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Research Article

Generalizations of n-ideals of Commutative Rings

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Abstract
Let R be a commutative ring with identity and $Id(R)$ denotes the set of all ideals of R. We will concerned in this study mainly with the generalizations of n-ideals in commutative rings via a function $\phi: Id(R) \rightarrow Id(R) \cup \{\emptyset\}$. Properties of this class of ideals will investigated in detail.

Keywords: n-ideal; $\phi - n$-ideal; ϕ −prime ideal; ϕ −primary ideal.

Değişmeli Halkalarda n-ideallerin Genelleştirmeleri

Öz
R değişmeli, birimli bir halka olsun ve $Id(R)$, R nin tüm ideallerinin kümesini göstersin. Bu çalışmada, esas olarak değişmeli halkalarda n −ideal kavramının bir $\phi: Id(R) \rightarrow Id(R) \cup \{\emptyset\}$ fonksiyonu aracıyla genelleştirmeleri üzerinde duracağız. Bu ideal sınıfının özellikleri detaylarıyla incelenecektir.

Anahtar Kelimeler: n-ideal; $\phi - n$-ideal; ϕ −asal ideal; ϕ −asalımsı ideal.

1. Preliminaries and Background

Throughout this paper, all rings are assumed to be commutative with nonzero identity and by $Id(R)$, we mean the set of all ideals of a ring R. Let I be a proper ideal of a ring R. The radical of I is given by $\sqrt{I} = \{r \in R : r^k \in I \text{ for some } k \in \mathbb{N}\}$. In particular, the set of the nilpotent elements of R is $\sqrt{0}$, that is $\{r \in R : r^k = 0 \text{ for some } k \in \mathbb{N}\}$. For an element $r \in R$, the ideal $\{s \in R: rs \in I\}$ is denoted by $(I: r)$.

Since prime ideals have an important role in ring theory, several authours generalized these concept in different ways. Please see Anderson and Smith (2003), Bataineh (2006), Atani and Farzalipour (2005), Badawi (2007), Badawi and Darani (2013), Anderson and Badawi (2011) and Badawi et al. (2014). Later, the concepts of ϕ −prime and ϕ −primary ideals are introduced in (Anderson and Batanieh 2008, Darani 2012). Let $\phi : Id(R) \rightarrow Id(R) \cup \{\emptyset\}$ be a function and $\emptyset \neq I \in Id(R)$. Then I is said to be a *ϕ −prime (resp. ϕ −primary) ideal* of R if whenever $r, s \in R$ and $rs \in I - \phi(I)$, then $r \in I$ or $s \in I$ (resp. $r \in I$ or $s \in \sqrt{I}$). Recall from Khaksari (2015) that I is called a *ϕ − 2-absorbing ideal* of R if whenever $r, s, t \in R$ and $rst \in I - \phi(I)$, then either $rs \in I$ or $st \in I$ or $rt \in I$. The concept of ϕ −2-

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absorbing primary ideals is first introduced and studied in Badawi et al. (2016): I is called a $\phi-2$-*absorbing primary ideal of* R if whenever $r,s,t \in R$ and $rst \in I-\phi(I)$, then either $rs \in I$ *or* $st \in \sqrt{I}$ or $rt \in \sqrt{I}$. In a recent study U. Tekir et al. (2017), n-ideals are defined as following: I is an n-*ideal* if $r,s \in R$ and $rs \in I$ and $r \notin \sqrt{0}$, then $s \in I$. In this study, we generalize the concept of n-ideals in a commutative ring via a function $\phi: Id(R) \to Id(R) \cup \{\emptyset\}$. We investigate the properties of $\phi-n$-ideals in detail.

We give some notations and state the necessary lemmas which will be used in the sequel. Let R be a commutative ring and M an R-module. Then the idealization, $R(+)M = \{(r,m) : r \in R, m \in M\}$ is a commutative ring with componentwise addition and multiplication $(r,m)(s,n) = (rs, rn+sm)$ for each $r,s \in R$ and $m,n \in M$. Moreover, J is an ideal of $R(+)M$ if and only if $J = I(+)N$ where $I = \{r \in R: (r,m) \in J$ for some $m \in M\}$ an ideal of R, and $N = \{n \in M: (r,n) \in J$ for some $r \in R\}$ a submodule of M satisfying $IM \subseteq N$ (Huckaba 1988). As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo n, respectively.

