$

The proof is complete.

(iii) Note that

$$D_1(A)=\left\{\langle x, M_{D_1}(A)\,, N_{D_1}(A)\rangle : x\in X\right\},$$
$$M_{D_1}(A)=\left[M_{AL}(x),\left(M_{AU}(x)^{\delta}+\pi_{AL}(x)^{\delta}\ \right)^{\frac{1}{\delta}}\right],$$

$$= \left[M_{AL}(x), \left(1 - N_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

$$N_{D_1}(A) = \left[N_{AL}(x), \left(N_{AU}(x)^{\delta} + (1-1)\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

$$= [N_{AL}(x), N_{AU}(x)],$$

then

$$D_1(A) = \left\{\langle x, [M_{AL}(x), (1 - N_{AU}(x)^{\delta})^{\frac{1}{\delta}}], [N_{AL}(x), N_{AU}(x)]\rangle : x \in X\right\} = \lozenge A.$$

The proof is complete. Proofs of (iv) and (v) are obvious.

Definition 3.2. Let $A \in GIVIFS_B$ and $\alpha, \beta \in [0,1]$, where $0 \leq \alpha + \beta \leq 1$, we define the operator of $F_{\alpha,\beta}(A, \delta)$ (in summary, we will show as $F_{\alpha,\beta}(A)$) as follows

$$F_{\alpha,\beta}(A) = \left\{\langle x, M_{F_{\alpha}}(A), N_{F_{\beta}}(A)\rangle : x \in X\right\},$$

$$M_{F_{\alpha}}(A) = [M_{AL}(x), (M_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}], \ N_{F_{\beta}}(A) = [N_{AL}(x), (N_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}].$$

Theorem 3.2. For every $A \in GIVIFS_B$ and $\alpha, \beta, \gamma \in [0,1]$, where $0 \leq \alpha + \beta \leq 1$, it holds that

i. $F_{\alpha,\beta}(A) \in GIVIFS_B$,
ii. $0 \leq \gamma \leq \alpha \Rightarrow F_{\gamma,\beta}(A) \subset F_{\alpha,\beta}(A)$, $0 \leq \gamma + \beta \leq 1$,
iii. $0 \leq \gamma \leq \beta \Rightarrow F_{\alpha,\beta}(A) \subset F_{\alpha,\gamma}(A)$, $0 \leq \alpha + \gamma \leq 1$,
iv. $D_{\alpha}(A) = F_{\alpha,1-\alpha}(A)$,
v. $\square A = F_{0,1}(A)$,
vi. $\lozenge A = F_{1,0}(A)$,
vii. $F_{\alpha,\beta}(\overline{A}) = \overline{F_{\beta,\alpha}(A)}$,
viii. $F_{0,0}(A) = A$,
ix. $D_{\alpha}(A) \subset F_{\alpha,\beta}(A)$.

Proof. (i) Follows since

$$M_{F_{\alpha}(A)U}(x)^{\delta} + N_{F_{\beta}(A)U}(x)^{\delta}$$

$$= [(M_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}]^{\delta} + [(N_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}]^{\delta},$$
$$= M_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta} + N_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta},$$
$$= M_{AU}(x)^{\delta} + N_{AU}(x)^{\delta} + (\alpha+\beta)\pi_{AL}(x)^{\delta},$$
$$\leq M_{AU}(x)^{\delta} + N_{AU}(x)^{\delta} + \pi_{AL}(x)^{\delta} = 1.$$

Proofs of (ii) and (iii) are obvious.

(iv) Follows since

$$M_{F_{\alpha}}(A) = \left[M_{AL}(x), \left(M_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right], N_{F_{1-\alpha}}(A) = \left[N_{AL}(x), \left(N_{AU}(x)^{\delta} + (1-\alpha)\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

then

$$F_{\alpha,1-\alpha}(A) = \left\{\langle x, M_{F_{\alpha}}(A), N_{F_{1-\alpha}}(A)\rangle : x \in X\right\} = D_{\alpha}(A).$$

(v) Since $D_0(A) = F_{0,1}(A)$ by using Theorem3.1 it follows that $F_{0,1}(A) = \square A$.

(vi) Since $D_1(A) = F_{1,0}(A)$ by using Theorem3.1 it follows that $F_{1,0}(A) = \Diamond A$.

