




k -Congruences and the Zariski topology in semirings

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Abstract

The purpose of this paper is to study topological properties of both the set of all k -prime ideals and the set of all k -prime congruences for any commutative semiring with zero and identity. We first prove that the k -prime spectrum, i.e. the set of all k -prime ideals equipped with the Zariski topology is a spectral space, and then prove that the set of all k -prime congruences is homeomorphic to the k -prime spectrum with respect to their Zariski topologies.

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1. Introduction

The notion of semirings was introduced by Vandiver [17] in 1934. Semirings are algebraic systems that generalize both rings and distributive lattices and have many applications in diverse branches of mathematics and computer science [5, 7, 13]. Semirings have two binary operations of addition and multiplication which are connected by the ring-like distributive laws. But unlike in rings, subtraction is not allowed in semirings that are not rings. For this reason, there are considerable differences between ring theory and semiring theory and many results in ring theory have no analogues in semirings. In order to narrow the gap, Henriksen [8] in 1958 defined k -ideals in semirings. From the fact that k -ideals coincide with ring ideals in any ring, k -ideals in semirings have been a very interesting subject of research [6, 12, 15, 18, 19].

The objective of the present paper is to study topological properties of both the set of all k -prime ideals with the Zariski topology and the set of all k -prime congruences with the Zariski topology for any commutative semiring with zero and identity. For this purpose, we first investigate topological properties of the k -prime spectrum, i.e. the set of all k -prime ideals equipped with the Zariski topology for any commutative semiring with zero and identity, and then prove that the k -prime spectrum and the set of all k -prime congruences are homeomorphic with respect to their Zariski topologies.

It is well known that the prime spectrum of a commutative ring with identity plays an important role in commutative algebra, algebraic geometry and lattice theory [2, 3].

For any commutative semiring with zero and identity, Golan [5] proved that the prime spectrum, i.e. the set of all prime ideals with the Zariski topology is a compact T_0 -space, and Pena, Ruza and Vielma [14] proved that the prime spectrum is a spectral space.

It is shown below that the k -prime spectrum is a subspace of the prime spectrum for any commutative semiring with zero and identity. However, it has not been known even whether the k -prime spectrum is compact for any commutative semiring with zero and identity. In previous researches, Atani and Atani [1] considered topological properties of the prime spectrum of a commutative k -semiring, i.e. a commutative semiring with zero and identity whose ideals are all k -ideals. But one can see that this is a special case where the k -prime spectrum coincides with the prime spectrum. And Lescot [11] proved that the set of all saturated prime ideals with the Zariski topology is a spectral space for a characteristic one semiring. Note that a characteristic one semiring is an additively idempotent commutative semiring with zero and identity and saturated ideals are just k -ideals. Summing up, topological properties of the k -prime spectra have not been studied much for general commutative semirings with zero and identity.

The rest of this paper is as follows. Section 2 is for preliminaries. In Section 3, for any semiring, we define a k -congruence and show that k -congruences are in a one-to-one correspondence with k -ideals. In Section 4, for any commutative semiring with zero and identity, we study topological properties of the k -prime spectrum in detail and thus prove that the k -prime spectrum is a spectral space. In Section 5, for any commutative semiring with zero and identity, we introduce the Zariski topology on the set of all k -prime congruences and prove that the resulting topological space is homeomorphic to the k -prime spectrum.

2. Preliminaries

In this section, we recall some known definitions and facts [4–7].

Throughout this paper, R denotes a *semiring* $(R, +, \cdot)$, unless otherwise stated. That is to say, $(R, +)$ is a commutative semigroup, (R, \cdot) is a semigroup, and multiplication distributes over addition from either side. If R has an additively neutral element 0 and $0r = r0 = 0$ for all $r \in R$, then 0 is called a *zero* of R . We write $0 \in R$ when R has a zero 0 . If R has a multiplicatively neutral element 1 , then 1 is called an *identity* of R . R is called a *commutative* semiring if multiplication is commutative. Below, we assume that $|R| \geq 2$ to avoid trivial exceptions.

A nonempty subset A of R is called an *ideal* of R if $a + b \in A$ and $ra, ar \in A$ for all $a, b \in A$ and $r \in R$. An ideal A of R is said to be *proper* if $A \neq R$. R and $\{0\}$ (if $0 \in R$) are said to be *trivial* ideals of R . Denote by $\mathcal{J}(R)$ the family of all ideals of R . For an ideal A of R , the set

$$\overline{A} = \{x \in R \mid x + a = b \text{ for some } a, b \in A\}$$

is called the *subtractive closure* or *k -closure* of A in R . Then \overline{A} is an ideal of R and it holds that $A \subseteq \overline{A}$ and $\overline{(\overline{A})} = \overline{A}$. A is said to be *k -closed* in R if $A = \overline{A}$. For ideals A and B of R , $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$. An ideal A of R is called a *subtractive ideal* or *k -ideal* of R if $A = \overline{A}$. If A is an ideal of R , then \overline{A} is a k -ideal of R . R is a k -ideal of itself, and $\{0\}$ is also a k -ideal of R if $0 \in R$. Denote by $\mathcal{KJ}(R)$ the family of all k -ideals of R .

An equivalence relation \equiv on R is called a *congruence* on R if for any $a, b, c \in R$, $a \equiv b$ implies $a + c \equiv b + c$, $ac \equiv bc$ and $ca \equiv cb$. The family of all congruences on R is denoted by $\mathcal{C}(R)$. Given a congruence θ on R , the quotient set $R/\theta = \{[x] \mid x \in R\}$ consisting of all the congruence classes forms a semiring under the operations defined as $[x] + [y] = [x + y]$ and $[x] \cdot [y] = [xy]$ for $x, y \in R$. The semiring R/θ is called the *quotient semiring* of R by θ . If $0 \in R$, then $[0]$ is a zero of R/θ .