Lemma 1.1. Darani (2012) Let R be a commutative ring, and let $\phi: Id(R) \to Id(R) \cup \{\emptyset\}$ be a function. Then every ϕ-prime ideal of R is ϕ-primary.

Lemma 1.2. Tekir et. al. (2017) Let R be a commutative ring and $J \subseteq I$ be two ideals of R. If I is an n-ideal of R, then I/J is an n-ideal of R/J.

2. $\phi-n$-ideals of Commutative Rings

In this section, we are going to intoduce $\phi-n$-ideals in commutative rings and present many the properties of them.

Definition 2.1. Let R be a commutative ring, I a proper ideal of R. Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function. We call I a $\phi-n$-ideal of R if whenever $r,s \in R$ and $rs \in I - \phi(I)$, then either r is nilpotent or $s \in I$.

Let I be a $\phi-n$-ideal of R. Then define:

(1) *If* $\phi(J) = \emptyset$ for all $J \in Id(R)$, then we say that $\phi = \phi_\emptyset$ and I is called a $\phi_\emptyset$-*n-ideal*, and hence I is an n-*ideal* of R.

(2) *If* $\phi(J) = 0$ for all $J \in Id(R)$, then we say that $\phi = \phi_0$ and I is called a ϕ_0-*n-ideal (weakly* n-*ideal*) of R.

(3) *If* $\phi(J) = J$ for all $J \in Id(R)$, then we say that $\phi = \phi_1$ and I is called a ϕ_1-*n-ideal* (any ideal) of R.

(4) *If* $k \geq 2$ and $\phi(J) = J^k$ for all $J \in Id(R)$, then we say that $\phi = \phi_k$ and I is called a ϕ_k-*n-ideal (k-almost* n-*ideal)* of R. In special, if $k = 2$, then we call I an *almost* n-*ideal* of R.

(5) If $\phi(J) = \bigcap_{i=1}^{\infty} J^i$ for all $J \in Id(R)$, then we say that $\phi = \phi_\omega$ *and* I *is called a* $\phi_\omega-n$-*ideal (*$\omega-n-$*ideal) of* R.

Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function. Observe that $I - \phi(I) = I - (I \cap \phi(I))$. So without loss of generality, assume throughout that $\phi(I) \subseteq I$. If ψ_1 and ψ_2 are two functions $\psi_{1,2}: Id(R) \to Id(R) \cup \{\emptyset\}$, then we say $\psi_1 \leq \psi_2$ if $\psi_1(J) \subseteq \psi_2(J)$ for all $J \in Id(R)$.

We give the following examples which show that the concept of ϕ-n-ideals and n-ideals are different:

Example 2.2.

(1) For every ring R, the zero ideal is a $\phi_k - n$-ideal of R for all $k \geq 0$. However, it may not be an n-ideal. Consider the ring $\mathbb{Z}_6$. Since $\bar{2} \cdot \bar{3} \in (\bar{0})$ but neither $\bar{2} \in \sqrt{0}$ nor $\bar{3} \in (\bar{0})$. So $(\bar{0})$ is not an n-ideal ($\phi_\emptyset$-n-ideal) of $\mathbb{Z}_6$.

(2) Consider the ideal $A = \{\bar{0}, \bar{9}, \overline{18}, \overline{27}\}$ of $\mathbb{Z}_{36}$. Let $R = \mathbb{Z}_{36}(+)A$ and $I = \{(\bar{0}, \bar{0}), (\bar{0}, \overline{18})\}$. Then I is a $\phi_2 - n$-ideal of R which is not an n-ideal. Indeed, since there is no $r, s \in R$ with $rs \in I - \phi_2(I) = I - I^2 = (\bar{0}, \overline{18})$, I is clearly a $\phi_2 - n$-ideal. However, since $(\bar{4}, \bar{0}) \cdot (\bar{9}, \bar{0}) \in I$ but neither $(\bar{4}, \bar{0}) \in \sqrt{0}$ nor $(\bar{9}, \bar{0}) \in I$, it is not an n-ideal.

Theorem 2.3. For any $I \in Id(R)$, the following statements hold:

(1) Let ψ_1 and ψ_2 are two functions $\psi_{1,2}$: $Id(R) \rightarrow Id(R) \cup \{\emptyset\}$ such that $\psi_1 \leq \psi_2$. If I is a $\psi_1 - n$-ideal, then I is a $\psi_2 - n$-ideal.

