Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (2) (2020), 843 – 853
DOI : 10.15672/hujms.479184

Research Article

On submanifolds of Kenmotsu manifold with Torqued vector field

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Abstract

In this paper, we consider the submanifold M of a Kenmotsu manifold $\tilde{M}$ endowed with torqued vector field $\mathcal{T}$. Also, we study the submanifold M admitting a Ricci soliton of both Kenmotsu manifold $\tilde{M}$ and Kenmotsu space form $\tilde{M}(c)$. Indeed, we provide some necessary conditions for which such a submanifold M is an $\eta-$Einstein. We have presented some related results and classified. Finally, we obtain an important characterization which classifies the submanifold M admitting a Ricci soliton of Kenmotsu space form $\tilde{M}(c)$.

Mathematics Subject Classification (2010). 53C25, 53C40.

Keywords. Kenmotsu manifold, Ricci soliton, Torqued vector field.

1. Introduction

Hamilton introduced the concept of Ricci soliton, which is a natural generalization of Einstein manifold, in 1982 [11]. This notion actually corresponds to the self-similar solution of Hamilton's Ricci flow: $\frac{\partial}{\partial t}g = -2\tilde{S}$, viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphims and scaling, (for details, see [12]).

In the framework of the contact geometry, Sharma started the studying of the problem of the Ricci solitons in K-contact manifolds in [18]. After this work, Ricci solitons have been investiaged in some different classes of contact geometry. For instance, it is proved by Ghosh that the constant curvature of a Kenmotsu 3−manifold as Ricci soliton is −1 in [10]. Then, Perktaş and Keleş proved that if a 3−dimensional normal almost paracontact metric manifold admits a Ricci soliton then it is shrinking in [17]. For more details, see ([1, 2, 8, 9, 16, 19, 21]).

Consider the following equation on a Riemannian manifold $(\tilde{M}, g)$

$$(\pounds_V g)(X, Y) + 2\tilde{S}(X, Y) + 2\lambda g(X, Y) = 0, \tag{1.1}$$

where $\pounds_V g$ is the Lie-derivative of the metric tensor g in the direction vector field V, $\tilde{S}$ is the Ricci tensor of $\tilde{M}$ and λ is a constant. $(\tilde{M}, g)$ is called a *Ricci soliton* if the equation (1.1) holds for vector fields X, Y on $\tilde{M}$. The vector field V is called the potential field of Ricci soliton $(\tilde{M}, g)$. If $\pounds_V g = \rho g$, then potential field V is said to be conformal Killing,

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Received: 06.11.2018; Accepted: 17.04.2019

where ρ is a function. If ρ vanishes identically, then V is said to be Killing. Also, if V is zero or Killing in (1.1), then the Ricci soliton is called trivial and in this case, the metric is an Einstein. In addition, a Ricci soliton is called a gradient if the potential field V is the gradient of a potential function $-f$ (i.e., $V = -\nabla f$) and is called shrinking, steady or expanding depending on $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively.

On the other hand, Riemannian manifolds which admit torqued vector fields (as a combination of concurrent and recurrent vector fields) were first defined by Chen in [6]. According to this definition, a nowhere zero vector field $\mathcal{T}$ on a Riemannian manifold $(\tilde{M}, g)$ is called torqued vector field, if it satisfies the following two conditions

$$\tilde{\nabla}_X \mathcal{T} = fX + \alpha(X)\mathcal{T} \quad \text{and} \quad \alpha(\mathcal{T}) = 0, \tag{1.2}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$, for any $X \in \Gamma(T\tilde{M})$. The function f is called the torqued function and 1−form α is called the torqued form of $\mathcal{T}$. Here, Chen characterized rectifying submanifolds for a Riemannian manifold endowed with torqued vector field in [6]. Then, Chen proved that every Ricci soliton with torqued potential field is an almost quasi-Einstein under some conditions (see [7]).

The paper is organized as follows:

In Section 2, we recall some basic notions which are going to be needed.

In Section 3, we consider the submanifold M of Kenmotsu manifold $\tilde{M}$ endowed with a torqued vector field $\mathcal{T}$ and find that the characteristic vector field ξ of $\tilde{M}$ is never torqued on the ambient space $\tilde{M}$. Also, we give a necessary and sufficient condition for which the tangential part $\mathcal{T}^\top$ of $\mathcal{T}$ is torse-forming on M.