An ideal A of R defines a congruence κ_A on R by

$$x \kappa_A y \iff x + a = y + b \text{ for some } a, b \in A.$$

Then the *k*-closure \overline{A} of A is a zero of the quotient semiring R/κ_A . In addition, $\kappa_A = \kappa_{\overline{A}}$.

A *k*-semiring is a semiring whose ideals are all *k*-ideals [1].

A proper *k*-ideal A of a semiring R is said to be *k*-maximal in R if for each *k*-ideal B of R , $A \subset B \subseteq R$ implies $B = R$ [15, 18]. Denote by $\text{Max}_k(R)$ the family of all *k*-maximal ideals of R .

Lemma 2.1 ([15, 18]). *If R is a semiring with zero and identity, then each proper *k*-ideal of R is contained in a *k*-maximal ideal of R .*

A proper ideal A of a commutative semiring R is said to be *prime* if $xy \in A$ with $x, y \in R$ implies $x \in A$ or $y \in A$ [15]. The family of all prime ideals of R is denoted by $\text{Spec}(R)$. A *k*-ideal A of R is said to be *k*-prime or *subtractive prime* if it is a prime ideal of R . The family of all *k*-prime ideals of R is denoted by $\text{Spec}_k(R)$.

Lemma 2.2 ([12, 15]). *If R is a commutative semiring with zero and identity, then $\emptyset \neq \text{Max}_k(R) \subseteq \text{Spec}_k(R)$.*

A non-zero element f in a commutative semiring R with identity 1 is called a *semiunit* in R if $1 + rf = sf$ for some $r, s \in R$ [1], which is equivalent to the condition that $Rf = R$.

For an ideal A of a semiring R , the set

$$\sqrt{A} = \{a \in R \mid a^n \in A \text{ for some } n \in \mathbb{N}\}$$

is called the *radical* of A . Especially, if R has zero 0, then $\text{Nil}(R) = \sqrt{\{0\}}$ is called the *nilradical* of R [5].

Let R be a commutative semiring with zero and identity. For each nonempty subset S of R , put $U(S) = \{P \in \text{Spec}(R) \mid S \subseteq P\}$. Then the collection $\{U(S) \mid \emptyset \neq S \subseteq R\}$ satisfies all the axioms of closed sets on a topology. The resulting topology is called the *Zariski topology* on $\text{Spec}(R)$ and the topological space $\text{Spec}(R)$ is called the *prime spectrum* of R [5].

3. *k*-Congruences on semirings

In this section, for any semiring R , we introduce the *k*-congruences, establish criteria for *k*-congruences, show that there is an inclusion-preserving bijection between the *k*-congruences and *k*-ideals, and present an equivalent condition for the existence of a zero with the help of *k*-congruences.

Lemma 3.1. *If θ is a congruence on R and R/θ has a zero O , then O is a *k*-ideal of R .*

Proof. Show that O is an ideal of R . If $a, b \in O$, then $[a + b] = [a] + [b] = O + O = O$ and so $a + b \in O$. If $a \in O$ and $r \in R$, then $[ar] = [a] \cdot [r] = O \cdot [r] = O$ and similarly $[ra] = O$, thus $ar, ra \in O$. Show that O is *k*-closed in R . If $x + b \in O$ and $b \in O$, then $[x] = [x] + O = [x] + [b] = [x + b] = O$ and so $x \in O$. \square

Definition 3.2. A congruence θ on R is called a *subtractive congruence* or *k*-congruence if there is an ideal A of R such that $\theta = \kappa_A$.

Denote by $\mathcal{KC}(R)$ the family of all *k*-congruences on R .

Theorem 3.3. *If θ is a congruence on R , then the following conditions are equivalent.*

- (1) θ is a *k*-congruence.
- (2) R/θ has a zero O and $\theta \subseteq \kappa_O$.
- (3) R/θ has a zero O and $\theta = \kappa_O$.

Proof. (1) \Rightarrow (2) Suppose that $\theta = \kappa_A$ for an ideal A of R . Then the *k*-closure \overline{A} is a zero of R/θ . If $x\theta y$, then $x + a = y + b$ for some $a, b \in A$ while $A \subseteq \overline{A}$ and so $x\kappa_{\overline{A}}y$.

(2) \Rightarrow (3) By Lemma 3.1, O is an ideal of R . Show that $\kappa_O \subseteq \theta$. If $x\kappa_O y$, then $x + a = y + b$ for some $a, b \in O$ and $[x] = [x] + O = [x] + [a] = [x + a] = [y + b] = [y] + [b] = [y] + O = [y]$, thus $x\theta y$. Hence $\theta = \kappa_O$.

(3) \Rightarrow (1) This follows from Lemma 3.1. □

Example 3.4. Let $R = \{0, a, b, c, d, 1\}$ be the lattice with the Hasse graph shown in Figure 3.1, which is not a distributive lattice. Define a multiplication \cdot on R by

$$x \cdot y = \begin{cases} d, & \text{if } x, y \in \{1, b, c, d\} \\ 0, & \text{otherwise.} \end{cases}$$

Then (R, \vee, \cdot) is a semiring and $A = \{0, a\}$ is an ideal of (R, \vee, \cdot) . Thus κ_A is a k -congruence on (R, \vee, \cdot) with congruence classes $\{0, a\}$ and $\{1, b, c, d\}$.

