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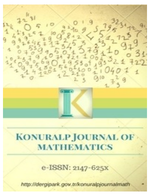

Perrin Octonions and Perrin Sedenions

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Abstract

In this study, we introduce new classes of octonion and sedenion numbers associated with Perrin numbers. We define Perrin octonions and Perrin sedenions by using the Perrin numbers. We give some relationship between Perrin octonions, Perrin sedenions and Perrin numbers. Moreover we obtain the generating functions, Binet formulas and sums formulas of them.

Keywords: *Perrin numbers, Perrin octonions, Perrin sedenions.*
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1. Introduction

Octonion algebra is 8-dimensional, non-commutative, non-associative and normed division algebra over the real numbers. Sedenions are obtained by applying the Cayley-Dickson construction to the octonions and form a 16-dimensional non-associative and non-commutative algebra over the set of real numbers.

Many different classes of octonion and sedenion number sequences such as Fibonacci octonion and sedenion, Lucas octonion and sedenion, Pell octonion and sedenion have been obtained by a number of authors in many different ways. In addition, generating functions, Binet formulas and some identities for these octonions and sedenions have been presented ([1, 2, 3, 4, 7, 10]).

Let O be the octonion algebra over the real number field $\mathbb{R}$. It is known, by the Cayley-Dickson process that any $p \in O$ can be written as

$$p = p' + p''e$$

where $p', p'' \in H = \{a_0 + a_1 i + a_2 j + a_3 k : i^2 = j^2 = k^2 = -1, ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$, the real quaternion division algebra. The addition and multiplication of any two octonions, $p = p' + p''e, q = q' + q''e$ are defined by

$$p + q = (p' + q') + (p'' + q'')e$$

and

$$pq = (p'q' - \overline{q''}p'') + (q''p' + p''\overline{q'})e$$

where $\overline{q'}$, $\overline{q''}$ denote the conjugates of the quaternions q', q'' respectively. Thus, O is an eight-dimensional non-associative division algebra over the real numbers $\mathbb{R}$. A natural basis of this algebra as a space over $\mathbb{R}$ is formed by the elements

$$e_0 = 1, e_1 = i, e_2 = j, e_3 = k, e_4 = e, e_5 = ie, e_6 = je, e_7 = ke.$$

The multiplication table for the basis of O is

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$\cdot$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Table 1

Under this notation, all octonions take the form

$$p = \sum_{s=0}^{7} p_s e_s$$

where the coefficients p_s are real. Also, every $p \in O$ can be simply written as $p = Re(p) + Im(p)$, where $Re(p) = p_0$ and $Im(p) = \sum_{s=1}^{7} p_s e_s$ are called the real and imaginary parts, respectively. The conjugate of p is defined to be

$$\overline{p} = \overline{p'} - p''e = Re(p) - Im(p).$$

This operation satisfies

$$\overline{\overline{p}} = p, \quad \overline{(p+q)} = \overline{p} + \overline{q}, \quad \overline{pq} = \overline{q}\ \overline{p}$$

for all $p, q \in O$. The norm of p is defined to be

$$N_p = p\overline{p} = \overline{p}p = \sum_{s=0}^{7} p_s^2$$

The inverse of non-zero octonion $p \in O$ is

$$p^{-1} = \frac{p}{N_p}.$$

For all $p, q \in O$

$$N_{pq} = N_p N_q,$$

$$(pq)^{-1} = q^{-1} p^{-1}$$

O is non-commutative, non-associative but it is alternative

$$p(pq) = p^2 q, \quad (qp)p = qp^2, \quad (pq)p = p(qp) := pqp,$$

([11],[8]) On the other hand, a sedenion S can be written as

$$S = \sum_{i=0}^{15} a_i e_i$$

where $a_0, a_1, ..., a_{15}$ are real numbers. Imaeda and Imaeda ([6]) defined a sedenion by

$$S = (O_1; O_2) \in S, \quad O_1, O_2 \in O$$

where O is the octonion algebra over the reals. As a sedenion is an ordered pair of two octonions, the conjugate of a sedenion $S = (O_1; O_2)$ is defined by $\overline{S} = (O_1; -O_2)$. Under the Cayley-Dickson process, the product of two sedenions $S_1 = (O_1; O_2)$ and $S_2 = (O_3; O_4)$ is

$$S_1 S_2 = (O_1 O_3 + \rho \overline{O_4} O_2; O_2 \overline{O_3} + O_4 O_1).$$

After choosing the field parameter $\rho = -1$ and the generator e_8, Imaeda and Imaeda examined the sedenions. By setting $i \equiv e_i$, where $i = 0, 1, ..., 15$, Cawagas ([5]) constructed the following multiplication table for the basis of S.
Multiplication table for the basis of S is

