Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (2) (2020), 835 – 842
DOI : 10.15672/hujms.522814

Research Article

Lorentz-Schatten classes of direct sum of operators

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Abstract

In this paper, the relations between Lorentz-Schatten property of the direct sum of operators and Lorentz-Schatten property of its coordinate operators are studied. Then, the results are supported by applications.

Mathematics Subject Classification (2010). 47A05, 47A10

Keywords. direct sum of Hilbert spaces and operators, compact operators, Lorentz-Schatten operator classes

1. Introduction

The general theory of singular numbers and operator ideals was given by Pietsch [13,14] and the case of linear compact operators was investigated by Gohberg and Krein [5]. However, the first result in this area can be found in the works of Schmidt [16] and Schatten, von Neumann [15]. They used these concepts in the theory of non-selfadjoint integral equations.

Later on, the main aim of mini-workshop held in Oberwolfach (Germany) was to present and discuss some modern applications of the functional-analytic concepts of $s-$numbers and operator ideals in areas like numerical analysis, theory of function spaces, signal processing, approximation theory, probability of Banach spaces and statistical learning theory (see [3]).

Let $\mathcal{H}$ be a Hilbert space, $S_\infty(\mathcal{H})$ be a class of linear compact operators in $\mathcal{H}$ and $s_n(T)$ be the $n-th$ singular numbers of the operator $T \in S_\infty(\mathcal{H})$. The Lorentz-Schatten operator ideals are defined as

$$S_{p,q}(\mathcal{H}) = \left\{ T \in S_\infty(\mathcal{H}) : \sum_{n=1}^{\infty} n^{\frac{q}{p}-1} s_n^q(T) < \infty \right\},\ 0 < p \leq \infty,\ 0 < q < \infty$$

and

$$S_{p,\infty}(\mathcal{H}) = \left\{ T \in S_\infty(\mathcal{H}) : \sup_{n\geq 1} n^{\frac{1}{p}} s_n(T) < \infty \right\},\ 0 < p \leq \infty$$

in [1,13,14,17].

Let α be a positive real number. If $s_n(T) \sim cn^{-\alpha},\ c > 0,\ n \to \infty$ for any linear compact operator T in a Hilbert space $\mathcal{H}$, then for each $p \in \left(\frac{1}{\alpha}, \infty\right]$ and $q \in (0,\infty),\ T \in S_{p,q}(\mathcal{H})$. In

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Received: 05.02.2019; Accepted: 10.04.2019

this case, the necessary and sufficient condition for the series $\sum_{n=1}^{\infty} n^{\frac{q}{p}-1-\alpha q}$ to be convergent is $p > \dfrac{1}{\alpha}$. Moreover, the necessary and sufficient condition for $T \in S_{p,\infty}(\mathcal{H})$ is $p \in \left[\frac{1}{\alpha}, \infty\right]$.

The infinite direct sum of Hilbert spaces and the infinite direct sum of operators have been studied in [4]. Namely, the infinite direct sum of Hilbert spaces $H_n,\ n \geq 1$ and the infinite direct sum of operators A_n in $H_n,\ n \geq 1$ are defined as

$$H = \bigoplus_{n=1}^{\infty} H_n = \left\{ u = (u_n) : u_n \in H_n,\ n \geq 1,\ \sum_{n=1}^{\infty} \|u_n\|_{H_n}^2 < +\infty \right\},$$

$$A = \bigoplus_{n=1}^{\infty} A_n,$$

$$D(A) = \{u = (u_n) \in H : u_n \in D(A_n),\ n \geq 1,\ Au = (A_n u_n) \in H\}.$$

Recall that H is a Hilbert space with the norm induced by the inner product

$$(u, v)_H = \sum_{n=1}^{\infty} (u_n, v_n)_{H_n},\ u, v \in H.$$

Our aim in this paper is to study the relations between Lorentz-Schatten property of the direct sum of operators and Lorentz-Schatten property of its coordinate operators.

It should be noted that the analogous problems in special cases have been investigated in [8].

The problem of belonging to the Schatten-von Neuman classes of the resolvent operators of the normal extensions of the minimal operator generated by the direct sum of differential-operator expression for first order with suitable operator coefficients in the direct sum of Hilbert spaces in finite interval has been studied in [7].

In [6,9], the same problem for normal and hyponormal extensions of the minimal operators generated by corresponding differential-operator expressions under some conditions to operator coefficients in a finite interval has been investigated.