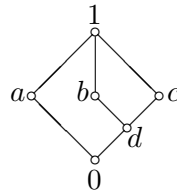


Figure 3.1

Example 3.5. Let $R = \{0, a, b, c, 1\}$ be the distributive lattice with the Hasse graph shown in Figure 3.2. Then the congruence θ on (R, \vee, \wedge) with congruence classes $\{0\}$ and $\{1, a, b, c\}$ is not a k -congruence. In fact, the zero of R/θ is $O = \{0\}$ and $a\theta b$ while $a \vee 0 \neq b \vee 0$, which means that $\theta \notin \kappa_O$.

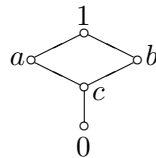


Figure 3.2

By Lemma 3.1 and Theorem 3.3, we can define a function $\iota : \mathcal{KC}(R) \rightarrow \mathcal{KJ}(R)$ by $\iota(\theta) = O_{R/\theta}$, where $O_{R/\theta}$ is a zero of R/θ . Also we define a function $\kappa : \mathcal{J}(R) \rightarrow \mathcal{C}(R)$ by $\kappa(A) = \kappa_A$.

Then we obtain the following conclusion from Theorem 3.3.

Lemma 3.6. *If θ is a k -congruence on R , then $\kappa(\iota(\theta)) = \theta$.*

Lemma 3.7. *An ideal A of R is a k -ideal of R iff $\iota(\kappa(A)) = A$.*

Proof. This follows from the fact that $\iota(\kappa(A)) = \iota(\kappa_A) = O_{R/\kappa_A} = \overline{A}$. □

Theorem 3.8. (1) *The restriction of κ to $\mathcal{KJ}(R)$ is an inclusion-preserving bijection from $\mathcal{KJ}(R)$ onto $\mathcal{KC}(R)$.*

(2) *If $\mathcal{KJ}(R) \subseteq \mathcal{F} \subseteq \mathcal{J}(R)$ and κ is injective on \mathcal{F} , then $\mathcal{F} = \mathcal{KJ}(R)$.*

Proof. (1) This follows from Lemmas 3.6 and 3.7.

(2) If $A \in \mathcal{F}$, then $\overline{A} \in \mathcal{KJ}(R) \subseteq \mathcal{F}$. Since $\kappa(A) = \kappa_A = \kappa_{\overline{A}} = \kappa(\overline{A})$ and κ is injective on \mathcal{F} , $A = \overline{A}$ and thus $A \in \mathcal{KJ}(R)$. □

Remark 3.9. Let R be an additively idempotent semiring, i.e. $r + r = r$ for all $r \in R$. If A is an ideal of R and $x, y \in R$, then the condition $x + a = y + b$ for some $a, b \in A$ implies that $x + c = y + c$ for some $c \in A$. In fact, putting $c = a + b \in A$ gives $x + c = x + a + b = x + a + a + b = y + b + a + b = y + a + b = y + c$.

Remark 3.10. A characteristic one semiring is a commutative semiring with identity 1 such that $1 + 1 = 1$ [10, 12]. Obviously, characteristic one semirings are additively idempotent.

Lescot [10] showed the existence of a bijection between the saturated ideals and excellent congruences in a characteristic one semiring. Actually, the saturated ideals are *k*-ideals and the excellent congruences are *k*-congruences. And Zhou and Yao [20] proved the existence of a one-to-one correspondence between the ideals which are lower sets and regular congruences in any additively idempotent semiring. By Han [6], the ideals which are lower sets are just *k*-ideals and every regular congruence is a *k*-congruence by Remark 3.9. Hence Theorem 3.8(1) is a generalization of Theorem 3.8 in [10] and Theorem 5 in [20].

Remark 3.11. $R \times R$ is always a *k*-congruence on R because $\kappa_R = R \times R$. But id_R is not necessarily a *k*-congruence on R .

Theorem 3.12. R has a zero 0 iff id_R is a *k*-congruence on R . In this case, $\kappa_{\{0\}} = \text{id}_R$.

Proof. (\Rightarrow) If R has a zero 0, then $\{0\}$ is an ideal of R and obviously $\kappa_{\{0\}} = \text{id}_R$.

(\Leftarrow) If id_R is a *k*-congruence on R , then each of the congruence classes is a singleton. By Theorem 3.3, R/id_R has a zero $[a] = \{a\}$, where $a \in R$. Show that a is a zero of R . For any $x \in R$, $[x + a] = [x] + [a] = [x]$ and so $x + a = x$. Also $[xa] = [x] \cdot [a] = [a]$ and similarly $[ax] = [a]$, thus $xa = a = ax$. \square

Theorem 3.12 enables us to give the following definition.

Definition 3.13. R is said to be *k*-congruence-simple if it has no *k*-congruences other than $R \times R$ and id_R .

R is said to be *k*-simple or simple if it has no nontrivial *k*-ideals [3, 16]. R is said to be congruence-simple if it has just two congruences [4]. R is said to be ideal-free if it has no nontrivial ideals [16]. Obviously, every congruence-simple semiring is *k*-congruence-simple and every ideal-free semiring is *k*-simple.

Theorem 3.14. R is *k*-simple iff it is *k*-congruence-simple.

Proof. By Remark 3.11, $\kappa(R) = \kappa_R = R \times R$. By Theorem 3.12, $\kappa(\{0\}) = \kappa_{\{0\}} = \text{id}_R$ if $0 \in R$. Hence nontrivial *k*-ideals of R bijectively correspond to *k*-congruences other than $R \times R$ and id_R under κ by Theorem 3.8(1). This completes the proof. \square

Corollary 3.15. If R is a ring, then the following statements are equivalent.