In Section 4, we deal with Kenmotsu space form $\tilde{M}(c)$ endowed with a torqued vector field $\mathcal{T}$ and give some characterizations on a submanifold admitting a Ricci soliton of $\tilde{M}(c)$.

The last section is devoted to conclusion. Here, we present our results which are obtained in this paper.

2. Preliminaries

In this section, we shall review some basic definitions and formulas of almost contact metric manifolds from [3,4,15,20] and [22].

Let $\tilde{M}$ be an $(2n+1)-$dimensional almost contact metric manifold with an almost contact metric structure (φ, ξ, η, g) such that φ is a tensor field of type $(1,1)$, ξ is a vector field (called the characteristic vector field) of type $(0,1)$, $1-$ form η is a tensor field of type $(1,0)$ on $\tilde{M}$ and the Riemannian metric g satisfies the following relations:

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(X) = g(X, \xi) \tag{2.1}$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\varphi X, Y) = -g(X, \varphi Y) \tag{2.2}$$

for any $X, Y \in \Gamma(T\tilde{M})$.

If the following condition is satisfied for an almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, g)$, then it is called a Kenmotsu manifold

$$(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{2.3}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$, for any $X, Y \in \Gamma(T\tilde{M})$. From (2.3), for a Kenmotsu manifold we also have

$$\tilde{\nabla}_X \xi = X - \eta(X)\xi. \tag{2.4}$$

On the other hand, a Kenmotsu manifold $\tilde{M}$ with constant $\varphi-$sectional curvature c is said to be a Kenmotsu space form and it is denoted by $\tilde{M}(c)$. The curvature tensor $\tilde{R}$ of a Kenmotsu space form is given by

$$\begin{aligned}\tilde{R}(X,Y)Z &= \frac{c-3}{4}\Big\{g(Y,Z)X - g(X,Z)Y\Big\}\\ &+\frac{c+1}{4}\Big\{\Big[\eta(X)Y - \eta(Y)X\Big]\eta(Z)\\ &+\Big[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\Big]\xi\\ &+g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z\Big\} \quad (2.5)\end{aligned}$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$.

Let M be isometrically immersed submanifold of Kenmotsu manifold $\tilde{M}$. For any $X, Y \in \Gamma(TM)$, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad (2.6)$$

where $\tilde{\nabla}$ and ∇ stand for the Levi-Civita connections on $\tilde{M}$ and M, respectively. Then, the equality (2.6) is called the Gauss formula and h is called the second fundamental form of M. Also, if the second fundamental form h vanishes identically in (2.6), then the submanifold M is called totally geodesic. Similarly, one has

$$\tilde{\nabla}_U V = -A_V U + \nabla^{\perp}_U V, \quad (2.7)$$

where A_V and $\nabla^{\perp}$ denote the shape operator and the normal connection of M in the ambient space $\tilde{M}$, respectively, for any $U \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$. Using (2.4) and (2.6), it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \quad (2.8)$$

$$h(X,\xi) = 0, \quad (2.9)$$

where ∇ is the Levi-Civita connection of M.

Also, it is well known that the relation between second fundamental form h and the shape operator A_V are related by

$$g(A_V X, Y) = g(h(X,Y), V) \quad (2.10)$$

for any $X, Y \in \Gamma(TM)$. Here, we denote by the same symbol g the Riemannian metric induced by g on $\tilde{M}$.

The equation of Gauss is given by

$$\begin{aligned}g(R(X,Y)Z,W) &= g(\tilde{R}(X,Y)Z,W) + g(h(X,W),h(Y,Z))\\ &\quad -g(h(X,Z),h(Y,W)) \quad (2.11)\end{aligned}$$

for any $X, Y, Z, W \in \Gamma(TM)$.

We denote by H the mean curvature vector, that is,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where $\{e_1, e_2, ..., e_n = \xi\}$ is an orthonormal basis of the tangent space T_pM, $p \in M$. As it is known, M is called minimal if H vanishes identically.

The submanifold M is $\omega-$umbilical with respect to a normal vector field ω if its shape operator satisfies $A_\omega = \mu I$, where μ is a function on M and I is the identity map.

Furthermore, the submanifold M is said to be totally umbilical if and only if one has

$$h(X,Y) = g(X,Y)H \quad (2.12)$$

for any $X, Y \in \Gamma(TM)$, where h and H denote the second fundamental form and the mean curvature vector, respectively.