(2) I is a n-ideal $\Rightarrow$ I is a weakly n-ideal $\Rightarrow$ I is a $\phi_\omega - n$-ideal $\Rightarrow$ I is a $\phi_{k+1} - n$-ideal for every $k \geq 2$ $\Rightarrow$ I is a $\phi_k - n$-ideal for every $n \geq 2$ $\Rightarrow$ I is an almost ideal.

(3) I is a $\phi - n$-ideal $\Rightarrow$ I is a ϕ $-$primary ideal $\Rightarrow$ I is a $\phi - 2$-absorbing primary ideal.

(4) I is an idempotent ideal of R $\Rightarrow$ I is an $\phi_k - n$-ideal of R for every $k \geq 1$.

(5) I is a $\phi_k - n$-ideal for all $k \geq 2 \Leftrightarrow I$ is a $\phi_\omega - n$-ideal**.**

Proof: (1) Straightforward.

(2) It is clear that there is a linear ordering: $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{k+1} \leq \phi_k \leq \cdots \leq \phi_2 \leq \phi_1$. So we obtain the result.

(3) It is clear.

(4) Since I is idempotent, clearly $I^k = I^2 = I$ for all $k \geq 2$. Hence $\phi_k(I) = I$ for all $k \geq 1$, we are done.

(5) It is clear by *(2)*.

Lemma 2.4. If I is an ideal of which elements are nilpotent, then I is $\phi - n$-ideal if and only if I is ϕ $-$primary. Moreover, if R is a ring of which elements are nilpotent, then the concepts of a ϕ $-$primary and a $\phi - n$-ideal coincide.

Proof: Suppose that I is $\phi - n$-ideal of R. Then it is ϕ $-$primary by *Theorem 2.3. (3)*. Conversely, suppose that $r, s \in R$ with $rs \in I - \phi(I)$ and r is non-nilpotent. Since $\sqrt{I} = \sqrt{0}$ and I is assumed to be ϕ-primary, we have $s \in I$. Thus I is a $\phi - n$-ideal of R. The "moreover" part is obvious.

Theorem 2.5. If $I = \sqrt{0}$, then all of the following cases are equivalent:

(1) I is $\phi - n$-ideal.

(2) I is ϕ $-$prime.

(3) I is ϕ $-$primary.

(4) I is $\phi - 2$-absorbing primary.

(5) I is $\phi - 2$-absorbing.

Proof: **(1)** $\Rightarrow$ **(2)** Let $r, s \in R$ with $rs \in I - \phi(I)$ and $r \notin I = \sqrt{0}$. Since I is $\phi - n$-ideal, we conclude that $s \in I$.

(2) $\Rightarrow$ **(3)** From *Lemma 1.1*, it is clear.

(3) $\Rightarrow$ **(4)** It is obvious.

(4) $\Rightarrow$ **(5)** Since $I = \sqrt{I} = \sqrt{0}$, the result is clear.

(5) $\Rightarrow$ **(1)** Let $r, s \in R$ with $rs \in I - \phi(I)$ and $r \notin I$. Then $r \cdot 1 \cdot s \in I - \phi(I)$ and $r \cdot 1 \notin I$ and $r \cdot s \notin I$. Since I is $\phi - 2$-absorbing, we have $s = s \cdot 1 \in I = \sqrt{0}$, we are done.

Theorem 2.6. For any $R \neq I \in Id(R)$, the following statements hold:

(1) *If I is a $\phi - n$*-ideal of R, *then $I/\phi(I)$ is a weakly $\phi - n$*-ideal of $R/\phi(I)$.

(2) *If $I/\phi(I)$ is a weakly n*-ideal of $R/\phi(I)$ and $\sqrt{\phi(I)} = \sqrt{0}$, then *$I$ is a $\phi - n$*-ideal of R.

Proof: **(1)** Let $0 \neq \big(r + \phi(I)\big)\big(s + \phi(I)\big) \in I/\phi(I)$ and $\big(r + \phi(I)\big)$ be a non-nilpotent element of $R/\phi(I)$. Hence $rs \in I - \phi(I)$ and r is a non-nilpotent element of R. Since I is $\phi - n$-ideal, we have $s \in I$; so $s + \phi(I) \in I/\phi(I)$, as needed.