Later on, some more general Schatten-von Neumann classes of compact operators in Hilbert spaces have been defined and characterized in [10] in terms of Berezin symbols. In [2], the question raised by Nordgren and Rosenthal about the Schatten-von Neumann class membership of operators in standard reproducing kernel Hilbert spaces in terms of their Berezin symbols has been answered.

2. Lorentz-Schatten property of block diagonal operator matrices

Let H_n be a Hilbert space, $A_n \in L(H_n)$ for $n \geq 1$ and

$$H = \bigoplus_{n=1}^{\infty} H_n,\ A = \bigoplus_{n=1}^{\infty} A_n.$$

Recall that, in order to $A \in L(H)$ the necessary and sufficient condition is $\sup_{n\geq 1} \|A_n\| < \infty$. Moreover, $\|A\| = \sup_{n\geq 1} \|A_n\|$ (see [11]).

It is known that if $A_n \in S_\infty(H_n)$ for $n \geq 1$, then the necessary and sufficient condition for $A \in S_\infty(H)$ is $\lim_{n\to\infty} \|A_n\| = 0$ (see [12]).

The following result on singular numbers of the operator $A \in S_\infty(H)$

$$\{s_m(A) : m \geq 1\} = \bigcup_{n=1}^{\infty} \{s_m(A_n) : m \geq 1\}$$

can be found in [8].

Throughout this paper, for the simplicity we assume that:

(1) for any $n,\ k \geq 1$ with $n \neq k$, $\{s_m(A_n) : m \geq 1\} \cap \{s_m(A_k) : m \geq 1\} = \emptyset$ or $\{0\}$;
(2) for any $n \geq 1$ in the sequence $(s_m(A_n))$, if for some $k > 1,\ s_k(A_n) > 0$, then $s_k(A_n) < s_{k-1}(A_n)$.

Proposition 2.1. *For $n \geq 1$ there is a strongly increasing sequence $k_m^{(n)} : \mathbb{N} \to \mathbb{N}$ such that $s_{k_m^{(n)}}(A) = s_m(A_n)$ holds for $m \geq 1$ and $\bigcup\limits_{n=1}^{\infty} \bigcup\limits_{m=1}^{\infty} \left\{k_m^{(n)}\right\} = \mathbb{N}$. Moreover, it is clear that $k_m^{(n)} \geq m$ for $n, m \geq 1$.*

Indeed, in the Hilbert space $H = \bigoplus\limits_{n=1}^{\infty} H_n = l_2(\mathbb{R})$, where $H_n = (\mathbb{R}, |\cdot|)$, consider the following infinite matrices with reel entries in forms

$$A = \begin{pmatrix} a_1 & & & & & \\ & a_2 & & & & \\ & & a_3 & & 0 & \\ & & & \ddots & & \\ & 0 & & & a_n & \\ & & & & & \ddots \end{pmatrix} : H \to H$$

and

$$B = \begin{pmatrix} b_1 & & & & & \\ & b_2 & & & & \\ & & b_3 & & 0 & \\ & & & \ddots & & \\ & 0 & & & b_n & \\ & & & & & \ddots \end{pmatrix} : H \to H,$$

where for any $n,\ m \geq 1,\ n \neq m,\ a_n \neq a_m,\ a_n > 0$ and $b_n = \dfrac{a_n + a_{n+1}}{2}$ with property $\lim\limits_{n\to\infty} a_n = 0$.

In this case, $A,\ B \in S_\infty(H)$ and the singular numbers of the operators $A,\ B$ are given in the following forms

$$\{s_m(A_n) : m \geq 1\} = \{a_n : n \geq 1\},$$
$$\{s_m(B_n) : m \geq 1\} = \{b_n : n \geq 1\},$$

respectively. Then, by [12] it implies that $T = A \oplus B \in S_\infty(H \oplus H)$ and $\{s_m(T) : m \geq 1\} = \{a_n, b_n : n \geq 1\}$. In this case, it is easy to see that

$$k_m^{(1)} = 2m - 1, m \geq 1,$$
$$k_m^{(2)} = 2m, m \geq 1.$$

Theorem 2.2. *Let $0 < p, q < \infty$. $A \in S_{p,q}(H)$ if and only if the series*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(k_m^{(n)}\right)^{\frac{q}{p}-1} s_m^q(A_n)$$

is convergent.