The scalar curvature r of (M, g) is defined by

$$r = \sum_{i=1}^{n} S(e_i, e_i),$$

where $\{e_1, e_2, ..., e_n = \xi\}$ is an orthonormal frame of TM and S is the Ricci tensor of M.

Now, we recall some definitions from ([7, 14, 22]), as follows:

A Riemannian manifold $(\tilde{M}, g)$ is called $\eta-$Einstein if there exists two real constants a and b such that the Ricci curvature tensor field $\tilde{S}$ satisfies

$$\tilde{S}(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(T\tilde{M})$. If the constant b is equal to zero, then $\tilde{M}$ becomes an Einstein.

The Ricci tensor $\tilde{S}$ of a Kenmotsu manifold $(\tilde{M}, g)$ is called $\eta-$parallel if it satisfies

$$(\tilde{\nabla}_X \tilde{S})(\varphi Y, \varphi Z) = 0$$

such that

$$(\tilde{\nabla}_X \tilde{S})(\varphi Y, \varphi Z) = \tilde{\nabla}_X \tilde{S}(\varphi Y, \varphi Z) - \tilde{S}(\tilde{\nabla}_X \varphi Y, \varphi Z) - \tilde{S}(\varphi Y, \tilde{\nabla}_X \varphi Z)$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$.

A vector field v on a Riemannian manifold $(\tilde{M}, g)$ is called torse-forming if it satisfies

$$\tilde{\nabla}_X v = fX + \alpha(X)v, \tag{2.13}$$

where f is a function, α is a $1-$form and $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$, for any $X \in \Gamma(T\tilde{M})$. The $1-$form α is called the generating form and the function f is called the conformal scalar of v.

If the $1-$form α in (2.13) vanishes identically, then the vector field v is called concircular [5]. If $f = 1$ and $\alpha = 0$, then the vector field v is called concurrent [23]. The vector field v is called recurrent if it satisfies (2.13) with $f = 0$. Also, if $f = \alpha = 0$, the vector field v in (2.13) is called parallel.

Let $\tilde{M}$ be a Kenmotsu manifold endowed with a torqued vector field $\mathcal{T}$ and $\phi : M \to \tilde{M}$ be an isometric immersion. Then, we get

$$\mathcal{T} = \mathcal{T}^{\top} + \mathcal{T}^{\perp}, \tag{2.14}$$

where $\mathcal{T}^{\top}$ and $\mathcal{T}^{\perp}$ the tangential and normal components of $\mathcal{T}$ on $\tilde{M}$, respectively.

3. The submanifolds admitting Ricci soliton of Kenmotsu manifolds

In this section, we deal with the submanifold M of Kenmotsu manifold $\tilde{M}$ endowed with torqued vector field $\mathcal{T}$.

From now on, we make the following:

Assumption. Throughout the paper, we suppose that the characteristic vector field ξ is tangent to M.

Theorem 3.1. *Let $\tilde{M}$ be a Kenmotsu manifold endowed with a torqued vector field $\mathcal{T}$. Then, the characteristic vector field ξ is never torqued vector field on $\tilde{M}$.*

Proof. Since $\mathcal{T}$ is a torqued vector field on $\tilde{M}$, then we have

$$\tilde{\nabla}_X \mathcal{T} = fX + \alpha(X)\mathcal{T} \qquad \text{and} \qquad \alpha(\mathcal{T}) = 0, \tag{3.1}$$

where $\tilde{\nabla}$ stands for the Levi-Civita connection on $\tilde{M}$, for any $X \in \Gamma(T\tilde{M})$.

Suppose that ξ is a torqued vector field on $\tilde{M}$. Using ξ instead of $\mathcal{T}$ in equation (3.1), one has

$$\tilde{\nabla}_X \xi = fX + \alpha(X)\xi \qquad \text{and} \qquad \alpha(\xi) = 0. \tag{3.2}$$

Also, taking the inner product of (3.2) with ξ, we have

$$\alpha(X) = -f\eta(X).$$

Therefore, the equation (3.2) reduces to

$$\nabla_X \xi = f(X - \eta(X)\xi). \tag{3.3}$$

It follows from (2.8) and (3.3),

$$f = 1 \qquad \text{and} \qquad \alpha(X) = -\eta(X) \tag{3.4}$$

are found.