(vii) We have

$$F_{\beta,\alpha}(A) = \left\{\langle x, M_{F_\beta}(A), N_{F_\alpha}(A)\rangle : x \in X\right\},$$
$$M_{F_\beta}(A) = [M_{AL}(x), (M_{AU}(x)^\delta + \beta\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}],\ N_{F_\alpha}(A) = \left[N_{AL}(x), \left(N_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right],$$

and

$$F_{\alpha,\beta}(\overline{A}) = \left\{\langle x, M_{F_\alpha}(\overline{A}), N_{F_\beta}(\overline{A})\rangle : x \in X\right\},$$
$$M_{F_\alpha}(\overline{A}) = [N_{AL}(x), (N_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}],\ N_{F_\beta}(\overline{A}) = \left[M_{AL}(x), \left(M_{AU}(x)^\delta + \beta\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right],$$

Finally, we have $F_{\alpha,\beta}(\overline{A}) = \overline{F_{\beta,\alpha}(A)}$. The proof of part (viii) is straightforward.

(ix) It follows from the fact that $\beta \leq 1 - \alpha$.

Definition 3.3. Let $A \in GIVIFS_B$ and $\alpha, \beta \in [0,1]$, we define the operator of $J_{\alpha,\beta}(A, \delta)$ (in summary, we will show as $J_{\alpha,\beta}(A)$) as follows

$$J_{\alpha,\beta}(A) = \left\{\langle x, M_{J_{\alpha,\beta}}(A), N_{J_{\alpha,\beta}}(A)\rangle : x \in X\right\},$$
$$M_{J_{\alpha,\beta}}(A) = [M_{AL}(x), \left(M_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}],\ N_{J_{\alpha,\beta}}(A) = [\beta^{\frac{1}{\delta}} N_{AL}(x), \beta^{\frac{1}{\delta}} N_{AU}(x)].$$

Theorem 3.3. For every $A \in GIVIFS_B$ and $\alpha, \beta, \gamma \in [0,1]$, it holds that

i. $J_{\alpha,\beta}(A) \in GIVIFS_B$,
ii. $\alpha \leq \gamma \Rightarrow J_{\alpha,\beta}(A) \subset J_{\gamma,\beta}(A)$,
iii. $\beta \leq \gamma \Rightarrow J_{\alpha,\gamma}(A) \subset J_{\alpha,\beta}(A)$,
iv. $\Diamond A = J_{1,1}(A)$,
v. $A = J_{0,1}(A)$,
vi. $A \subset J_{\alpha,\beta}(A)$.

Proof. (i) Follows since

$$\begin{aligned} M_{J_{\alpha,\beta}(A)U}(x)^\delta + N_{J_{\alpha,\beta}(A)U}(x)^\delta &= \left(M_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}})^\delta + \left(\beta^{\frac{1}{\delta}} N_{AU}(x)\right)^\delta, \\ &= \left(M_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta\right) + \beta N_{AU}(x)^\delta, \\ &\leq M_{AU}(x)^\delta + \pi_{AL}(x)^\delta + N_{AU}(x)^\delta = 1. \end{aligned}$$

(ii) Since $\alpha \leq \gamma$ then it is clear that

$$[M_{AL}(x), \left(M_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}] \leq [M_{AL}(x), \left(M_{AU}(x)^\delta + \gamma\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}].$$

Finally we have $J_{\alpha,\beta}(A) \subset J_{\gamma,\beta}(A)$. This completes the proof.

The proof of (iii) is similar to that of (ii). Proofs of (iv), (v) and (vi) are obvious.

Definition 3.4. Let $\alpha \in [0,1]$ and $A \in GIVIFS_B$, we define the operator of $d_\alpha(A,\delta)$ (in summary, we will show as $d_\alpha(A)$) as follows

$$d_\alpha(A) = \left\{\langle x, M_{d_\alpha}(A), N_{d_\alpha}(A)\rangle : x \in X\right\},$$
$$M_{d_\alpha}(A) = \left[N_{AL}(x), \left(N_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right], N_{d_\alpha}(A) = [M_{AL}(x), (M_{AU}(x)^\delta + (1-\alpha)\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}].$$

It can be easily shown that $\pi_{d_\alpha(A)L}(x) = 0$.

Theorem 3.4. For every $A \in GIVIFS_B$ and $\alpha \in [0,1]$, it holds that

i. $d_\alpha(A) \in GIVIFS_B$,
ii. $d_0(A) = \overline{\Diamond A}$,
iii. $d_1(A) = \overline{\Box A}$,
iv. $d_\alpha(\overline{A}) = \overline{d_{1-\alpha}(A)} = D_\alpha(A)$,
v. $d_\alpha\left(d_\alpha(A)\right) = D_{1-\alpha}(A)$.

Proof. The proof of (i) is obvious.