Proof. If $A \in S_{p,q}(H)$, it is clear that the series

$$\sum_{m=1}^{\infty} m^{\frac{q}{p}-1} s_m^q(A)$$

is convergent. From the structure of the set of the singular numbers of the operator A and the important theorem on the convergent of the rearrangement series it is obtained that the series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(k_m^{(n)}\right)^{\frac{q}{p}-1}s_m^q(A_n)$$

is convergent. □

Conversely, if the series in the theorem is convergent, then $\sum\limits_{m=1}^{\infty} m^{\frac{q}{p}-1}s_m^q(A)$, which is the rearrangement of the above series, is convergent. So, $A \in S_{p,q}(H)$.

Now, in Theorem 2.3-2.5, we will investigate the problem of belonging to Lorentz-Schatten classes of its coordinate operators, if the direct sum of operators belongs to Lorentz-Schatten classes.

Theorem 2.3. *Let $A \in S_\infty(H)$ and $0 < p \le q < \infty$. If $A \in S_{p,q}(H)$, then $A_n \in S_{p,q}(H_n)$ for $n \ge 1$.*

Proof. In the special case $0 < p = q < \infty$, the result has been proved in [8].

In the case of $p < q$, we have

$$m \le k_m^{(n)} \text{ and } s_{k_m^{(n)}}(A) = s_m(A_n)$$

for $n, m \ge 1$. Consequently, for $n \ge 1$ we get

$$\begin{aligned}\sum_{m=1}^{\infty} m^{\frac{q}{p}-1}s_m^q(A_n) &\le \sum_{m=1}^{\infty}\left(k_m^{(n)}\right)^{\frac{q}{p}-1}s_m^q(A_n)\\ &\le \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(k_m^{(n)}\right)^{\frac{q}{p}-1}s_m^q(A_n)\\ &= \sum_{m=1}^{\infty} m^{\frac{q}{p}-1}s_m^q(A) < \infty.\end{aligned}$$

Hence, $A_n \in S_{p,q}(H_n)$ for $n \ge 1$. □

Theorem 2.4. *Let $0 < q < p < \infty$ and for $n \ge 1$, $\sup\limits_{m\ge1}\left(\frac{k_m^{(n)}}{m}\right) \le \gamma < \infty$. If $A \in S_{p,q}(H)$, then $A_n \in S_{p,q}(H_n)$ for $n \ge 1$.*

Proof. Under the assumptions in the theorem, we have

$$\begin{aligned}\sum_{m=1}^{\infty} m^{\frac{q}{p}-1}s_m^q(A_n) &= \sum_{m=1}^{\infty}\left(\frac{m}{k_m^{(n)}}\right)^{\frac{q}{p}-1}\left(k_m^{(n)}\right)^{\frac{q}{p}-1}s_m^q(A_n)\\ &\le \sup_{m\ge1}\left(\frac{k_m^{(n)}}{m}\right)^{1-\frac{q}{p}}\sum_{m=1}^{\infty}\left(k_m^{(n)}\right)^{\frac{q}{p}-1}s_m^q(A_n)\\ &\le \gamma^{1-\frac{q}{p}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(k_m^{(n)}\right)^{\frac{q}{p}-1}s_m^q(A_n)\\ &= \gamma^{1-\frac{q}{p}}\sum_{j=1}^{\infty} j^{\frac{q}{p}-1}s_j^q(A) < \infty.\end{aligned}$$

Therefore, $A_n \in S_{p,q}(H_n)$ for $n \ge 1$. □

Now, we will investigate the case of $q = \infty$.

Theorem 2.5. *Let $0 < p \leq \infty$. If $A \in S_{p,\infty}(H)$, then $A_n \in S_{p,\infty}(H_n)$ for $n \geq 1$.*

Proof. Since $A \in S_{p,\infty}(H)$, we have $\sup_{m\geq 1} m^{\frac{1}{p}} s_m(A) < \infty$. Hence, $\sup_{m\geq 1} \left(k_m^{(n)}\right)^{\frac{1}{p}} s_m(A_n) < \infty$. On the other hand, we get

$$\begin{aligned}\sup_{m\geq 1} m^{\frac{1}{p}} s_m(A_n) &= \sup_{m\geq 1} \left(k_m^{(n)}\right)^{\frac{1}{p}} s_m(A_n) \left(\frac{m}{k_m^{(n)}}\right)^{\frac{1}{p}} \\ &\leq \sup_{m\geq 1} \left(k_m^{(n)}\right)^{\frac{1}{p}} s_m(A_n) < \infty.\end{aligned}$$

Then, $A_n \in S_{p,\infty}(H_n)$ for $n \geq 1$. □

Now, in Theorem 2.6-2.8, we will investigate the problem of belonging to Lorentz-Schatten classes of the direct sum of operators, if its coordinate operators belong to Lorentz-Schatten classes.