On the other hand, if we take the characteristic vector field $X = \xi$ in (3.4), then we find

$$\alpha(\xi) = -1 \tag{3.5}$$

which is a contradiction. Hence, ξ is never torqued vector field on Kenmotsu manifold $\tilde{M}$. □

The next example supports Theorem 3.1, as follows:

Example 3.2. ([13]). We consider the three-dimensional Riemannian manifold

$$\tilde{M} = \{(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq (0, 0, 0)\},$$

and the linearly independent vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = -z\frac{\partial}{\partial z},$$

where (x, y, z) are the Cartesian coordinates in $\mathbb{R}^3$. Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_i, e_i) &= 1 \\ g(e_i, e_j) &= 0 \qquad for \qquad i \neq j. \end{aligned}$$

and is given by

$$g = \frac{1}{z^2}\Big\{dx \otimes dx + dy \otimes dy + dz \otimes dz\Big\}.$$

Also, let η, φ be the $1-$ form and the $(1, 1)-$tensor field, respectively defined by

$$\eta(Z, e_3) = 1, \quad \varphi(e_1) = -e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = 0$$

for any $Z \in \Gamma(T\tilde{M})$. Hence, $(\tilde{M}, \varphi, \xi, \eta, g)$ becomes an almost contact metric manifold with the characteristic vector field $e_3 = \xi$.

By direct calculations, we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1 \quad and \quad [e_2, e_3] = e_2.$$

On the other hand, using Koszul's formula for the Riemannian metric g, we have:

$$\tilde{\nabla}_{e_1} e_3 = e_1, \qquad \tilde{\nabla}_{e_2} e_3 = e_2, \qquad \tilde{\nabla}_{e_3} e_3 = 0 \tag{3.6}$$

and others

$$\tilde{\nabla}_{e_1}e_2 = \tilde{\nabla}_{e_2}e_1 = \tilde{\nabla}_{e_3}e_1 = \tilde{\nabla}_{e_3}e_2 = 0, \quad \tilde{\nabla}_{e_1}e_1 = \tilde{\nabla}_{e_2}e_2 = -e_3. \tag{3.7}$$

Therefore, the manifold $\tilde{M}$ is a $3-$dimensional Kenmotsu manifold. Now, we suppose that $e_3 = \xi$ is a torqued vector field on $\tilde{M}$. Then,

$$\tilde{\nabla}_{e_1}\xi = fe_1 + \alpha(e_1)\xi \quad \text{and} \quad \alpha(\xi) = 0 \tag{3.8}$$

$$\tilde{\nabla}_{e_2}\xi = fe_2 + \alpha(e_2)\xi \quad \text{and} \quad \alpha(\xi) = 0 \tag{3.9}$$

$$\tilde{\nabla}_{e_3}\xi = fe_3 + \alpha(e_3)\xi \quad \text{and} \quad \alpha(\xi) = 0. \tag{3.10}$$

are satisfied. From (3.6), (3.8), (3.9) and (3.10), we get

$$f = 1 \quad \text{and} \quad \alpha(e_3) = \alpha(\xi) = -1 \neq 0, \tag{3.11}$$

which is a contradiction. Therefore, $e_3 = \xi$ is never torqued vector field on Kenmotsu manifold $\tilde{M}$.

Considering Theorem 3.1, we get the following:

Remark 3.3. Let M be a submanifold endowed with a torqued vector field $\mathcal{T}$ of a Kenmotsu manifold $\tilde{M}$. Then, the characteristic vector field ξ is never torqued on M.

Next, we have the following theorem.

Theorem 3.4. *Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ endowed with a torqued vector field $\mathcal{T}$. The submanifold M is totally geodesic if and only if the tangential component $\mathcal{T}^\top$ of $\mathcal{T}$ is a torse-forming vector field on M whose conformal scalar is the restriction of the torqued function and whose generating form is the restriction of the torqued function of $\mathcal{T}$ on M.*