(ii) Follows since

$$\begin{aligned}
M_{d_0}(A) &= \left[N_{AL}(x), \left(N_{AU}(x)^\delta + 0\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right],\\
&= [N_{AL}(x), N_{AU}(x)],\\
N_{d_0}(A) &= \left[M_{AL}(x), \left(M_{AU}(x)^\delta + (1-0)\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right],\\
&= \left[M_{AL}(x), \left(M_{AU}(x)^\delta + \pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right],\\
&= \left[M_{AL}(x), \left(1 - N_{AU}(x)^\delta\right)^{\frac{1}{\delta}}\right].
\end{aligned}$$

then

$$d_0(A) = \left\{\langle x, [N_{AL}(x), N_{AU}(x)], \left[M_{AL}(x), \left(1 - N_{AU}(x)^\delta\right)^{\frac{1}{\delta}}\right]\rangle : x \in X\right\} = \overline{\Diamond A}.$$

(iii) Follows since

$$\begin{aligned}
d_1(A) &= \left\{\langle x, M_{d_1}(A), N_{d_1}(A)\rangle : x \in X\right\},\\
N_{d_1}(A) &= [M_{AL}(x), \left(M_{AU}(x)^\delta + (1-1)\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}],\\
&= [M_{AL}(x), M_{AU}(x)],\\
M_{d_1}(A) &= [N_{AL}(x), (N_{AU}(x)^\delta + \pi_{AL}(x)^\delta)^{\frac{1}{\delta}}],\\
&= [N_{AL}(x), (1 - M_{AU}(x)^\delta)^{\frac{1}{\delta}}],\\
d_1(A) &= \left\{\langle x, [N_{AL}(x), (1 - M_{AU}(x)^\delta)^{\frac{1}{\delta}}], [M_{AL}(x), M_{AU}(x)]\rangle : x \in X\right\} = \overline{\Box A}.
\end{aligned}$$

This completes the proof.

(iv) The proof of this paper is analogous to the proof of part (vi) in Theorem 3.2. The proof of part (v) is straightforward.

Definition 3.5. Let $A \in GIVIFS_B$ and $\alpha, \beta \in [0,1]$, where $0 \leq \alpha + \beta \leq 1$, we define the operator of $f_{\alpha,\beta}(A, \delta)$ (in summary, we will show as $f_{\alpha,\beta}(A)$) as follows

$$f_{\alpha,\beta}(A) = \left\{\langle x, M_{f_\alpha}(A), N_{f_\beta}(A)\rangle : x \in X\right\},$$
$$M_{f_\alpha}(A) = [N_{AL}(x), (N_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}],\ N_{f_\beta}(A) = [M_{AL}(x), (M_{AU}(x)^\delta + \beta\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}].$$

Theorem 3.5. For every $GIVIFS_B \in A$ and $\alpha, \beta, \gamma \in [0,1]$, where $0 \leq \alpha + \beta \leq 1$, it holds that

i. $f_{\alpha,\beta}(A) \in GIVIFS_B$,
ii. $0 \leq \gamma \leq \alpha \Rightarrow f_{\gamma,\beta}(A) \subset f_{\alpha,\beta}(A)$, $\quad 0 \leq \gamma + \beta \leq 1$,
iii. $0 \leq \gamma \leq \beta \Rightarrow f_{\alpha,\beta}(A) \subset f_{\alpha,\gamma}(A)$, $\quad 0 \leq \alpha + \gamma \leq 1$,
iv. $f_{\alpha,1-\alpha}(A) = d_\alpha(A)$,
v. $f_{0,1}(A) = \overline{\lozenge A}$,
vi. $f_{1,0}(A) = \overline{\square A}$,
vii. $\overline{f_{\alpha,\beta}(\overline{A})} = f_{\beta,\alpha}(A)$,
viii. $f_{0,0}(A) = \overline{A}$,
ix. $d_\alpha(A) \subset f_{\alpha,\beta}(A)$.

Proof. (i) Follows since

$$\begin{aligned} M_{f_{\alpha,\beta}(A)U}(x)^\delta + N_{f_{\alpha,\beta}(A)U}(x)^\delta \\ &= [(N_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}]^\delta + [(M_{AU}(x)^\delta + \beta\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}]^\delta, \\ &= N_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta + M_{AU}(x)^\delta + \beta\pi_{AL}(x)^\delta, \\ &= M_{AU}(x)^\delta + N_{AU}(x)^\delta + (\alpha + \beta)\pi_{AL}(x)^\delta, \\ &\leq M_{AU}(x)^\delta + N_{AU}(x)^\delta + \pi_{AL}(x)^\delta = 1. \end{aligned}$$

Proofs of (ii) and (iii) are obvious.

(iv) Since

$$M_{f_\alpha}(A) = \left[N_{AL}(x), (N_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}\right], N_{f_{1-\alpha}}(A) = \left[M_{AL}(x), (M_{AU}(x)^\delta + (1-\alpha)\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}\right],$$

then

$$f_{\alpha,1-\alpha}(A) = \{\langle x, M_{f_\alpha}(A), N_{f_{1-\alpha}}(A)\rangle : x \in X\} = d_\alpha(A).$$

(v) Since $f_{0,1}(A) = d_0(A)$by using Theorem 3.4 it follows that $f_{0,1}(A) = \overline{\lozenge A}$.