- (1) R is ideal-simple as a ring.
- (2) R is ideal-free as a semiring.
- (3) R is *k*-simple as a semiring.
- (4) R is congruence-simple as a ring.
- (5) R is congruence-simple as a semiring.
- (6) R is *k*-congruence-simple as a semiring.

Proof. Proposition 1.2 in [4] shows that (1), (2), (4) and (5) are equivalent. For any ring, a semiring ideal is a ring ideal iff it is a *k*-ideal. Thus (1) and (3) are equivalent. Hence Theorem 3.14 completes the proof. \square

4. *k*-Prime spectrum

In this section, for any commutative semiring R with zero and identity, we prove that the *k*-prime spectrum is a compact subspace of the prime spectrum, and give a necessary and sufficient condition for the *k*-prime spectrum to be a T_1 -space as well as necessary and sufficient conditions for the *k*-prime spectrum and its subsets to be irreducible. Finally, we prove that the *k*-prime spectrum is a spectral space.

Below, R denotes any commutative semiring with zero 0 and identity 1, and $X = \text{Spec}_k(R)$. Lemma 2.2 shows that $X \neq \emptyset$.

If $A \in \mathcal{J}(R)$, then $\sqrt{A} = \bigcap_{A \subseteq P \in \text{Spec}(R)} P$ (see [5]). The following lemma gives a *k*-version of the radical of a *k*-ideal.

Lemma 4.1. *Let $X = \text{Spec}_k(R)$. If $A \in \mathcal{KJ}(R)$, then $\sqrt{A} = \bigcap_{A \subseteq P \in X} P$.*

Proof. Show $\sqrt{A} \subseteq \bigcap_{A \subseteq P \in X} P$. If $a \in \sqrt{A}$, then $a^n \in A$ for some $n \in \mathbb{N}$. For every k -prime ideal P with $A \subseteq P$, $a^n \in P$ and so $a \in P$. Hence $a \in \bigcap_{A \subseteq P \in X} P$.

Show the reverse inclusion. If $a \notin \sqrt{A}$, then $a^n \notin A$ for all $n \in \mathbb{N}$. Let Σ be the family of all such $B \in \mathcal{KJ}(R)$ that $A \subseteq B$ and $a^n \notin B$ for all $n \in \mathbb{N}$. Then $A \in \Sigma$ and Σ is a poset under set-inclusion. For each chain $\{B_\alpha \mid \alpha \in \Lambda\}$ in Σ , $\bigcup_{\alpha \in \Lambda} B_\alpha$ is an upper bound of the chain in Σ . By Zorn's lemma, Σ has a maximal element Q . Now show that Q is a prime ideal. In fact, if $xy \in Q$ for some $x, y \in R \setminus Q$, then the maximality of Q implies that $\overline{Q + Rx} \notin \Sigma$ and $\overline{Q + Ry} \notin \Sigma$. Thus $a^m \in \overline{Q + Rx}$ and $a^n \in \overline{Q + Ry}$ for some $m, n \in \mathbb{N}$. There are $p_1, p_2, q_1, q_2 \in Q$ and $r_1, r_2, s_1, s_2 \in R$ such that

$$a^m + p_1 + r_1x = p_2 + r_2x, \tag{4.1}$$

$$a^n + q_1 + s_1y = q_2 + s_2y. \tag{4.2}$$

Multiplying two sides of Eq. 4.1 by y gives

$$a^m y + p_1 y + r_1 x y = p_2 y + r_2 x y,$$

and $a^m y \in Q$ since $xy \in Q$. Multiplying two sides of Eq. 4.2 by a^m gives

$$a^{m+n} + q_1 a^m + s_1 a^m y = q_2 a^m + s_2 a^m y,$$

and so $a^{m+n} \in Q$, which contradicts $Q \in \Sigma$. Hence $Q \in X$. Meanwhile, since $Q \in \Sigma$, $a \notin Q$ and thus $a \notin \bigcap_{A \subseteq P \in X} P$. □

Lemma 4.1 is a generalization of Theorem 5.1 and Proposition 5.5 in [11].

Corollary 4.2 ([12]). *For $A \in \mathcal{KJ}(R)$, $A = \sqrt{A}$ iff A is an intersection of some k -prime ideals in R .*

The following example shows that the assumption “ A is a k -ideal” cannot be omitted in Lemma 4.1.

Example 4.3. Let B_1 be the 2-element Boolean lattice and $R = B_1[x]$ be the polynomial semiring in the indeterminate x over B_1 . Consider the ideal

$$A = \langle x^2 + x, x^3 + x \rangle = \{(x^2 + x)f(x) + (x^3 + x)g(x) \mid f, g \in R\}$$

in R . Obviously, $x^n \notin A$ for all $n \in \mathbb{N}$ and so $x \notin \sqrt{A}$. Since $x + (x^2 + x) = x^2 + x$, $x \in \overline{A}$ and thus A is not a k -ideal. If P is a k -prime ideal of R and $A \subseteq P$, then $\overline{A} \subseteq P$ and so $x \in P$. Hence $x \in \bigcap_{A \subseteq P \in \text{Spec}_k(R)} P$.

For each nonempty subset S of R , let $V(S) = \{P \in X \mid S \subseteq P\}$.

Lemma 4.4. (1) $V(\{0\}) = X$ and $V(R) = \emptyset$.

(2) If $\emptyset \neq S \subseteq T \subseteq R$, then $V(T) \subseteq V(S)$.

(3) If $\emptyset \neq S \subseteq R$, then $V(S) = V(\langle S \rangle) = V(\overline{\langle S \rangle})$.