(2) Let $r, s \in R$ with $rs \in I - \phi(I)$. Hence $0 \neq \big(r + \phi(I)\big)\big(s + \phi(I)\big) \in I/\phi(I)$. Since $I/\phi(I)$ is a weakly n-ideal, we conclude that $r + \phi(I) \in \sqrt{0_{R/\phi(I)}}$ or $s + \phi(I) \in I/\phi(I)$. Therefore, $r \in \sqrt{\phi(I)} = \sqrt{0}$ or $s \in I$. Consequently, I is a ϕ –n-ideal of R.

Theorem 2.7. For any $R \neq I \in Id(R)$, the following conditions are equivalent:

(1) I is a $\phi - n$-ideal of R.

(2) $(I : r) = I \cup (\phi(I) : r)$ for every non-nilpotent element r of R.

(3) $(I : r) = I$ or $(I : r) = (\phi(I) : r)$ for every non-nilpotent element r of R.

(4) For every ideals J and K of R, $JK \subseteq I$ and $JK \nsubseteq \phi(I)$ imply $J \subseteq \sqrt{0}$ or $K \subseteq I$.

Proof: **(1)** $\Rightarrow$ **(2)** Since $I \subseteq (I : r)$ and $(\phi(I) : r) \subseteq (I : r)$, we need to show that $(I : r) \subseteq I \cup (\phi(I) : r)$. Let $s \in (I : r)$. Then $rs \in I$. If $rs \in \phi(I)$, then $s \in (\phi(I) : r)$. Now suppose that $rs \notin \phi(I)$. Since I *is* $\phi - n$-ideal and r is non-nilpotent, we conclude that $s \in I$. Thus we conclude $s \in I \cup (\phi(I) : r)$, as needed.

(2) $\Rightarrow$ **(3).** It is clear.

(3) $\Rightarrow$ **(4).** Suppose that J and K are ideals of R with $JK \subseteq I$ but $J \nsubseteq \sqrt{0}$, $K \nsubseteq I$. Let $j \in J$. Then j is a nilpotent element or not.

Case I. Suppose that j is non-nilpotent. Hence $jK \subseteq I$ which means $K \subseteq (I : j)$. On the other hand, we have $(I : j) = I$ or $(I : j) = (\phi(I) : j)$ by *(3)*. Since our assumption $K \nsubseteq I$, we conclude $K \subseteq (\phi(I) : j)$, i.e. $jK \subseteq \phi(I)$.

Case II. Suppose that j is a nilpotent element. Since $J \nsubseteq \sqrt{0}$, there exists a non-nilpotent element x in J. Then it is clear that $(j + x)$ is a non-nilpotent element of J. From *(3)*, we have $K \subseteq (\phi(I) : x)$ and $K \subseteq (\phi(I) : (j + x))$. Let $k \in K$. Now we conclude $jk = (j + x)k - xk \in \phi(I)$. Consequently, $JK \subseteq \phi(I)$, we are done.

(4) $\Rightarrow$ **(1).** Let $r, s \in R$ and $rs \in I - \phi(I)$. Put $J = (r)$, $K = (s)$ in *(4)*. Then the result is clear.

Definition 2.8. Let R be a commutative ring, I a proper ideal of R. Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function. We call I *a strongly* ϕ*–*n*-ideal of* R if whenever $JK \subseteq I$ and $JK \nsubseteq \phi(I)$ for some ideals J, K of R, then $I \subseteq \sqrt{0}$ or $K \subseteq I$.

So we conclude the following corollary:

Corollary 2.9. For a proper ideal I of R, I is a $\phi - n$-ideal of R if and only if I is a strongly $\phi - n$-ideal of R.

Theorem 2.10. Let I be a $\phi - n$-ideal of R. Then $I - \phi(I) \subseteq \sqrt{0}$.

Proof: Assume that $I - \phi(I) \nsubseteq \sqrt{0}$. Then there is a non-nilpotent element with $r \in I - \phi(I)$. Since $r = r \cdot 1 \in I - \phi(I)$ and I is $\phi - n$-ideal, this implies that $1 \in I$, a contadiction. Thus $I - \phi(I) \subseteq \sqrt{0}$.

