Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (3) (2020), 984 – 997
DOI : 10.15672/hujms.507488

Research Article

Asymptotically isometric copies of $\ell^{1\boxplus 0}$

Veysel Nezir

Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Turkey

Abstract

Using James' Distortion Theorems, researchers have inquired relations between spaces containing nice copies of c_0 or ℓ^1 and the failure of the fixed point property for nonexpansive mappings especially after the fact that every classical nonreflexive Banach space contains an isometric copy of either ℓ^1 or c_0. For instance, finding asymptotically isometric (ai) copies of ℓ^1 or c_0 inside a Banach space reveals the space's failure of the fixed point property for nonexpansive mappings. There has been many researches done using these tools developed by James and followed by Dowling, Lennard, and Turett mainly to see if a Banach space can be renormed to have the fixed point property for nonexpansive mappings when there is failure.

In this paper, we introduce the concept of Banach spaces containing ai copies of $\ell^{1\boxplus 0}$ and give alternative methods of detecting them. We show the relations between spaces containing these copies and the failure of the fixed point property for nonexpansive mappings. Finally, we give some remarks and examples pointing our vital result: if a Banach space contains an ai copy of $\ell^{1\boxplus 0}$, then it contains an ai copy of ℓ^1 but the converse does not hold.

Mathematics Subject Classification (2010). 46B45, 47H09, 46B42, 46B10

Keywords. fixed point property, nonexpansive mapping, renorming, asymptotically isometric copy of c_0, asymptotically isometric copy of ℓ^1

1. Introduction

A Banach space $(X, \|.\|)$ is said to have the fixed point property for nonexpansive mappings if every self-map T of any closed, bounded, and convex domain C in that space satisfying the condition $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$ has a fixed point.

It is a fact that either c_0 or ℓ^1 is almost isometrically embedded in any nonreflexive Banach spaces with an unconditional basis (see e.g. [7]). Thus, all of the classical nonreflexive spaces fail the fixed point property for nonexpansive mappings; that is, there exists a closed, bounded, and convex subset, and a nonexpansive self-map T defined on that set such that T is fixed point free. This result depends on well-known facts (Theorems 1.c.12 in [10] and 1.c.5 in [11]) stated by the following: a Banach lattice or a Banach space with an unconditional basis is reflexive if and only if it contains no isomorphic copies of c_0 or ℓ^1. Hence, if it can be shown that neither c_0 nor ℓ^1 can be renormed to have the fixed point property, it would follow that the fixed point property in either a Banach lattice or in a Banach space with an unconditional basis would imply reflexivity.

Email address: veyselnezir@yahoo.com
Received: 03.01.2019; Accepted: 18.06.2019

On this matter, for many years, researchers have asked the question whether or not either ℓ^1 or c_0 can be renormed so that the resulting space has the fixed point property for nonexpansive mappings. In the case of ℓ^1, there is a fact suggesting the contrary by Lin [9]. In [7], James showed that if a Banach space contains an isomorphic copy of ℓ^1 (respectively, c_0), then it contains almost isometric copies of ℓ^1 (respectively, c_0) and then he provided a tool that helped researches investigate the question of whether ℓ^1 or c_0 can be renormed to have the fixed point property for nonexpansive mappings. Using and strengthening this tool, Dowling, Lennard, and Turett, in several articles, have inquired relations between spaces containing nice copies of c_0 or ℓ^1 and the failure of the fixed point property for nonexpansive mappings.

Ai copies of the classical Banach spaces ℓ^1 and c_0 have applications in metric fixed point theory because they arise naturally in many places. For example, every non-reflexive subspace of $L^1[0,1]$, every closed infinite dimensional subspace of ℓ^1, and every equivalent renorming of ℓ^∞ contains an ai copy of ℓ^1 and so all of these spaces fail the fixed point property for nonexpansive mappings [3–5]. The concept of containing an ai copy of ℓ^1 also arises in the isometric theory of Banach spaces in an intriguing way: a Banach space X contains an asymptotically isometric copy ℓ^1 if and only if X^* contains an isometric copy of $L^1[0,1]$ [2].