(vi) Since $f_{1,0}(A) = d_1(A)$ by using Theorem 3.4 it follows that $f_{1,0}(A) = \overline{\square A}$.

(vii) Since

$$f_{\beta,\alpha}(A) = \left\{\langle x, M_{f_\beta}(A), N_{f_\alpha}(A)\rangle : x \in X\right\},$$
$$M_{f_\beta}(A) = [N_{AL}(x), (N_{AU}(x)^\delta + \beta\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}],\ N_{f_\alpha}(A) = [M_{AL}(x), (M_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}],$$

and

$$f_{\alpha,\beta}(\overline{A}) = \left\{\langle x, M_{f_\alpha}(\overline{A}), N_{f_\beta}(\overline{A})\rangle : x \in X\right\},$$

$M_{f_\alpha}(\bar{A}) = [M_{AL}(x), (M_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}]$, $N_{f_\beta}(\bar{A}) = [N_{AL}(x), (N_{AU}(x)^\delta + \beta\pi_{AL}(x)^\delta)^{\frac{1}{\delta}}]$,

hence

$$\overline{f_{\alpha,\beta}(\bar{A})} = \left\{\langle x, N_{f_\beta}(\bar{A}), M_{f_\alpha}(\bar{A})\rangle : x \in X\right\}.$$

Finally, we have $\overline{f_{\alpha,\beta}(\bar{A})} = f_{\beta,\alpha}(A)$. The proof of part (viii) is straightforward.

(ix) It follows from the fact that $\beta \leq 1 - \alpha$.

Theorem 3.6. For every $A \in GIVIFS_B$ and $\alpha, \beta \in [0,1]$, where $0 \leq \alpha + \beta \leq 1$, it holds that

i. $D_\alpha(\Box A) = F_{\alpha,\beta}(\Box A) = \Box A$,
ii. $D_\alpha(\Diamond A) = F_{\alpha,\beta}(\Diamond A) = \Diamond A$,
iii. $d_\alpha(\Box A) = f_{\alpha,\beta}(\Box A) = \overline{\Box A}$,
iv. $d_\alpha(\Diamond A) = f_{\alpha,\beta}(\Diamond A) = \overline{\Diamond A}$.

Proof. Proof of the theorem is obtained directly from the definitions.

Theorem 3.7. For every $A \in GIVIFS_B$ and $\alpha_1, \alpha_2, \eta, \gamma \in [0,1]$, where $0 \leq \eta + \gamma \leq 1$, it holds that

i. $d_{\alpha_1}\left(d_{\alpha_2}(A)\right) = D_{1-\alpha_2}(A)$,
ii. $D_{\alpha_1}\left(D_{\alpha_2}(A)\right) = D_{\alpha_2}(A)$,
iii. $d_{\alpha_1}\left(D_{\alpha_2}(A)\right) = d_{1-\alpha_2}(A)$,
iv. $D_{\alpha_1}\left(d_{\alpha_2}(A)\right) = d_{\alpha_2}(A)$,
v. $F_{\eta,\gamma}\left(D_{\alpha_2}(A)\right) = D_{\alpha_2}(A)$,
vi. $F_{\eta,\gamma}\left(d_{\alpha_2}(A)\right) = d_{\alpha_2}(A)$,
vii. $f_{\eta,\gamma}\left(D_{\alpha_2}(A)\right) = d_{1-\alpha_2}(A)$,
viii. $f_{\eta,\gamma}\left(d_{\alpha_2}(A)\right) = D_{1-\alpha_2}(A)$.

Proof. Proof of the theorem is obtained directly from the definitions.

It can be easily shown that $F_{\eta,\gamma}\left(D_\alpha(A)\right) = \overline{F_{\eta,\gamma}\left(d_{1-\alpha}(A)\right)}$, $f_{\eta,\gamma}\left(d_{1-\alpha}(A)\right) = F_{\eta,\gamma}\left(D_\alpha(A)\right)$, and $d_{\alpha_1}\left(D_{1-\alpha_2}(A)\right) = D_{\alpha_1}\left(d_{\alpha_2}(A)\right)$.