Remark 2.11. (1) If (R, M) is a local ring with unique prime ideal, then every ideal is a $\phi - n$-ideal for all ϕ.

(2) Let R be an integral domain. Then zero ideal is a $\phi - n$-ideal for all ϕ.

There are some rings which have no $\phi - n$-ideal for $\phi \neq \phi_1$.

Example 2.12. Consider the ring $R = \mathbb{Z}_{p_1 p_2 \cdots p_t}$ for some distinct prime integers $p_1, \dots, p_t$. Then there is no $\phi - n$-ideal for $\phi \neq \phi_1$.

A ring R is called a reduced ring if there is no nonzero nilpotent element of R.

Theorem 2.13. Let R be a reduced ring which is not an integral domain. Then R has no $\phi - n$-ideal for $\phi \neq \phi_1$.

Proof: Assume on the contary that J is a ϕ–n-ideal of R. From *Theorem 2.10*, we conclude $J - \phi(J) \subseteq \sqrt{0} = 0$. Thus $\phi(J) = J$, and so $\phi = \phi_1$. Thus R has no $\phi - n$-ideal for $\phi \neq \phi_1$.

Corollary 2.14. Let R be a reduced ring and $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function such that $\phi \neq \phi_1$. Then the following statements are equivalent:

(1) R is a integral domain.

(2) 0 is a $\phi - n$-ideal of R.

Proof: (1) ⇒ (2). Since R is an integral domain, $\sqrt{0} = 0$ is a prime ideal, so it is ϕ −prime. Thus 0 is a $\phi - n$-ideal of R by *Theorem 2.5.*

(2) ⇒ (1). It is clear by *Theorem 2.13*.

Theorem 2.14. Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ a function and $I - \phi(I)$ is a prime ideal of R. Then the following two conditions are equivalent:

(1) I is $\phi - n$-ideal.

(2) $I - \phi(I) = \sqrt{0}$.

Proof: (1) ⇒ (2) From *Theorem 2.10*, we have $I - \phi(I) \subseteq \sqrt{0}$ as I is assumed to be a $\phi - n$-ideal of R. The inverse inclusion is clear as $I - \phi(I)$ is prime, so we have the equality.

(2) ⇒ (1) Let $r, s \in R$ with $rs \in I - \phi(I) = \sqrt{0}$ and r is non-nilpotent. Thus we conclude $b \in I - \phi(I) \subseteq I$, as needed.

The next two theorems give the conditions for a $\phi - n$-ideal to be an n-ideal of R.

Theorem 2.16. If $I\sqrt{0} \nsubseteq \phi(I)$ for a $\phi-n$-ideal of I of R, then I is an n-ideal of R.

Proof: Let $r, s \in R$ and $rs \in I$. If $rs \notin \phi(I)$, then we are done. So suppose that $rs \in \phi(I)$. Here there are three cases:

Case I. Let $rI \nsubseteq \phi(I)$. Then there exists $x \in I$ such that $rx \notin \phi(I)$. Since $r(x+s) = rx + rs \in I - \phi(I)$, we conclude that either r is nilpotent or $x + s \in I$, i.e. $r \in \sqrt{0}$ or $s \in I$.

Case II. Let $s\sqrt{0} \nsubseteq \phi(I)$. Then there is a nilpotent element $y \in R$ satisfying $sy \notin \phi(I)$. Since $(y+r)s = ys + rs \in I - \phi(I)$, we conclude that either $y + r \in \sqrt{0}$ or $s \in I$, that is, $r \in \sqrt{0}$ or $s \in I$.

Case III. Let $rI \subseteq \phi(I)$ and $s\sqrt{0} \subseteq \phi(I)$. Since $I\sqrt{0} \nsubseteq \phi(I)$, there exists $i \in I$ and $z \in \sqrt{0}$ such that $iz \notin \phi(I)$. Hence $(r+z)(s+i) = rs + ri + zs + zi \in I - \phi(I)$ which implies that $(r+z) \in \sqrt{0}$ or $(s+i) \in I$. Therefore, $r \in \sqrt{0}$ or $s \in I$. Thus I is an n-ideal of R.

Corollary 2.17. Let I be a $\phi-n$-ideal which is not an n-ideal of R. Then $I\sqrt{0} \subseteq \phi(I)$.