Proof. Since $\mathcal{T}$ is a torqued vector field on the ambient space $\tilde{M}$, it follows from (1.2), (2.14) and the formulas of Gauss and Weingarten, one has

$$\nabla_X \mathcal{T}^\top + h(X, \mathcal{T}^\top) - A_{\mathcal{T}^\perp}X + \nabla^\perp_X \mathcal{T}^\perp = fX + \alpha(X)\mathcal{T}^\top + \alpha(X)\mathcal{T}^\perp, \tag{3.12}$$

where ∇ stands for the Levi-Civita connection on M, for any $X \in \Gamma(TM)$. By comparing the tangential and normal components of (3.12), we get

$$\begin{aligned} h(X, \mathcal{T}^\top) + \nabla^\perp_X \mathcal{T}^\perp &= \alpha(X)\mathcal{T}^\perp, \\ \nabla_X \mathcal{T}^\top - A_{\mathcal{T}^\perp}X &= fX + \alpha(X)\mathcal{T}^\top. \end{aligned} \tag{3.13}$$

If M is a totally geodesic submanifold of $\tilde{M}$, then the equation (3.13) becomes

$$\nabla_X \mathcal{T}^\top = fX + \alpha(X)\mathcal{T}^\top, \tag{3.14}$$

which implies that $\mathcal{T}^\top$ is a torse-forming on M. The proof of the converse part is straightforward. □

Considering the equality (3.13), we have the following cases:

From now on, we suppose that the submanifold M admits a Ricci soliton in Theorem 3.4.

Case I: If we take $\mathcal{T}^\top \in \Gamma(D)$, then from (2.4), (2.9), (2.10) and (3.13) we get

$$g(\nabla_X \mathcal{T}^\top, \xi) = g(fX, \xi), \tag{3.15}$$

where $TM = D \oplus Span\{\xi\}$, for any $X \in \Gamma(TM)$. Since the Riemannian metric g is non-degenere, we have

$$\nabla_X \mathcal{T}^\top = fX, \tag{3.16}$$

which shows that the vector field $\mathcal{T}^\top$ is a concircular on M.

On the other hand, from the definition of Lie-derivative and (3.16) one has

$$\begin{aligned}(\pounds_{\mathcal{T}^\top} g)(X,Y) &= g(\nabla_X \mathcal{T}^\top, Y) + g(\nabla_Y \mathcal{T}^\top, X) \\ &= 2fg(X,Y) \end{aligned} \tag{3.17}$$

for any $X, Y \in \Gamma(TM)$, which means that the vector field $\mathcal{T}^\top$ is a conformal Killing. Also, from (1.1) and (3.17), we obtain

$$S(X,Y) = -(\lambda + f)g(X,Y),$$

where S is the Ricci tensor of M. Hence, M is an Einstein.

Case II: If we take $\mathcal{T}^\top \in \Gamma(D)$, then it follows from (3.15), we have

$$g(\nabla_X \mathcal{T}^\top, \xi) = 0 \tag{3.18}$$

for any $X \in \Gamma(D)$. As a consequence of the equation (3.18), $\mathcal{T}^\top$ is a parallel vector field on distribution D and thus, $\mathcal{T}^\top$ is a $D-$Killing vector field.

On the other side, using (1.1) and (3.18) the Ricci tensor S^D of the distribution D

$$S^D(X,Y) = -\lambda g(X,Y)$$

is found. Therefore, the distribution D is an Einstein.

Case III: If we use ξ instead of $\mathcal{T}^\top$ in (3.14), we have

$$\nabla_X \xi = fX + \alpha(X)\xi \tag{3.19}$$

for any $X \in \Gamma(TM)$. Taking the inner product of (3.19) with ξ, we get

$$g(\nabla_X \xi, \xi) = f\eta(X) + \alpha(X)$$

which yields

$$\alpha(X) = -f\eta(X).$$

It is easy to see that $\alpha(\xi) \neq 0$. So, ξ is a torse-forming on M.

Using the equality (3.13), we have the following:

Corollary 3.5. *Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ endowed with a torqued vector field $\mathcal{T}$. If M is $\mathcal{T}^\perp-$umbilical, then $\mathcal{T}^\top$ is a torse-forming on M.*

The next theorem gives a characterization as follows:

Theorem 3.6. *Let $\tilde{M}$ be a Kenmotsu manifold endowed with a torqued vector field $\mathcal{T}$ and M be a submanifold admitting a Ricci soliton of $\tilde{M}$. Then, (M, g, ξ, λ) is an $\eta-$Einstein.*