(4) If $A \in \mathcal{J}(R)$, then $V(A) = V(\sqrt{A})$.

(5) If $A, B \in \mathcal{J}(R)$, then $V(A) \cup V(B) = V(AB) = V(A \cap B)$.

(6) If $\{A_i \mid i \in \Lambda\} \subseteq \mathcal{J}(R)$, then $V(\sum_{i \in \Lambda} A_i) = \bigcap_{i \in \Lambda} V(A_i)$.

(7) If $A \in \mathcal{KJ}(R)$, $B \in \mathcal{J}(R)$ and $V(A) \subseteq V(B)$, then $B \subseteq \sqrt{A}$.

(8) For $A, B \in \mathcal{KJ}(R)$, $V(A) = V(B)$ iff $\sqrt{A} = \sqrt{B}$.

Proof. (1) through (3) are obvious. (4) through (6) are verified similarly to the case of $\text{Spec}(R)$.

(7) If $A \subseteq P \in X$, then $P \in V(A) \subseteq V(B)$ and so $B \subseteq P$. Thus $B \subseteq \bigcap_{A \subseteq P \in X} P$.

Lemma 4.1 shows that $B \subseteq \sqrt{A}$.

(8) (\Rightarrow) follows from Lemma 4.1.

(\Leftarrow) follows from (4). □

By Lemma 4.4(1)(3)(5)(6), the collection $\{V(S) \mid \emptyset \neq S \subseteq R\}$ satisfies all the axioms of closed sets on a topology. The resulting topology is called the *Zariski topology* and the topological space $X = \text{Spec}_k(R)$ is called the *k*-prime spectrum of R . In fact, the *k*-prime spectrum $X = \text{Spec}_k(R)$ is a subspace of the prime spectrum $\text{Spec}(R)$.

For each nonempty subset S of R , denote by X_S the open subset $X \setminus V(S)$ of X . Then $X_S = \{P \in X \mid S \not\subseteq P\}$. In particular, for $a \in R$, $X_{\{a\}}$ is denoted simply by X_a and called a *basic open subset* of X . Lemma 4.4(3) shows that $V(\langle a \rangle) = V(\{a\}) = X \setminus X_a$. Obviously, $X_S = \bigcup_{a \in S} X_a$ and so we obtain the following conclusion.

Lemma 4.5. *The collection $\{X_a \mid a \in R\}$ of all basic open subsets is a basis for open subsets for the Zariski topology on $X = \text{Spec}_k(R)$.*

Lemma 4.5 is a generalization of Proposition 6.2(2) in [11].

Theorem 4.6. *For each $a \in R$, X_a is a compact subset of $X = \text{Spec}_k(R)$. Especially, $X = X_1$ is a compact topological space.*

Proof. By Lemma 4.5, it suffices to verify that every covering of X_a consisting of basic open subsets has a finite subcovering.

Let $X_a \subseteq \bigcup_{i \in \Lambda} X_{a_i}$ and $A = \langle \{a_i \mid i \in \Lambda\} \rangle$. Lemma 4.4(3)(6) shows that

$$V(\langle a \rangle) \supseteq \bigcap_{i \in \Lambda} V(\langle a_i \rangle) = V\left(\sum_{i \in \Lambda} \langle a_i \rangle\right) = V\left(\overline{\sum_{i \in \Lambda} \langle a_i \rangle}\right) = V(A),$$

so $\langle a \rangle \subseteq \sqrt{A}$ by Lemma 4.4(7) and hence $a \in \sqrt{A}$. Then $a^l \in A$ for some $l \in \mathbb{N}$ and again

$$a^l + t_1 a_{i_1} + \dots + t_m a_{i_m} = t_{m+1} a_{i_{m+1}} + \dots + t_n a_{i_n}$$

for some $i_1, \dots, i_m, i_{m+1}, \dots, i_n \in \Lambda$ and some $\{t_i \mid 1 \leq i \leq n\} \subseteq R$. Hence $a \in \sqrt{\langle a_{i_1}, \dots, a_{i_n} \rangle}$. By Lemma 4.4(3)(4)(6),

$$\begin{aligned} V(\langle a \rangle) &\supseteq V\left(\sqrt{\langle a_{i_1}, \dots, a_{i_n} \rangle}\right) = V(\langle a_{i_1}, \dots, a_{i_n} \rangle) \\ &= V(\langle a_{i_1}, \dots, a_{i_n} \rangle) = \bigcap_{k=1}^n V(\langle a_{i_k} \rangle) \end{aligned}$$

and so $X_a \subseteq \bigcup_{k=1}^n X_{a_{i_k}}$. □

Theorem 4.6 is a generalization of Theorem 6.1 and Proposition 6.2(1) in [11].

Corollary 4.7. *An open subset of $X = \text{Spec}_k(R)$ is compact iff it is a union of a finite number of basic open subsets.*

Remark 4.8. By Theorem 7.20 in [5], the prime spectrum $\text{Spec}(R)$ is a T_0 -space. Since $\text{Spec}_k(R)$ is a subspace of $\text{Spec}(R)$, it is also a T_0 -space. However, $X = \text{Spec}_k(R)$ is not necessarily a T_1 -space.

The following theorem characterizes a semiring for which the *k*-prime spectrum is a T_1 -space.

Theorem 4.9. *$X = \text{Spec}_k(R)$ is a T_1 -space iff *k*-prime ideals are all *k*-maximal ideals in R .*

Proof. (\Rightarrow) Suppose that X is a T_1 -space and let P be any *k*-prime ideal in R . A singleton subset $\{P\}$ is closed in X and so $\{P\} = V(S)$ for some nonempty subset S of R . Since $P \neq R$, Lemma 2.1 says that $P \subseteq M$ for some *k*-maximal ideal M . Since M is a *k*-prime ideal by Lemma 2.2, $M \in V(S)$ and so $P = M$, a *k*-maximal ideal.