.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	-2	5	-4	-7	6	9	-8	-11	10	-13	12	15	-14
2	2	-3	0	1	6	7	-4	-5	10	11	-8	-9	-14	-15	12	13
3	3	2	-1	0	7	-6	5	-4	11	-10	9	-8	-15	14	-13	12
4	4	-5	-6	-7	0	1	2	3	12	13	14	15	-8	-9	-10	-11
5	5	4	-7	6	-1	0	-3	2	13	-12	15	-14	9	-8	11	-10
6	6	7	4	-5	-2	3	0	-1	14	-15	-12	13	10	-11	-8	9
7	7	-6	5	4	-3	-2	1	0	15	14	-13	-12	11	10	-9	-8
8	8	-9	-10	-11	-12	-13	-14	-15	0	1	2	3	4	5	6	7
9	9	8	-11	-10	-13	12	15	-14	-1	0	-3	2	-5	4	7	-6
10	10	11	8	-9	-14	-15	12	13	-2	3	0	-1	-6	-7	4	5
11	11	-10	9	8	-15	14	-13	12	-3	-2	1	0	-7	6	-5	4
12	12	13	14	15	8	-9	-10	-11	-4	5	6	7	0	-1	-2	-3
13	13	-12	15	-14	9	8	11	-10	-5	-4	7	-6	1	0	3	-2
14	14	-15	-12	13	10	-11	8	9	-6	-7	-4	5	2	-3	0	1
15	15	14	-13	-12	11	10	-9	8	-7	6	-5	-4	3	2	-1	0

Table 2

The Perrin sequence is the sequence of integers P_n defined by the initial values $P_0 = 3$, $P_1 = 0$, $P_2 = 2$ and the recurrence relation

$$P_n = P_{n-2} + P_{n-3} \tag{1.1}$$

for all $n \geq 3$. The first few values of P_n are

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, ...$$

The characteristic equation associated with Perrin sequence is $x^3 - x - 1 = 0$ with r_1, r_2, r_3, in which $r_1 = \alpha \simeq 1,324718$ is called plastic number and

$$\lim_{n \to \infty} \frac{P_{n+1}}{P_n} = \alpha.$$

The Binet's formula of Perrin sequence is

$$P_n = r_1^n + r_2^n + r_3^n$$

where r_1, r_2 and r_3 are the roots of the equation $x^3 - x - 1 = 0$. ([9])

In this paper, we introduce new classes of octonion and sedenion numbers associated with the Perrin numbers. We define Perrin octonions and sedenions numbers by using recurrence relation $P_n = P_{n-2} + P_{n-3}$ of the Perrin sequence is defined by the initial values $P_0 = 3$, $P_1 = 0$, $P_2 = 2$ for all $n \geq 3$. We give the Binet formulas given nth general term of these octonions and sedenions are found by using recurrence relation of the new defined Perrin octonions and Perrin sedenions. Also, we obtain the generating functions, sums formulas and some basic identities for these octonions and sedenions.

2. Main Results

Firstly we give the Perrin octonions.

2.1. Perrin Octonions

Definition 2.1. *For $n \geq 0$, the nth Perrin octonion is defined by*

$$OP_n = \sum_{i=0}^{7} P_{n+i} e_i$$

where P_n is the nth Perrin number and $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is the standard octonion basis.

We get the following theorem for the Perrin octonions from equation (1.1) and Definition 2.1

Theorem 2.2. *Let OP_n be the nth Perrin octonion. Then we give the following recurrence relation:*

$$OP_n = OP_{n-2} + OP_{n-3}$$

with inital conditions $OP_0 = \sum_{i=0}^{7} P_i e_i, OP_1 = \sum_{i=0}^{7} P_{1+i} e_i, OP_2 = \sum_{i=0}^{7} P_{2+i} e_i$.