Definition 3.6. Let $A \in GIVIFS_B$ and $\alpha, \beta \in [0,1]$, we define the operator of $j_{\alpha,\beta}(A, \delta)$ (in summary, we will show as $j_{\alpha,\beta}(A)$) as follows

$$j_{\alpha,\beta}(A) = \left\{\langle x, M_{j_{\alpha,\beta}}(A), N_{j_{\alpha,\beta}}(A)\rangle : x \in X\right\},$$
$$M_{j_{\alpha,\beta}}(A) = [N_{AL}(x), \left(N_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}],\ N_{j_{\alpha,\beta}}(A) = [\beta^{\frac{1}{\delta}} M_{AL}(x), \beta^{\frac{1}{\delta}} M_{AU}(x)].$$

Theorem 3.8. For every $A \in GIVIFS_B$ and $\alpha, \beta, \gamma \in [0,1]$, it holds that

i. $j_{\alpha,\beta}(A) \in GIVIFS_B$,

ii. $\alpha \leq \gamma \Rightarrow j_{\alpha,\beta}(A) \subset j_{\gamma,\beta}(A)$,
iii. $\beta \leq \gamma \Rightarrow j_{\alpha,\gamma}(A) \subset j_{\alpha,\beta}(A)$,
iv. $j_{1,1}(A) = \overline{\Box A}$,
v. $j_{0,1}(A) = \overline{A}$,
vi. $j_{\alpha,\beta}(\overline{A}) = J_{\alpha,\beta}(A)$,
vii. $\overline{A} \subset j_{\alpha,\beta}(A)$.

Proof. (i) Note that

$$\begin{aligned} M_{j_{\alpha,\beta}(A)U}(x)^{\delta} + N_{j_{\alpha,\beta}(A)U}(x)^{\delta} &= \left(N_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}})^{\delta} + \left(\beta^{\frac{1}{\delta}} M_{AU}(x)\right)^{\delta}, \\ &= \left(N_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta}\right) + \beta M_{AU}(x)^{\delta}, \\ &\leq N_{AU}(x)^{\delta} + \pi_{AL}(x)^{\delta} + M_{AU}(x)^{\delta} = 1. \end{aligned}$$

Finally, it can be concluded that $j_{\alpha,\beta}(A) \in GIVIFS_B$.

(ii) Since $\alpha \leq \gamma$ then it is clear that

$$[N_{AL}(x), \left(N_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}] \leq [N_{AL}(x), \left(N_{AU}(x)^{\delta} + \gamma\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}].$$

Finally we have $j_{\alpha,\beta}(A) \subset j_{\gamma,\beta}(A)$.

The proof of (iii) is similar to that of (ii). Proofs of (iv), (v), (vi) and (vii) are obvious.

Definition 3.7. Let $A \in GIVIFS_B$ and $\alpha, \beta \in [0,1]$, we define the operator of $H_{\alpha,\beta}(A,\delta)$ (in summary, we will show as $H_{\alpha,\beta}(A)$) as follows

$$H_{\alpha,\beta}(A) = \left\{\langle x, M_{H_{\alpha,\beta}}(A), N_{H_{\alpha,\beta}}(A)\rangle : x \in X\right\},$$
$$M_{H_{\alpha,\beta}}(A) = [\alpha^{\frac{1}{\delta}} M_{AL}(x), \alpha^{\frac{1}{\delta}} M_{AU}(x)],\ N_{H_{\alpha,\beta}}(A) = [(N_{AL}(x), (N_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}].$$

Theorem 3.9. For every $A \in GIVIFS_B$ and $\alpha, \beta, \gamma \in [0,1]$ it holds that

i. $H_{\alpha,\beta}(A) \in GIVIFS_B$,
ii. $\alpha \leq \gamma \Rightarrow H_{\gamma,\beta}(A) \subset H_{\alpha,\beta}(A)$,
iii. $\beta \leq \gamma \Rightarrow H_{\alpha,\gamma}(A) \subset H_{\alpha,\beta}(A)$,
iv. $H_{1,0}(A) = A$,
v. $H_{1,1}(A) = \Box A$,
vi. $H_{\alpha,\beta}(A) \subset A$.

Proof. (i) Follows since

$$\begin{aligned} M_{H_{\alpha,\beta}(A)U}(x)^{\delta} + N_{H_{\alpha,\beta}(A)U}(x)^{\delta} &= \left(\alpha^{\frac{1}{\delta}} M_{AU}(x)\right)^{\delta} + \left(N_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}})^{\delta}, \\ &= \alpha M_{AU}(x)^{\delta} + \left(N_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta}\right), \\ &\leq M_{AU}(x)^{\delta} + N_{AU}(x)^{\delta} + \pi_{AL}(x)^{\delta} = 1. \end{aligned}$$

Proofs of (ii), (iii), (iv), (v) and (vi) are obvious.