Proof. If we take ξ instead of $\mathcal{T}^\top$ in (3.13), we have

$$\nabla_X \xi - A_{\mathcal{T}^\perp} X = fX + \alpha(X)\xi. \tag{3.20}$$

From the equalities (2.4), (2.6) and (3.20), we get

$$A_{\mathcal{T}^\perp} X = (1 - f)X - (\eta(X) + \alpha(X))\xi. \tag{3.21}$$

Also, if we use the relations (2.1), (2.10) and (3.21), one has

$$g(h(X,Y), \mathcal{T}^\perp) = (1 - f)g(X,Y) - (\eta(X) + \alpha(X))\eta(Y). \tag{3.22}$$

Interchanging the roles of X and Y in (3.22) gives

$$g(h(Y,X), \mathcal{T}^\perp) = (1 - f)g(Y,X) - (\eta(Y) + \alpha(Y))\eta(X). \tag{3.23}$$

Since h and g are symmetric, from (3.22) and (3.23) we have

$$\begin{aligned} 2g(h(X,Y), \mathcal{T}^\perp) &= 2(1 - f)g(X,Y) - 2\eta(X)\eta(Y) \\ &\quad -\alpha(X)\eta(Y) - \alpha(Y)\eta(X) \end{aligned} \tag{3.24}$$

for any $X, Y \in \Gamma(TM)$.

On the other hand, from the definition of Lie-derivative and (2.1), (2.10), (3.20) and (3.24), we obtain

$$\begin{aligned}(\pounds_\xi g)(X,Y) &= g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)\\ &= g(fX+\alpha(X)\xi+A_{\mathfrak{T}^\perp}X,Y)\\ &\quad+g(fY+\alpha(Y)\xi+A_{\mathfrak{T}^\perp}Y,X)\\ &= 2g(X,Y)-2\eta(X)\eta(Y). \end{aligned} \tag{3.25}$$

Since M is a submanifold admitting a Ricci soliton and from the equalities (1.1) and (3.25), the Ricci tensor S of M

$$S(X,Y)=-(\lambda+1)g(X,Y)+\eta(X)\eta(Y) \tag{3.26}$$

is satisfied. This means M is an η-Einstein. □

As a consequence of Theorem 3.6, we can state the followings:

Corollary 3.7. *Let $\tilde{M}$ be a Kenmotsu manifold endowed with a torqued vector field $\mathfrak{T}$ and M be a submanifold admitting a Ricci soliton as its potential field ξ of $\tilde{M}$. Then, M has $\eta-$parallel Ricci tensor.*

Corollary 3.8. *Let $\tilde{M}$ be a Kenmotsu manifold endowed with a torqued vector field $\mathfrak{T}$ and M be a $n-$dimensional submanifold admitting a Ricci soliton as its potential field ξ of $\tilde{M}$. Then, M has constant scalar curvature r given by*

$$r=1-n(\lambda+1).$$

4. Ricci solitons in Kenmotsu space form with torqued vector field

In this section, we investigate the submanifolds admitting a Ricci soliton of Kenmotsu space form $\tilde{M}(c)$ endowed with torqued vector field $\mathfrak{T}$.

Now, we are ready to give the next theorem as follows:

Theorem 4.1. *Let $\tilde{M}(c)$ be a Kenmotsu space form and M be a $n-$dimensional submanifold of $\tilde{M}(c)$. If M is totally umbilical and the mean curvature $\|H\|$ is constant, then M is $\eta-$Einstein.*

Proof. Let $\{e_1,e_2,...,e_{n-1},\xi\}$ be an orthonormal basis of T_pM, $p\in M$. From the definition of the Ricci tensor, we have

$$S(Y,Z)=\sum_{i=1}^{n-1}g(R(e_i,Y)Z,e_i)+g(R(\xi,Y)Z,\xi), \tag{4.1}$$

where R is the Riemann curvature tensor of the submanifold M.

If we put $X = W = e_i$ in (2.11) and use the equalities (2.1), (2.2), (2.5), (2.9) and (2.12), then one has