Proof. By the equation (1.1) and Definiton 2.1 we have

$$\begin{aligned} OP_{n-2}+OP_{n-3} &= \sum_{i=0}^{7} P_{n-2+i}e_i+\sum_{i=0}^{7} P_{n-3+i}e_i \\ &= \sum_{i=0}^{7}(P_{n-2+i}+P_{n-3+i})e_i \\ &= \sum_{i=0}^{7} P_{n+i}e_i \\ &= OP_n. \end{aligned}$$

So proof is completed. □

The following theorem is related with the generating function of the Perrin octonions.

Theorem 2.3. *The generating function for the Perrin octonions OP_n is*

$$f(x)=\frac{OP_0+OP_1x+(OP_2-OP_0)x^2}{1-x^2-x^3}.$$

Proof. Let

$$f(x)=\sum_{n=0}^{\infty} OP_nx^n=OP_0+OP_1x+OP_2x^2+\cdots+OP_nx^n+\cdots$$

be generating function of the Perrin octonions. On the other hand, since the orders of OP_{n-2} and OP_{n-3} are 2 and 3 less than the order of OP_n we can obtain $x^2f(x)$ and $x^3f(x)$:

$$x^2f(x)=OP_0x^2+OP_1x^3+OP_2x^4+OP_3t^5+\cdots+OP_{n-2}x^n+\cdots$$

$$x^3f(x)=OP_0x^3+OP_1x^4+OP_2x^5+OP_3x^6+\cdots+OP_{n-3}x^n+\cdots$$

Then we write

$$(1-x^2-x^3)f(x)=OP_0+OP_1x+(OP_2-OP_0)x^2+(OP_3-OP_1-OP_0)x^3+\ldots+(OP_n-OP_{n-2}-OP_{n-3})x^n+\ldots$$

Note that the sequence $\{OP_n\}$ of Perrin octonions satisfies following second-order recurrence relation

$$OP_n=OP_{n-2}+OP_{n-3}$$

with inital conditions $OP_0=\sum_{i=0}^{7}P_ie_i, OP_1=\sum_{i=0}^{7}P_{1+i}e_i, OP_2=\sum_{i=0}^{7}P_{2+i}e_i$. Then we obtain

$$f(x)=\frac{OP_0+OP_1x+(OP_2-OP_0)x^2}{1-x^2-x^3}.$$

So proof is completed. □

The Binet's formula known as the general formula allows us to easily find any Perrin octonions without having to know all the terms before it. That is, Binet formula give us to find the nth Perrin octonion without using Definition 2.1. Now, we produce the Binet formula for the Perrin octonions.

Theorem 2.4. *For $n\geq 0$, the Binet's formula for the Perrin octonions is as follows*

$$OP_n=\alpha r_1^n+\beta r_2^n+\gamma r_3^n$$

where r_1, r_2 and r_3 are the roots of the equation $x^3-x-1=0$ and

$$\alpha=\sum_{i=0}^{7} r_1^ie_i, \beta=\sum_{i=0}^{7} r_2^ie_i, \gamma=\sum_{i=0}^{7} r_3^ie_i.$$

Proof. Using Binet's formula of Perrin sequence, we have

$$P_n=r_1^n+r_2^n+r_3^n$$

where r_1, r_2 and r_3 are the roots of the equation $x^3-x-1=0$. On the other hand, from Definition 2.1 we obtain

$$\begin{aligned} OP_n &= \sum_{i=0}^{7} P_{n+i}e_i \\ &= P_n+P_{n+1}e_1+P_{n+2}e_2+P_{n+3}e_3+P_{n+4}e_4+P_{n+5}e_5+P_{n+6}e_6+P_{n+7}e_7. \end{aligned}$$

Then we get

$$\begin{aligned} OP_n &= \sum_{i=0}^{7} P_{n+i} e_i \\ &= \sum_{i=0}^{7} [r_1^{n+i} + r_2^{n+i} + r_3^{n+i}] e_i \\ &= \alpha r_1^n + \beta r_2^n + \gamma r_3^n \end{aligned}$$

where $\alpha = \sum_{i=0}^{7} r_1^i e_i, \beta = \sum_{i=0}^{7} r_2^i e_i, \gamma = \sum_{i=0}^{7} r_3^i e_i$. So, we obtain the desired result. □