Definition 3.8. Let $A \in GIVIFS_B$ and $\alpha, \beta \in [0,1]$, we define the operator of $h_{\alpha,\beta}(A,\delta)$ (in summary, we will show as $h_{\alpha,\beta}(A)$) as follows

$$h_{\alpha,\beta}(A) = \left\{ \langle x, M_{h_{\alpha,\beta}}(A), N_{h_{\alpha,\beta}}(A)\rangle : x \in X \right\},$$
$$M_{h_{\alpha,\beta}}(A) = [\alpha^{\frac{1}{\delta}} N_{AL}(x), \alpha^{\frac{1}{\delta}} N_{AU}(x)],\ N_{h_{\alpha,\beta}}(A) = [(M_{AL}(x), (M_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}].$$

Theorem 3.10. For every $A \in GIVIFS_B$ and $\alpha, \beta, \gamma \in [0,1]$, it holds that

i. $h_{\alpha,\beta}(A) \in GIVIFS_B$,
ii. $\alpha \le \gamma \Rightarrow h_{\gamma,\beta}(A) \subset h_{\alpha,\beta}(A)$,
iii. $\beta \le \gamma \Rightarrow h_{\alpha,\gamma}(A) \subset h_{\alpha,\beta}(A)$,
iv. $h_{\alpha,\beta}(\overline{A}) = H_{\alpha,\beta}(A)$,
v. $h_{1,0}(A) = \overline{A}$,
vi. $h_{1,1}(A) = \overline{\Diamond A}$,
vii. $h_{\alpha,\beta}(A) \subset \overline{A}$.

Proof. (i) Follows since

$$\begin{aligned} M_{h_{\alpha,\beta}(A)U}(x)^{\delta} + N_{h_{\alpha,\beta}(A)U}(x)^{\delta} &= \left(\alpha^{\frac{1}{\delta}} N_{AU}(x)\right)^{\delta} + ((M_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}})^{\delta}, \\ &= \alpha N_{AU}(x)^{\delta} + (M_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta}), \\ &\le N_{AU}(x)^{\delta} + M_{AU}(x)^{\delta} + \pi_{AL}(x)^{\delta} = 1. \end{aligned}$$

Proofs of (ii), (iii), (iv), (v), (vi) and (vii) are obvious.

Theorem 3.11. For every $A \in GIVIFS_B$ and $\alpha, \beta, \gamma \in [0,1]$, it holds that

i. $H_{\alpha,\beta}(D_{\alpha}(A)) \subset D_{\alpha}(A) \subset J_{\alpha,\beta}(D_{\alpha}(A))$,
ii. $H(d_{\alpha}(A)) \subset d_{\alpha}(A) \subset J_{\alpha,\beta}(d_{\alpha}(A))$,
iii. $h(d_{\alpha}(A)) \subset D_{1-\alpha}(A) \subset j_{\alpha,\beta}(d_{\alpha}(A))$,
iv. $h(D_{\alpha}(A)) \subset d_{1-\alpha}(A) \subset j_{\alpha,\beta}(D_{\alpha}(A))$.

Proof. Proof of the theorem is obtained directly from the definitions.

Theorem 3.12. For every $A \in GIVIFS_B$ and $\alpha, \beta, \in [0,1]$, where $0 \le \alpha + \beta \le 1$, $\delta_1 \le \delta_2$, it holds that

i. $D_{\alpha}(A,\delta_2) \subset_{\Box} D_{\alpha}(A,\delta_1)$ and $D_{\alpha}(A,\delta_1) \subset_{\Diamond} D_{\alpha}(A,\delta_2)$,
ii. $F_{\alpha,\beta}(A,\delta_2) \subset_{\Box} F_{\alpha,\beta}(A,\delta_1)$ and $F_{\alpha,\beta}(A,\delta_1) \subset_{\Diamond} F_{\alpha,\beta}(A,\delta_2)$,
iii. $J_{\alpha,\beta}(A,\delta_2) \subset J_{\alpha,\beta}(A,\delta_1)$,
iv. $d_{\alpha}(A,\delta_2) \subset_{\Box} d_{\alpha}(A,\delta_1)$ and $d_{\alpha}(A,\delta_1) \subset_{\Diamond} d_{\alpha}(A,\delta_2)$,
v. $f_{\alpha,\beta}(A,\delta_2) \subset_{\Box} f_{\alpha,\beta}(A,\delta_1)$ and $f_{\alpha,\beta}(A,\delta_1) \subset_{\Diamond} f_{\alpha,\beta}(A,\delta_2)$,
vi. $j_{\alpha,\beta}(A,\delta_2) \subset j_{\alpha,\beta}(A,\delta_1)$,
vii. $H_{\alpha,\beta}(A,\delta_1) \subset H_{\alpha,\beta}(A,\delta_2)$,
viii. $h_{\alpha,\beta}(A,\delta_1) \subset h_{\alpha,\beta}(A,\delta_2)$.

Proof. It follows from the fact that $g_1(\delta) = (a^{\delta} + \alpha b^{\delta})^{\frac{1}{\delta}}$ is decreasing and $g_2(\delta) = a^{\frac{1}{\delta}}$ is increasing.