$$\begin{aligned}
\sum_{i=1}^{n-1} g(R(e_i,Y)Z,,e_i) &= \sum_{i=1}^{n-1} g(\tilde{R}(e_i,Y)Z,e_i) - g(h(e_i,e_i),h(Y,Z)) \\
&\quad + g(h(e_i,Z),h(Y,e_i)) \\
&= \sum_{i=1}^{n-1} \frac{c-3}{4}\Big\{g(Y,Z)g(e_i,e_i) - g(e_i,Z)g(Y,e_i)\Big\} \\
&\quad + \frac{c+1}{4}\Big\{3g(e_i,\varphi Y)g(\varphi Z,e_i) - \eta(Y)\eta(Z)g(e_i,e_i)\Big\} \\
&\quad + \sum_{i=1}^{n-1}\Big(g(e_i,Z)g(Y,e_i) - (g(e_i,e_i)g(Y,Z)\Big)\|H\|^2 \\
&= \frac{c-3}{4}\Big\{(n-2)g(Y,Z) + \eta(Y)\eta(Z)\Big\} \\
&\quad + \frac{c+1}{4}\Big\{3g(Y,Z) - (n+2)\eta(Y)\eta(Z)\Big\} \\
&\quad + \Big((n-2)g(Y,Z)) + \eta(Y)\eta(Z)\Big)\|H\|^2 . \qquad (4.2)
\end{aligned}$$

Similarly, taking $X = W = \xi$ in (2.11), we get

$$g(R(\xi,Y)Z,\xi) = g(\tilde{R}(\xi,Y)Z,\xi) = \eta(Y)\eta(Z) - g(Y,Z) \qquad (4.3)$$

for any $Y, Z \in \Gamma(TM)$. Then, using (4.2) and (4.3) in (4.1), the Ricci tensor S of M

$$\begin{aligned}
S(Y,Z) &= \Big(\frac{c(n+1)-3n+5}{4} + (n-2)\|H\|^2\Big)g(Y,Z) \\
&\quad - \Big(\frac{c(n+1)+n+1}{4} - \|H\|^2\Big)\eta(Y)\eta(Z) \qquad (4.4)
\end{aligned}$$

is obtained which means that M is an $\eta-$Einstein. This completes the proof. □

Theorem 4.2. *Let $\tilde{M}(c)$ be a Kenmotsu space form endowed with a torqued vector field $\mathcal{T}$ and M be an $n-$dimensional $(n > 1)$ totally umbilical submanifold admitting a Ricci soliton of $\tilde{M}$. Then, M has a constant mean curvature.*

Proof. If we put $Y = Z = \xi$ in (3.26) and using (2.1) and (2.2), we get

$$S(\xi,\xi) = -\lambda. \qquad (4.5)$$

Similarly, if we take $Y = Z = \xi$ in (4.4) and also using (2.1) and (2.2), then we have

$$S(\xi,\xi) = (1-n)(1-\|H\|^2). \qquad (4.6)$$

Since M is a Ricci soliton, from the equalities (4.5) and (4.6),

$$\|H\|^2 = 1 - \frac{\lambda}{n-1} \qquad (4.7)$$

is obtained which completes the proof of the theorem. □

Using the equality (4.7), we can state the following corollary:

Corollary 4.3. *Let $\tilde{M}(c)$ be a Kenmotsu space form endowed with a torqued vector field $\mathcal{T}$ and M be an $n-$dimensional $(n > 1)$ totally umbilical submanifold admitting a Ricci soliton of $\tilde{M}$. Then, we have the following:*
i) *If $\|H\| < 1$, then the Ricci soliton (M, g, ξ, λ) is expanding.*
ii) *If $\|H\| > 1$, then the Ricci soliton (M, g, ξ, λ) is shrinking.*
iii) *The Ricci soliton (M, g, ξ, λ) is steady if and only if $\|H\| = 1$.*

5. Conclusion

Ricci soliton is a natural generalization of Einstein manifold. This notion corresponds to the self-similar solution of Hamilton's Ricci flow. Over the last decades, the geometry of Ricci solitons has been studied by many mathematicians. In 2008, Sharma applied Ricci solitons to $K-$contact manifolds and launched the study of Ricci solitons. Since then, Ricci solitons have been studied. In this paper, we deal with the submanifold admitting a Ricci soliton of a Kenmotsu manifold endowed with torqued vector field $\mathfrak{T}$. We find that the characteristic vector field ξ is never torqued on submanifold M of Kenmotsu manifold $\tilde{M}$. We obtain a necessary and sufficient condition for the tangential part $\mathfrak{T}^{\top}$ of $\mathfrak{T}$ to be a torse-forming on M. Also, we prove that if M admits a Ricci soliton, then it is an $\eta-$Einstein. Finally, we study the submanifold M admitting a Ricci soliton of a Kenmotsu space form $\tilde{M}(c)$ endowed with a torqued vector field $\mathfrak{T}$ and obtain that if M admits a Ricci soliton as its potential field ξ, then it is an expanding.

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