Theorem 2.5. *Let OP_n be the nth Perrin octonion. Then we get the following sums formulas*

i. $\sum_{m=0}^{n} OP_m = OP_{n+3} + OP_{n+2} - OP_4$
ii. $\sum_{m=0}^{n} OP_{2m} = OP_{2n+3} - OP_1$
iii. $\sum_{m=0}^{n} OP_{2m+1} = OP_{2n+4} - OP_2$.

Proof. i. From Theorem 2.2, we can get the following relations:

$$\begin{aligned} OP_0 &= OP_3 - OP_1 \\ OP_1 &= OP_4 - OP_2 \\ OP_2 &= OP_5 - OP_3 \\ &\vdots \\ OP_{n-2} &= OP_{n+1} - OP_{n-1} \\ OP_{n-1} &= OP_{n+2} - OP_n \\ OP_n &= OP_{n+3} - OP_{n+1} \end{aligned}$$

we write

$$\begin{aligned} OP_0 + OP_1 + OP_2 + \ldots + OP_n &= OP_{n+3} + OP_{n+2} - OP_2 - OP_1 \\ \sum_{m=0}^{n} OP_m &= OP_{n+3} + OP_{n+2} - OP_4. \end{aligned}$$

ii. From Theorem 2.2, we can get the following relations:

$$\begin{aligned} OP_0 &= OP_3 - OP_1 \\ OP_2 &= OP_5 - OP_3 \\ OP_4 &= OP_7 - OP_5 \\ &\vdots \\ OP_{2n-4} &= OP_{2n-1} - OP_{2n-3} \\ OP_{2n-2} &= OP_{2n+1} - OP_{2n-1} \\ OP_{2n} &= OP_{2n+3} - OP_{2n+1} \end{aligned}$$

we write

$$\begin{aligned} OP_0 + OP_2 + \ldots + OP_{2n} &= OP_{2n+3} - OP_1 \\ \sum_{m=0}^{n} OP_{2m} &= OP_{2n+3} - OP_1. \end{aligned}$$

iii. From Theorem 2.2, we can get the following relations:

$$\begin{aligned} OP_1 &= OP_4 - OP_2 \\ OP_3 &= OP_6 - OP_4 \\ OP_5 &= OP_8 - OP_6 \\ &\vdots \\ OP_{2n-3} &= OP_{2n} - OP_{2n-2} \\ OP_{2n-1} &= OP_{2n+2} - OP_{2n} \\ OP_{2n+1} &= OP_{2n+4} - OP_{2n+2} \end{aligned}$$

we write

$$\begin{aligned} OP_1 + OP_3 + \ldots + OP_{2n+1} &= OP_{2n+4} - OP_2 \\ \sum_{m=0}^{n} OP_{2m+1} &= OP_{2n+4} - OP_2. \end{aligned}$$

So the proof is completed. □

2.2. Perrin Sedenions

Definition 2.6. *For $n \geq 0$, the nth Perrin sedenion is defined by*

$$SP_n = \sum_{i=0}^{15} P_{n+i} e_i$$

where P_n is the nth Perrin number and $(e_0, e_1, e_2, ..., e_{15})$ is the standard sedenion basis.
We get the following theorem for the Perrin sedenions from equation (1.1) and Definition 2.6.

Theorem 2.7. *Let SP_n be the nth Perrin sedenion. Then we give the following recurrence relation:*

$$SP_n = SP_{n-2} + OP_{n-3}.$$

with initial conditions $SP_0 = \sum_{i=0}^{15} P_i e_i$, $SP_1 = \sum_{i=0}^{15} P_{i+1} e_i$, $SP_2 = \sum_{i=0}^{15} P_{i+2} e_i$.

Proof. By the equation (1.1) and Definiton 2.6, we have

$$\begin{aligned} SP_{n-2} + SP_{n-3} &= \sum_{i=0}^{15} P_{n-2+i} e_i + \sum_{i=0}^{15} P_{n-3+i} e_i \\ &= \sum_{i=0}^{15} (P_{n-2+i} + P_{n-3+i}) e_i \\ &= \sum_{i=0}^{15} P_{n+i} e_i \\ &= SP_n. \end{aligned}$$

So proof is completed. □

Generating function for the Perrin sedenions is given in the next theorem.