(\Leftarrow) For any $P \in X$, P is a *k*-maximal ideal, so $\{P\} = V(P)$, a closed subset of X . Hence X is a T_1 -space. □

Example 4.10. For the semiring \mathbb{Z}_0^+ of all nonnegative integers together with the usual addition and multiplication, the k -prime spectrum $\text{Spec}_k(\mathbb{Z}_0^+)$ consists of $\{0\}$ and all possible $\langle p \rangle$, where p are primes, but it is not a T_1 -space since the k -prime ideal $\{0\}$ is not a k -maximal ideal.

Example 4.11. For the 2-element Boolean lattice $B_1 = \{0, 1\}$, $\{0\}$ is the unique k -prime ideal, thus the k -prime spectrum is $\text{Spec}_k(B_1) = \{\{0\}\}$ and so it is a T_1 -space.

A nonempty topological space Z is said to be *irreducible* if every pair of nonempty open subsets in Z has a nonempty intersection. A subset Y of a topological space Z is said to be *irreducible* if the subspace Y is irreducible. A subset Y of a topological space Z is irreducible iff for any closed subsets Y_1 and Y_2 in Z , $Y \subseteq Y_1 \cup Y_2$ implies either $Y \subseteq Y_1$ or $Y \subseteq Y_2$ [2].

Lemma 4.12. (1) If $f, e \in R$, then $X_f \cap X_e = X_{fe}$.

(2) $f \in \text{Nil}(R)$ iff $X_f = \emptyset$.

(3) $X_f = X$ iff f is a semiunit in R .

Proof. (1) If $P \in X_f \cap X_e$, then $f \notin P$ and $e \notin P$. Thus $fe \notin P$ and so $P \in X_{fe}$. Hence $X_f \cap X_e \subseteq X_{fe}$. The reverse inclusion is verified similarly.

(2) Lemma 4.1 shows that $\text{Nil}(R) = \bigcap_{P \in X} P$. If $f \in \text{Nil}(R)$, then for any $P \in X$, $f \in P$ and thus $X_f = \emptyset$. The converse is proved similarly.

(3) Let f be a semiunit in R . If there is a $P \in X$ such that $P \not\subseteq X_f$, then $f \in P$ and $\overline{Rf} \subseteq \overline{P} = P \neq R$, so $\overline{Rf} \neq R$, which contradicts that f is a semiunit. Hence $X \subseteq X_f$ and consequently $X = X_f$.

Conversely, suppose that $X_f = X$. Then for any $P \in X$, $P \in X_f$ and $f \notin P$. If f is not a semiunit in R , then $\overline{Rf} \neq R$, so by Lemma 2.1, there is a k -maximal ideal Q of R such that $\overline{Rf} \subseteq Q$. By Lemma 2.2, $Q \in X$ and so $Q \in X_f$, i.e. $f \notin Q$, which contradicts that $f \in \overline{Rf} \subseteq Q$. Therefore, f is a semiunit. \square

Lemma 4.12 is a generalization of Theorem 2.12 in [1].

Theorem 4.13. $X = \text{Spec}_k(R)$ is irreducible iff $\text{Nil}(R)$ is a k -prime ideal in R .

Proof. (\Rightarrow) Suppose that X is irreducible. By Lemma 4.1, $\text{Nil}(R)$ is a proper k -ideal in R . If $a, b \in R$ and $ab \in \text{Nil}(R)$, then $X_{ab} = \emptyset$ by Lemma 4.12(2). Again, $X_a \cap X_b = \emptyset$ by Lemma 4.12(1). Since X is irreducible, either $X_a = \emptyset$ or $X_b = \emptyset$, i.e. either $a \in \text{Nil}(R)$ or $b \in \text{Nil}(R)$. Hence $\text{Nil}(R)$ is prime.

(\Leftarrow) By Lemma 4.5, it suffices to show that every pair of nonempty basic open subsets has a nonempty intersection. If $X_a \cap X_b = \emptyset$, then $X_{ab} = \emptyset$ by Lemma 4.12(1), and so $ab \in \text{Nil}(R)$ by Lemma 4.12(2). By assumption, $\text{Nil}(R)$ is a prime ideal, thus either $a \in \text{Nil}(R)$ or $b \in \text{Nil}(R)$. Again either $X_a = \emptyset$ or $X_b = \emptyset$ by Lemma 4.12(2). Hence X is irreducible. \square

For $P \in X$, Lemma 4.4(3) shows that $\overline{\{P\}} = \bigcap_{P \in V(A)} V(A)$, where A are ideals in R .

For each nonempty subset Y of X , put $\tau(Y) = \bigcap_{P \in Y} P$. Then $\tau(Y)$ is a proper k -ideal of R .

Theorem 4.14. A nonempty subset Y of $X = \text{Spec}_k(R)$ is irreducible iff $\tau(Y)$ is a k -prime ideal of R .

Proof. (\Rightarrow) Suppose that Y is irreducible. If $ab \in \tau(Y)$, then $Y \subseteq V(\{ab\})$. By Lemma 4.4(3)(5), $V(\{ab\}) = V(\langle ab \rangle) = V(\langle a \rangle \langle b \rangle) = V(\langle a \rangle) \cup V(\langle b \rangle)$. Thus $Y \subseteq V(\langle a \rangle) \cup V(\langle b \rangle)$. Since Y is irreducible, either $V(\langle a \rangle) \supseteq Y$ or $V(\langle b \rangle) \supseteq Y$, and so either $a \in \tau(Y)$ or $b \in \tau(Y)$.