Theorem 2.18. Let I be a $\phi-n$-ideal of R. If $\phi(I)$ is n-ideal, then I is an n-ideal of R.

Proof: Suppose that $rs \in I$ for some $r, s \in R$ and r is non-nilpotent. If $rs \in \phi(I)$, then $s \in \phi(I) \subseteq I$ as $\phi(I)$ is n-ideal. If $rs \notin \phi(I)$, we conclude $s \in I$ as I is $\phi-n$-ideal.

Proposition 2.19. If I is a $\phi-n$-ideal of R with $\sqrt{\phi(I)} = \phi(\sqrt{I})$, then so is $\sqrt{I}$.

Proof: Assume that $rs \in \sqrt{I} - \phi(\sqrt{I})$ for some $r, s \in R$ and r is non-nilpotent. Then $(rs)^n = r^n s^n \in I$ for some positive integer n. Since $rs \notin \phi(\sqrt{I}) = \sqrt{\phi(I)}$, clearly it follows $r^n s^n \notin \phi(I)$. Now r^n is non-nilpotent and I is ϕ–n-ideal gives $s^n \in I$. Thus $s \in \sqrt{I}$, as required.

Let J be an ideal of R. Define $\phi_J : S\left(\frac{R}{J}\right) \to S\left(\frac{R}{J}\right) \cup \{\emptyset\}$ *by* $\phi_J(I/J) = (\phi(I) + J)/J$ for every ideal $J \supseteq I$ and $\phi_J(I/J) = \emptyset$ if $\phi(I) = \emptyset$. Observe that $\phi_J(I/J) \subseteq I/J$ and $(\phi_\propto)_J = \phi_\propto$ for $\propto \in \{\emptyset\} \cup \{0\} \cup \mathbb{N}$. As it is stated in Lemma 1.2 that if $J \in Id(R)$ and I is an n-ideal containing J, then I/J is also an n-ideal of R/J. The next theorem shows that if I is a $\phi-n$-ideal of R, then I/J is a $\phi_J - n$-ideal of R/J.

Theorem 2.20. Let I be $\phi-n$-ideal of R, and $I \subseteq J \in Id(R)$. Then I/J is a $\phi_J - n$-ideal of R/J.

Proof: Suppose that $(r+J)(s+J) \in I/J - \phi_J(I/J) = I/J - (\phi(I) + J)/J$ and $(r+J)$ is non-nilpotent element of R/J. Then $rs \in I - (\phi(I) + J)$ and clearly r is a non-nilpotent element of R. Hence $rs \in I - \phi(I)$. Since I is $\phi-n$-ideal, it follows $r \in I$. Thus $r + J \in I/J$, we are done.

Let T be a multiplicatively closed subset of a ring R. Then $T^{-1}R = \{\frac{r}{t} : r \in R, t \in T\}$. This localization is often written as R_T; we call it the localization of R at the multiplicatively closed set T. Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function. Then define $\phi_T : Id(R_T) \to Id(R_T) \cup \{\emptyset\}$ by $\phi_T(J) = \phi(J \cap R)_T$ for every ideal J of R_T and $\phi_T(J) = \emptyset$ if $\phi(J \cap R) = \emptyset$. Note that $\phi_T(J) \subseteq J$. The following theorem shows that If I is a $\phi-n$-ideal of R with $(\phi(I))_T \subseteq \phi_T(I_T)$, then I_T is a $\phi_T - n$-ideal of R_T.

Theorem 2.21. Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function, T a multiplicatively closed subset of R and $I \cap T = \emptyset$ for some $I \in Id(R)$. If I is a $\phi - n$-ideal of R and $(\phi(I))_T \subseteq \phi_T(I_T)$, then the following conditions are satisfied:

(1) I_T is a $\phi_T - n$-ideal of R_T.

(2) If $I_T \neq \phi(I)_T$, then $I_T \cap R \subseteq \sqrt{0}$.

Proof: (1) Suppose that $\frac{r}{h}\frac{s}{t} \in I_T - \phi_T(I_T)$ and $\frac{r}{h}$ is a non-nilpotent element of R_T. Then $urs \in I$ for some $u \in T$ and clearly r is a non-nilpotent element of R. Since $\frac{rs}{ht} \notin \phi_T(I_T)$, $rsv \notin \phi_T(I_T) \cap R$ for all $v \in T$. From our assumption $\phi(I)_T \subseteq \phi_T(I_T)$, we conclude that $rsv \notin \phi(I)$ for all $v \in T$. Thus $r(sv) \in I - \phi(I)$. Since I is assumed to be a $\phi - n$-ideal and r is non-nilpotent, it implies that $su \in I$, and so $\frac{s}{t} \in I_T$. Therefore I_T is a $\phi_T - n$-ideal of R_T.