Theorem 2.8. *The generating function for the Perrin sedenions SP_n is*

$$g(x) = \frac{SP_0 + SP_1 x + (SP_2 - SP_0) x^2}{1 - x^2 - x^3}.$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} SP_n x^n = SP_0 + SP_1 x + SP_2 x^2 + \cdots + SP_n x^n + \cdots$$

be generating function of the Perrin sedenions. On the other hand, since the orders of SP_{n-2} and SP_{n-3} are 2 and 3 less than the order of SP_n we can obtain $x^2 g(x)$ and $x^3 g(x)$

$$x^2 g(x) = SP_0 x^2 + SP_1 x^3 + SP_2 x^4 + \cdots + SP_{n-2} x^n + \cdots$$

$$x^3 g(x) = SP_0 x^3 + SP_1 x^4 + SP_2 x^5 + \cdots + SP_{n-3} x^n + \cdots$$

Then we write

$$(1 - x^2 - x^3) g(x) = SP_0 + SP_1 x + (SP_2 - SP_0) x^2 + (SP_3 - SP_1 - SP_0) x^3 + ... + (SP_n - SP_{n-2} - SP_{n-3}) x^n + ...$$

Note that the sequence $\{SP_n\}$ of the Perrin sedenions satisfies following second-order recurrence relation

$$SP_n = SP_{n-2} + SP_{n-3}$$

with inital conditions $SP_0 = \sum_{i=0}^{15} P_i e_i, SP_1 = \sum_{i=0}^{15} P_{1+i} e_i, SP_2 = \sum_{i=0}^{15} P_{2+i} e_i$. Then we obtain

$$g(x) = \frac{SP_0 + SP_1 x + (SP_2 - SP_0) x^2}{(1 - x^2 - x^3)}.$$

So proof is completed. □

The next theorem gives the Binet's formula for the Perrin sedenions.

Theorem 2.9. *For $n \geq 0$, the Binet's formula for the Perrin sedenions is as follows*

$$SP_n = \alpha r_1^n + \beta r_2^n + \gamma r_3^n$$

where r_1, r_2 and r_3 are the roots of the equation $x^3 - x - 1 = 0$ and

$$\alpha = \sum_{i=0}^{15} r_1^i e_i, \beta = \sum_{i=0}^{15} r_2^i e_i, \gamma = \sum_{i=0}^{15} r_3^i e_i.$$

Proof. Using Binet's formula of Perrin sequence, we have

$$P_n = r_1^n + r_2^n + r_3^n$$

where r_1, r_2 and r_3 are the roots of the equation $x^3 - x - 1 = 0$. On the other hand, from Definition 2.6 we obtain

$$SP_n = \sum_{i=0}^{15} P_{n+i} e_i = P_n + P_{n+1} e_1 + \ldots + P_{n+15} e_{15}.$$

Then we get

$$\begin{aligned} SP_n &= \sum_{i=0}^{15} P_{n+i} e_i \\ &= \sum_{i=0}^{15} [r_1^{n+i} + r_2^{n+i} + r_3^{n+i}] e_i \\ &= \alpha r_1^n + \beta r_2^n + \gamma r_3^n \end{aligned}$$

where $\alpha = \sum_{i=0}^{15} r_1^i e_i, \beta = \sum_{i=0}^{15} r_2^i e_i, \gamma = \sum_{i=0}^{15} r_3^i e_i$. So we obtain the desired result. □

Theorem 2.10. *Let SP_n be the nth Perrin sedenion. Then we get the following sums formulas:*

i. $\sum_{m=0}^{n} SP_m = SP_{n+3} + SP_{n+2} - SP_4$
ii. $\sum_{m=0}^{n} SP_{2m} = SP_{2n+3} - SP_1$
iii. $\sum_{m=0}^{n} SP_{2m+1} = SP_{2n+4} - SP_2$.

Proof. The proof is seen by using Definition 2.6 and Theorem 2.7. □

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