(\Leftarrow) If Y_1 and Y_2 are closed subsets in X with $Y \subseteq Y_1 \cup Y_2$, then $Y_1 = V(A_1)$ and $Y_2 = V(A_2)$ for some ideals A_1 and A_2 in R . By Lemma 4.4(5), $Y \subseteq V(A_1) \cup V(A_2) = V(A_1 A_2)$

and so $A_1A_2 \subseteq \tau(Y)$. Since $\tau(Y)$ is a prime ideal, either $A_1 \subseteq \tau(Y)$ or $A_2 \subseteq \tau(Y)$, i.e. either $Y \subseteq V(A_1) = Y_1$ or $Y \subseteq V(A_2) = Y_2$. \square

Corollary 4.15. *For each $P \in X = \text{Spec}_k(R)$, $V(P)$ is irreducible.*

Proof. In Theorem 4.14, putting $Y = V(P)$ gives $\tau(Y) = \bigcap_{Q \in V(P)} Q = P$, thus the conclusion follows. \square

Lemma 4.16. *If $P \in \text{Spec}_k(R)$, then $\overline{\{P\}} = V(P)$.*

Proof. Since $P \in V(P)$, we have $\overline{\{P\}} \subseteq V(P)$. If $Q \in V(P)$, then $P \subseteq Q$ and for any ideal A with $P \in V(A)$, $A \subseteq P \subseteq Q$ and so $Q \in V(A)$, thus $Q \in \bigcap_{P \in V(A)} V(A)$. Hence $V(P) \subseteq \overline{\{P\}}$. \square

Theorem 4.17. *If Y is an irreducible closed subset of $\text{Spec}_k(R)$, then $Y = \overline{\{P\}}$ for some $P \in \text{Spec}_k(R)$.*

Proof. Since Y is closed, $Y = V(S)$ for some nonempty subset S of R . Since Y is irreducible, $P = \tau(Y)$ is a k -prime ideal by Theorem 4.14.

Show that $V(P) = Y$. In fact, $S \subseteq \tau(Y) = P$ and $V(S) \supseteq V(P)$ by Lemma 4.4(2), so $V(P) \subseteq Y$. If $Q \in Y$, then $P = \tau(Y) \subseteq Q$ and so $Q \in V(P)$, thus $Y \subseteq V(P)$.

Lemma 4.16 is applied to show that $Y = \overline{\{P\}}$. \square

Let Z be a topological space. If Y is a closed subset of Z and there exists a y such that $Y = \overline{\{y\}}$, then y is called a *generic point* of Y . Z is called a *spectral space* if it is a compact T_0 -space, the compact open subsets form a basis for the topology and are closed under finite intersection, and each of the irreducible closed subsets has a generic point [9].

Theorem 4.18. *$\text{Spec}_k(R)$ with the Zariski topology is a spectral space.*

Proof. By Theorem 4.6, $\text{Spec}_k(R)$ is a compact space. Remark 4.8 shows that $\text{Spec}_k(R)$ is a T_0 -space. By Lemma 4.5 and Theorem 4.6, the compact open subsets form a basis for the Zariski topology in $\text{Spec}_k(R)$. By Corollary 4.7 and Lemma 4.12(1), the intersection of any finite number of compact open subsets is also a compact open subset in $\text{Spec}_k(R)$. By Theorem 4.17, every irreducible closed subset has a generic point in $\text{Spec}_k(R)$. Hence $\text{Spec}_k(R)$ is a spectral space. \square

Theorem 4.18 generalizes the part about saturated prime ideals in Corollary 6.3 in [11].

5. Zariski topology on k -prime congruences

In this section, we prove that for any commutative semiring R with zero 0 and identity 1, the set of all the k -prime congruences with the Zariski topology is homeomorphic to the k -prime spectrum and so it is also a spectral space.

Definition 5.1. A congruence θ on a semiring R is said to be *prime* if it satisfies the following two conditions:

- (1) $\theta \neq R \times R$,
- (2) $(ab)\theta 0$ implies either $a\theta 0$ or $b\theta 0$.

The collection of all k -prime congruences on R is denoted by $\text{CSpec}_k(R)$.

Lescot [10] defined the prime congruences in the setting of characteristic one semirings.

Lemma 5.2. (1) *A congruence θ on R is prime iff $O_{R/\theta}$ is a prime ideal in R .*

(2) *For any ideal P in R , κ_P is a prime congruence on R iff \overline{P} is a prime ideal in R .*

Proof. (1) If θ is prime, then $\theta \neq R \times R$ and so $O_{R/\theta} \neq R$. If $a, b \in R$ and $ab \in O_{R/\theta}$, then $(ab)\theta 0$, so either $a\theta 0$ or $b\theta 0$. Hence either $a \in O_{R/\theta}$ or $b \in O_{R/\theta}$.

Conversely, if $O_{R/\theta}$ is a prime ideal, then $O_{R/\theta} \neq R$ and so $\theta \neq R \times R$. If $a, b \in R$ and $(ab)\theta 0$, then $ab \in O_{R/\theta}$ and so either $a \in O_{R/\theta}$ or $b \in O_{R/\theta}$. Hence either $a\theta 0$ or $b\theta 0$.

(2) It follows from (1) since $O_{R/\theta} = \overline{P}$. \square

For each nonempty subset S of R , put

$$W(S) = \{\theta \in \text{CSpec}_k(R) \mid S \subseteq O_{R/\theta}\}.$$

Lemma 5.3. (1) $W(\{0\}) = \text{CSpec}_k(R)$ and $W(R) = \emptyset$.