Theorem 2.6. *Let $0 < q \leq p < \infty$. If $A_n \in S_{p,q}(H_n)$ for $n \geq 1$ and the series $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m^{\frac{q}{p}-1} s_m^q(A_n)$ is convergent, then $A \in S_{p,q}(H)$.*

Proof. For $0 < q \leq p < \infty$, we have

$$\begin{aligned}\sum_{m=1}^{\infty} m^{\frac{q}{p}-1} s_m^q(A) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(k_m^{(n)}\right)^{\frac{q}{p}-1} s^q_{k_m^{(n)}}(A) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{k_m^{(n)}}{m}\right)^{\frac{q}{p}-1} m^{\frac{q}{p}-1} s_m^q(A_n) \\ &\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m^{\frac{q}{p}-1} s_m^q(A_n) < \infty.\end{aligned}$$

This completes the proof. □

Theorem 2.7. *Let $0 < p < q < \infty$, for $n \geq 1$ $\sum_{m=1}^{\infty} m^{\frac{q}{p}-1} s_m^q(A_n) \leq \beta_n < \infty$, $\sup_{m\geq 1} \left(\frac{k_m^{(n)}}{m}\right) \leq \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^{\frac{q}{p}-1} \beta_n < \infty$. If $A_n \in S_{p,q}(H_n)$ for $n \geq 1$, then $A \in S_{p,q}(H)$.*

Proof. The validity of this claim is clear from the following inequality

$$\begin{aligned}\sum_{m=1}^{\infty} m^{\frac{q}{p}-1} s_m^q(A) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(k_m^{(n)}\right)^{\frac{q}{p}-1} s^q_{k_m^{(n)}}(A) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{k_m^{(n)}}{m}\right)^{\frac{q}{p}-1} m^{\frac{q}{p}-1} s_m^q(A_n) \\ &\leq \sum_{n=1}^{\infty} \left(\sup_{m\geq 1}\left(\frac{k_m^{(n)}}{m}\right)\right)^{\frac{q}{p}-1} \sum_{m=1}^{\infty} m^{\frac{q}{p}-1} s_m^q(A_n) \\ &\leq \sum_{n=1}^{\infty} \gamma_n^{\frac{q}{p}-1} \beta_n.\end{aligned}$$

□

Now, we will investigate in the case of $q = \infty$.

Theorem 2.8. *Let* $0 < p < \infty$, *for* $n \geq 1$ $\alpha_n = \sup_{m\geq 1}\left(\frac{k_m^{(n)}}{m}\right)^{\frac{1}{p}} < \infty$, $\gamma_n = \sup_{m\geq 1} m^{\frac{1}{p}} s_m(A_n)$ *and* $\sup_{n\geq 1}\alpha_n\gamma_n < \infty$. *If* $A_n \in S_{p,\infty}(H_n)$ *for* $n \geq 1$, *then* $A \in S_{p,\infty}(H)$.

Proof. This result is clear from the following relation

$$\begin{aligned}
\sup_{m\geq 1} m^{\frac{1}{p}} s_m(A) &= \sup_{n,m\geq 1}\left(k_m^{(n)}\right)^{\frac{1}{p}} s_{k_m^{(n)}}(A) \\
&= \sup_{n,m\geq 1}\left(k_m^{(n)}\right)^{\frac{1}{p}} s_m(A_n) \\
&\leq \sup_{n\geq 1}\left(\sup_{m\geq 1}\left(\frac{k_m^{(n)}}{m}\right)^{\frac{1}{p}} \sup_{m\geq 1} m^{\frac{1}{p}} s_m(A_n)\right) \\
&= \sup_{n\geq 1}\alpha_n\gamma_n < \infty.
\end{aligned}$$

□

Theorem 2.9. *Let* $0 < p_n, q_n < \infty$, $A_n \in S_{p_n,q_n}(H_n)$ *for* $n \geq 1$ *and* $p = \sup_{n\geq 1} p_n < \infty$, $q = \sup_{n\geq 1} q_n < \infty$. *Then,* $A \in S_{p,q}(H)$ *if and only if the series* $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(k_m^{(n)}\right)^{\frac{q}{p}-1} s_m^q(A_n)$ *is convergent.*

Proof. From the result in [1], we have $A_n \in S_{p,q}(H_n)$ for $n \geq 1$. Therefore, the validity of this claim is implied by Theorem 2.2. □