Corollary 3.1. For every $A \in GIVIFS_B$, where $M_{AU}(x)^{\delta} + N_{AU}(x)^{\delta} = 1$, it holds it

i. $D_\alpha(A) = F_{\alpha,\beta}(A) = A$,
ii. $d_\alpha(A) = f_{\alpha,\beta}(A) = \overline{A}$.

Corollary 3.2. For every $A \in GIVIFS_B$ and $\alpha_i, \beta_i \in [0,1]$, where $\alpha_1 \leq \alpha_2$, $\beta_2 \leq \beta_1$ and $0 \leq \alpha_i + \beta_i \leq 1$,i=1,2, it holds that

i. $D_{\alpha_1}(A) \subset D_{\alpha_2}(A)$,
ii. $d_{\alpha_1}(A) \subset d_{\alpha_2}(A)$,
iii. $F_{\alpha_1,\beta_1}(A) \subset F_{\alpha_2,\beta_2}(A)$,
iv. $J_{\alpha_1,\beta_1}(A) \subset J_{\alpha_2,\beta_2}(A)$,
v. $j_{\alpha_1,\beta_1}(A) \subset j_{\alpha_2,\beta_2}(A)$,
vi. $f_{\alpha_1,\beta_1}(A) \subset f_{\alpha_2,\beta_2}(A)$,
vii. $H_{\alpha_1,\beta_1}(A) \subset H_{\alpha_2,\beta_2}(A)$,
viii. $h_{\alpha_1,\beta_1}(A) \subset h_{\alpha_2,\beta_2}(A)$.

Corollary 3.3. For every $A \in GIVIFS_B$ and $\alpha, \beta \in [0,1]$, it holds that

i. $d_\alpha(A) \subset f_{\alpha,\beta}(A) \subset j_{\alpha,\beta}(A)$,
ii. $D_\alpha(A) \subset F_{\alpha,\beta}(A) \subset J_{\alpha,\beta}(A)$,
iii. $H_{\alpha,\beta}(A) \subset J_{\alpha,\beta}(A)$,
iv. $h_{\alpha,\beta}(A) \subset j_{\alpha,\beta}(A)$.

Remark 3.1. According to definition, the operators of $D_\alpha(A)$and $F_{\alpha,\beta}(A)$increases the membership and non-membership degreeA, the operators of $d_\alpha(A)$and $f_{\alpha,\beta}(A)$ increases the membership and non-membership degree $\overline{A}$, the operators of $h_{\alpha,\beta}(A)$ reduces the membership degree $\overline{A}$ and increases non-membership degree $\overline{A}$, the operators of $H_{\alpha,\beta}(A)$ reduces the membership degree A and increases non-membership degree A, the operators of $j_{\alpha,\beta}(A)$ increases the membership degree $\overline{A}$ and reduces non-membership degree $\overline{A}$, the operators of $J_{\alpha,\beta}(A)$ increases the membership degree A and reduces non-membership degree A.

Example 3.1. Let $A = \{\langle x_1, [0.2, 0.3], [0.1,0.2]\rangle\}$, $\delta = 0.5$, then

$\Box A = \{\langle x_1, [0.2,0.3], [0.1,0.204555]\rangle\}$,
$\Diamond A = \{\langle x_1, 0.2, 0.305573],0.10.2]\rangle\}$,
$\pi_{AL}(x_1)^{0.5} = 0.005064$,
$F_{\alpha,\beta}(A) = \{\langle x_1, [0.2, (\sqrt{0.3} + 0.005064\alpha)^2], [0.1, (\sqrt{0.2} + 0.005064\beta)^2]\rangle\}$,
$f_{\alpha,\beta}(A) = \{\langle x_1, [0.1, (\sqrt{0.2} + 0.005064\alpha)^2], [0.2, (\sqrt{0.3} + 0.005064\beta)^2]\rangle\}$,
$J_{\alpha,\beta}(A) = \{\langle x_1, [0.2, (\sqrt{0.3} + 0.005064\alpha)^2], [0.1\beta^2, 0.2\beta^2]\rangle\}$,
$j_{\alpha,\beta}(A) = \{\langle x_1, [0.1, (\sqrt{0.2} + 0.005064\alpha)^2], [0.2\beta^2, 0.3\beta^2]\rangle\}$,
$H_{\alpha,\beta}(A) = \{\langle x_1, [0.2\alpha^2, 0.3\alpha^2], [0.1, (\sqrt{0.2} + 0.005064\beta)^2]\rangle\}$,
$h_{\alpha,\beta}(A) = \{\langle x_1, [0.1\alpha^2, 0.2\alpha^2], [0.2, (\sqrt{0.3} + 0.005064\beta)^2]\rangle\}$.

Example 3.2. Let $A = \{\langle x_1, [0.2, 0.3], [0.1,0.2]\rangle\}$, $\delta = 0.5$, then operators due to Baloui Jamkhaneh [41] are as follows

$\pi_A(x_1) = [0.005064^2, 0.236559^2]$, $\lambda_A = 1.3032254$.