(2) If $\emptyset \neq S \subseteq T \subseteq R$, then $W(T) \subseteq W(S)$.

(3) If $\emptyset \neq S \subseteq R$, then $W(S) = W(\langle S \rangle) = W(\overline{\langle S \rangle})$.

(4) If $A \in \mathcal{J}(R)$, then $W(A) = W(\sqrt{A})$.

(5) If $A, B \in \mathcal{J}(R)$, then $W(A) \cup W(B) = W(AB) = W(A \cap B)$.

(6) If $\{A_i \mid i \in \Lambda\} \subseteq \mathcal{J}(R)$, then $\bigcap_{i \in \Lambda} W(A_i) = W(\bigcup_{i \in \Lambda} A_i) = W(\sum_{i \in \Lambda} A_i)$.

Proof. (1) Since $\{0\} \subseteq [0] = O_{R/\theta}$ for all $\theta \in \text{CSpec}_k(R)$, the first equality holds. For any $\theta \in \text{CSpec}_k(R)$, $\theta \neq R \times R$ and so $O_{R/\theta} \neq R$. Thus the second equality holds.

(2) is obvious.

(3) Since $S \subseteq \langle S \rangle \subseteq \overline{\langle S \rangle}$, $W(\overline{\langle S \rangle}) \subseteq W(\langle S \rangle) \subseteq W(S)$ by (2). Now let $\theta \in W(S)$. Then $S \subseteq O_{R/\theta}$ and by Lemma 3.1, $O_{R/\theta}$ is a k -ideal, so $\overline{\langle S \rangle} \subseteq O_{R/\theta}$. Thus $\theta \in W(\overline{\langle S \rangle})$.

(4) Since $A \subseteq \sqrt{A}$, $W(\sqrt{A}) \subseteq W(A)$ by (2). Now let $\theta \in W(A)$. Then $A \subseteq O_{R/\theta}$ and by Lemma 5.2(1), $O_{R/\theta}$ is a prime ideal of R because θ is a prime congruence. Since \sqrt{A} is the intersection of all prime ideals of R containing A , we have $\sqrt{A} \subseteq O_{R/\theta}$. Thus $\theta \in W(\sqrt{A})$.

(5) Since $AB \subseteq A \cap B \subseteq A$, $W(A) \subseteq W(A \cap B) \subseteq W(AB)$ by (2). Similarly, $W(B) \subseteq W(A \cap B) \subseteq W(AB)$ and so $W(A) \cup W(B) \subseteq W(A \cap B) \subseteq W(AB)$. Now let $\theta \in W(AB)$. Then $AB \subseteq O_{R/\theta}$ and by Lemma 5.2(1), $O_{R/\theta}$ is a prime ideal, so either $A \subseteq O_{R/\theta}$ or $B \subseteq O_{R/\theta}$. Thus either $\theta \in W(A)$ or $\theta \in W(B)$, i.e. $\theta \in W(A) \cup W(B)$.

(6) The first equality is obvious. Since $\langle \bigcup_{i \in \Lambda} A_i \rangle = \sum_{i \in \Lambda} A_i$, the second equality follows from (3). \square

Lemma 5.3(1)(3)(5)(6) shows that the family $\{W(S) \mid \emptyset \neq S \subseteq R\}$ satisfies all the axioms of closed sets on a topology. The resulting topology is called the *Zariski topology* on $\text{CSpec}_k(R)$.

Theorem 5.4. $\text{Spec}_k(R)$ is homeomorphic to $\text{CSpec}_k(R)$ with respect to the Zariski topologies.

Proof. Define $\kappa : \text{Spec}_k(R) \rightarrow \text{CSpec}_k(R)$ by $\kappa(P) = \kappa_P$ and define $\iota : \text{CSpec}_k(R) \rightarrow \text{Spec}_k(R)$ by $\iota(\theta) = O_{R/\theta}$. By Lemma 5.2, both κ and ι are well-defined, and

$$\kappa \circ \iota = \text{id}_{\text{CSpec}_k(R)}, \quad \iota \circ \kappa = \text{id}_{\text{Spec}_k(R)}.$$

We will show that κ and ι are continuous. If G is any closed subset in $\text{CSpec}_k(R)$, then $G = W(S)$ for some nonempty subset S of R . Since

$$\begin{aligned} P \in \kappa^{-1}(G) &\Leftrightarrow \kappa_P = \kappa(P) \in G = W(S) \\ &\Leftrightarrow S \subseteq O_{R/\kappa_P} = \overline{P} = P \Leftrightarrow P \in V(S), \end{aligned}$$

we have $\kappa^{-1}(G) = V(S)$, a closed subset in $\text{Spec}_k(R)$. Hence κ is continuous. If H is any closed subset in $\text{Spec}_k(R)$, then $H = V(S)$ for some nonempty subset S of R . Since

$$\begin{aligned} \theta \in \iota^{-1}(H) &\Leftrightarrow O_{R/\theta} = \iota(\theta) \in H = V(S) \\ &\Leftrightarrow S \subseteq O_{R/\theta} \Leftrightarrow \theta \in W(S), \end{aligned}$$

we have $\iota^{-1}(H) = W(S)$, a closed subset in $\text{CSpec}_k(R)$. Hence ι is continuous.

Therefore, κ and ι are both homeomorphisms. \square

By Theorems 4.18 and 5.4, we draw the following conclusion.

Theorem 5.5. $\text{CSpec}_k(R)$ with the Zariski topology is a spectral space.

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