Remark 2.10. Using this method, the analogous researches for the following operators

$$B = \begin{pmatrix} 0 & B_1 & & & & & \\ & 0 & B_2 & & & & \\ & & 0 & B_3 & & 0 & \\ & & & \ddots & \ddots & & \\ & 0 & & & 0 & B_n & \\ & & & & & \ddots & \ddots \end{pmatrix} : H = \bigoplus_{n=1}^{\infty} H_n \to H$$

and

$$C = \begin{pmatrix} 0 & & & & & \\ C_1 & 0 & & & & \\ & C_2 & 0 & & & 0 \\ & & \ddots & \ddots & & \\ & 0 & & C_n & 0 & \\ & & & & \ddots & \ddots \end{pmatrix} : H = \bigoplus_{n=1}^{\infty} H_n \to H$$

can be studied.

3. Examples

In this section, we provide some examples as applications of our theorems.

Example 3.1. In the Hilbert space $H = \bigoplus_{n=1}^{\infty} H_n = l_2(\mathbb{C})$, where $H_n := (\mathbb{C}, |\cdot|),\ n \geq 1$, consider the following diagonal infinite matrix with complex entries

$$A = \begin{pmatrix} a_1 & & & & & \\ & a_2 & & & & \\ & & a_3 & & 0 & \\ & & & \ddots & & \\ & 0 & & & a_n & \\ & & & & & \ddots \end{pmatrix} : H \to H$$

under the condition $|a_n| < r < 1,\ n \geq 1$. Then, $\lim_{n\to\infty} a_n = 0$. In this case, $A \in S_\infty(H)$. If we define $A_n := a_n$ for $n \geq 1$, then $s_m(A_n) = |\lambda(A_n)| = \{|a_n|,\ 0\},\ m \geq 1$.
Hence, the singular numbers of the operator A are given as

$$\{s_m(A) : m \geq 1\} = \{|a_n| : n \geq 1\}.$$

On the other hand, for $n \geq 1$ and $0 < q \leq p < \infty$ we get

$$\sum_{m=1}^{\infty} m^{\frac{q}{p}-1} s_m^q(A_n) = |a_n|^q.$$

Then, $A_n \in S_{p,q}(H_n),\ n \geq 1,\ 0 < q \leq p < \infty$. Therefore, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m^{\frac{q}{p}-1} s_m^q(A_n) = \sum_{n=1}^{\infty} |a_n|^q < \infty.$$

Hence, by Theorem 2.6, $A \in S_{p,q}(H)$.

Example 3.2. Let $H_n := (\mathbb{C}^2, |\cdot|_2),\ H := \bigoplus_{n=1}^{\infty} H_n = l_2(\mathbb{C}^2),\ A_n = \begin{pmatrix} 0 & \alpha^{2n-1} \\ \alpha^{2n} & 0 \end{pmatrix}$ for $n \geq 1$, $0 < |\alpha| < 1$ and $A = \bigoplus_{n=1}^{\infty} A_n : H \to H$. Then $A \in S_\infty(H)$ (see [12]).
In this case, for $n \geq 1$ we get

$$\|A_n\| = |\alpha|^{2n-1},$$
$$\{s_m(A_n) : m \geq 1\} = \{|\alpha|^{2n-1},\ |\alpha|^{2n}\}$$

and

$$\{s_m(A) : m \geq 1\} = \{|\alpha|^n :\ n \geq 1\}.$$

On the other hand, for $n \geq 1$ and $0 < q \leq p < \infty$ we obtain

$$\sum_{m=1}^{\infty} m^{\frac{q}{p}-1} s_m^q(A_n) = |\alpha|^{(2n-1)q} + 2^{\frac{q}{p}-1} |\alpha|^{2nq} < \infty.$$

Hence, $A_n \in S_{p,q}(H_n),\ n \geq 1,\ 0 < q \leq p < \infty$. Therefore, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m^{\frac{q}{p}-1} s_m^q(A_n) = \sum_{n=1}^{\infty} \left(|\alpha|^{(2n-1)q} + 2^{\frac{q}{p}-1} |\alpha|^{2nq}\right) = \frac{|\alpha|^q}{1 - |\alpha|^{2q}} \left(1 + 2^{\frac{q}{p}-1} |\alpha|^q\right) < \infty.$$

Hence, by Theorem 2.6, $A \in S_{p,q}(H)$.

Acknowledgment. The author would like to thank Professor Z. I. Ismailov (Karadeniz Technical University, Department of Mathematics, Turkey) for his various comments and suggestions.

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