$F_{\alpha,\beta}(A) = \left\{\langle x_1, [(\sqrt{0.2} + 0.236559\alpha)^2, (\sqrt{0.3} + 0.005064\alpha)^2], [(\sqrt{0.1} + 0.236559\beta)^2, (\sqrt{0.2} + 0.005064\beta)^2]\rangle\right\}, 0 \le \alpha \le 0.4341737, 0 \le \beta \le 0.5658262.$

$f_{\alpha,\beta}(A) = \left\{\langle x_1, \left[(\sqrt{0.1} + 0.236559\alpha)^2, (\sqrt{0.2} + 0.005064\alpha)^2\right], \left[(\sqrt{0.2} + 0.236559\beta)^2, (\sqrt{0.3} + 0.005064\beta)^2\right]\rangle\right\}, 0 \le \alpha \le 0.5658262, 0 \le \beta \le 0.4341737.$

$J_{\alpha,\beta}(A) = \left\{\langle x_1, [(\sqrt{0.2} + 0.236559\alpha)^2, (\sqrt{0.3} + 0.005064\alpha)^2], [0.1\beta^2, 0.2\beta^2]\rangle : x \in X\right\},$
$0 \le \alpha \le 0.4341737, 0 \le \beta \le 1.$

$j_{\alpha,\beta}(A) = \left\{\langle x_1, \left[(\sqrt{0.1} + 0.236559\alpha)^2, (\sqrt{0.2} + 0.005064\alpha)^2\right], [0.2\beta^2, 0.3\beta^2]\rangle\right\},$
$0 \le \alpha \le 0.5658262, 0 \le \beta \le 1.$

$H_{\alpha,\beta}(A) = \left\{\langle x_1, [0.2\alpha^2, 0.3\alpha^2], [(\sqrt{0.1} + 0.236559\alpha)^2, (\sqrt{0.2} + 0.005064\alpha)^2]\rangle\right\},$
$0 \le \alpha \le 1, 0 \le \beta \le 0.5658262.$

$h_{\alpha,\beta}(A) = \left\{\langle x_1, [0.1\alpha^2, 0.2\alpha^2], [(\sqrt{0.2} + 0.236559\alpha)^2, (\sqrt{0.3} + 0.005064\alpha)^2, (0.36 + 0.55\beta)^{\frac{1}{2}}]\rangle\right\},$
$0 \le \alpha \le 1, 0 \le \beta \le 0.4341737.$

Remark 3.2. According to definitions and examples, only the upper bound increases for any new operator that increases the degrees. Correspondingly, the upper and lower bound increases for any operator that increases the degrees in the operators of Baloui Jamkhaneh [41]. From these comparison results, it can be seen that the proposed operators have more general parameters.

4. CONCLUSIONS

We have introduced modified modal types of operators over Baloui's generalized interval valued intuitionistic fuzzy sets and their relationships are proved. We show that these operators are $GIVIFS_B$. Some proven relationships between operators are shown in Table 1. For example, cell (1,1) shows that $d_{\alpha_1}(D_{\alpha_2}(A)) = d_{1-\alpha_2}(A)$. An open problem is: definition of level operators, negation operators and other operators over $GIVIFS_B$ and the study of their properties.

Table 1. *Relation between operators*

	$D_{\alpha_2}(A)$	$d_{\alpha_2}(A)$	$\Box A$	$\lozenge A$
d_{α_1}	$d_{1-\alpha_2}(A)$	$D_{1-\alpha_2}(A)$	$\overline{\Box A}$	$\overline{\lozenge A}$
D_{α_1}	$D_{\alpha_2}(A)$	$d_{\alpha_2}(A)$	$\Box A$	$\lozenge A$
$F_{\eta,\gamma}$	$D_{\alpha_2}(A)$	$d_{\alpha_2}(A)$	$\Box A$	$\lozenge A$
$f_{\eta,\gamma}$	$d_{1-\alpha_2}(A)$	$D_{1-\alpha_2}(A)$	$\overline{\Box A}$	$\overline{\lozenge A}$

Table 2. *Special cases of operators*

	$D_\alpha(A)$	$F_{\alpha,\beta}(A)$	$d_\alpha(A)$	$f_{\alpha,\beta}(A)$
$\alpha = 0, \beta = 1$	$\Box A$	$\Box A$	$\overline{\lozenge A}$	$\overline{\lozenge A}$
$\alpha = 1, \beta = 0$	$\lozenge A$	$\lozenge A$	$\overline{\Box A}$	$\overline{\Box A}$

ACKNOWLEDGEMENTS

The author wishes to thank the editor and referees, for the careful reading of the paper and for the helpful suggestions and comments.

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

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