(2) Let $r \in I_T \cap R$. Then there is an element $u \in T$ such that $au \in I$. Now, if $ru \notin \phi(I)$, then r is nilpotent as $u \notin I$. If $ru \in \phi(I)$, then $r \in \phi(I)_T \cap R$. Thus $I_T \cap R \subseteq \sqrt{0} \cup (\phi(I)_T \cap R)$. From hypothesis $I_T \neq \phi(I)_T$, we conclude that $I_T \cap R \subseteq \sqrt{0}$.

Let $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function. It is said that ϕ preserves the order if $I \subseteq J$ for ideals I, J of R implies that $\phi(I) \subseteq \phi(J)$.

Proposition 2.22. Let $\{I_i\}_{i\in\Lambda}$ be a directed collection of $\phi - n$-ideals of R where ϕ preserves the order. Then $I = \cup_{i\in\Lambda} I_i$ is a $\phi - n$-ideal of R.

Proof: Suppose that $rs \in I - \phi(I)$ for some $r, s \in R$ where r is non-nilpotent. Then $rs \in I_k$ for some $k \in \Lambda$ but $rs \notin \phi(I_i)$ for all $i \in \Lambda$ as ϕ preserves the order. Since I_k is assumed to be a $\phi - n$-ideal, we get $s \in I_k \subseteq I$, as needed.

Proposition 2.23. Let A be a nonempty subset of R *and* $\phi : Id(R) \to Id(R) \cup \{\emptyset\}$ be a function which preserves order. If I is a $\phi - n$-ideal of R with $A \nsubseteq I$, then so is $(I : A)$.

Proof: Let $rs \in (I : A) - \phi(I : A)$ and r is a non-nilpotent element of R. Since ϕ preserves order, we have $\phi(I) \subseteq \phi(I : A)$. Hence $rsA \subseteq I - \phi(I)$. It follows $sA \subseteq I$ by *Theorem 2.7*. Thus $s \in (I : A)$, we are done.

Theorem 2.24. Let M an R-module. Let $\psi_1 : Id(R) \to Id(R) \cup \{\emptyset\}$ and $\psi_2 : Id(R(+)M) \to Id(R(+)M) \cup \{\emptyset\}$ be two functions satisfying $\psi_2(I(+)M) = \psi_1(I)(+)N$ for a proper ideal I of R. If $I(+)M$ is a $\psi_2 - n$-ideal of $R(+)M$, then I is a $\psi_1 - n$-ideal of R.

Proof: Let $r, s \in R$ with $rs \in I - \psi_1(I)$ and r be a non-nilpotent. Then $(r, 0)(s, 0) \in I(+)M - \psi_2(I(+)M)$ as $\psi_2(I(+)M) = \psi_1(I)(+)N$. It is not hard to see that $(r, 0)$ is non-nilpotent element of $R(+)M$. Therefore, it implies that $(s, 0) \in I(+)M$; and so $s \in I$. Thus I is a $\psi_1 - n$-ideal of R.

Remark 2.25. Let R_1 and R_2 be two commutative rings with nonzero identity and $R = R_1 \times R_2$. Let $\psi_1 : Id(R_1) \to Id(R_1) \cup \{\emptyset\}$, $\psi_2 : Id(R_2) \to Id(R_2) \cup \{\emptyset\}$ be two functions and $\phi = \psi_1 \times \psi_2$. Then R has no $\phi - n$-ideal. Indeed, if I is ϕ –n-ideal, then $I = I_1 \times I_2$ for some ideals I_1, I_2 of R_1, R_2 respectively. On the other hand, since $(1,0) \cdot (0,1) \in I$ but neither $(1,0)$ nor $(0,1)$ is a nilpotent element of R. It implies that $(0,1), (1,0) \in I$. Thus we conclude $1 \in I_2$

and $1 \in I_1$, so $I = R_1 \times R_2 = R$, a contradiction.

3. References

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