In this paper, we aim to obtain an alternative property for a Banach space to contain an ai copy of ℓ^1. In our recent study [14], we investigated a renorming of ℓ^1 and noticed that an equivalent renorming of ℓ^1 turns out to produce a degenerate ℓ^1-analog Lorentz-Marcinkiewicz space $\ell_{\delta,1}$, where the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}} = (2,1,1,1,\cdots)$ is a decreasing positive sequence in $\ell^\infty \backslash c_0$, rather than in $c_0\backslash\ell^1$ (the usual Lorentz case). Our aforementioned work inspired us to construct the notion of ai copy of $\ell^{1\boxplus 0}$ which involves the combination of the usual norms of ℓ^1 and c_0. Therefore, we prefer to use $\ell^{1\boxplus 0}$ notation.

Then, following the researches by Dowling, Lennard, and Turett, first, we introduce the concept of Banach spaces containing ai copies of $\ell^{1\boxplus 0}$. Next, we provide alternative methods of recognizing this property. Finally, we give some remarks and examples that point our vital result: if a Banach space contains an ai copy of $\ell^{1\boxplus 0}$, then it contains an ai copy of ℓ^1 but the converse does not hold.

2. Preliminaries

In this section, we recall James' Distortion Theorems and some of the results given by Dowling, Lennard, and Turett including their findings for the concept of Banach spaces containing ai copy of ℓ^1 and those containing ai copy of c_0. Next, we give the definition of our property. Then, in the following subsection, we show examples of Banach spaces where this new property naturally arises.

Throughout the paper our scalar field is $\mathbb{R}$, c_0 represents the Banach space of scalar sequences converging to 0 and ℓ^1 stands for the Banach space of absolutely summable sequences.

That is,

$$c_0 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \lim_{n\longrightarrow\infty} x_n = 0 \right\}$$

such that its usual norm is given by $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$, for all $x = (x_n)_{n\in\mathbb{N}} \in c_0$; and

$$\ell^1 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \|x\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty \right\}.$$

Furthermore, in the paper, we will be using canonical basis $(e_n)_{n\in\mathbb{N}}$, given by 1 in its nth coordinate, and 0 in all other coordinates for each $n \in \mathbb{N}$, which is an unconditional basis for both $(c_0, \|\cdot\|_\infty)$ and $(\ell^1, \|\cdot\|_1)$.

Theorem 2.1. [7] *If a Banach space $(X, \|\cdot\|)$ contains an isomorphic copy of ℓ^1, then there exists a sequence $(x_n)_n$ in X such that for every $\varepsilon > 0$ and for all $(a_n)_n \in \ell^1$,*

$$(1-\varepsilon)\sum_{n=1}^{\infty}|a_n| \le \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \le \sum_{n=1}^{\infty}|a_n| \ .$$

Theorem 2.2. [7] *If a Banach space $(X, \|\cdot\|)$ contains an isomorphic copy of c_0, then there exists a sequence $(x_n)_n$ in X such that for every $\varepsilon > 0$ and for all $(a_n)_n \in \ell^1$,*

$$(1-\varepsilon)\sup_n |a_n| \le \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \le \sup_n |a_n| \ .$$

Definition 2.3. [3] We call a Banach space $(X, \|\cdot\|)$ contains an ai copy of ℓ^1 if there exist a sequence $(x_n)_n$ in X and a null sequence $(\varepsilon_n)_n$ in $(0,1)$ so that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| \le \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \le \sum_{n=1}^{\infty}|a_n| \ ,$$

for all $(a_n)_n \in \ell^1$.

Definition 2.4. [4] We call a Banach space $(X, \|\cdot\|)$ contains an ai copy of c_0 if there exist a sequence $(x_n)_n$ in X and a null sequence $(\varepsilon_n)_n$ in $(0,1)$ so that

$$\sup_n (1-\varepsilon_n)|a_n| \le \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \le \sup_n |a_n| \ ,$$

for all $(a_n)_n \in c_0$.

Then we can give the following theorem as the summary of the results in papers [3,4].

Theorem 2.5. *If a Banach space $(X, \|\cdot\|)$ contains an ai copy of ℓ^1 or an ai copy of c_0, then X fails the fixed point property for nonexpansive mappings.*

The following is the definition of our construction.

Definition 2.6. We will say that a Banach space $(X, \|\cdot\|)$ contains an ai copy of $\ell^{1\boxplus 0}$ if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X such that

$$\sup_{n\in\mathbb{N}} (1-\varepsilon_n)\,|a_n| + \sum_{n=1}^{\infty}(1-\varepsilon_n)\,|a_n| \le \left\|\sum_{n=1}^{\infty} a_n\, x_n\right\| \le \sup_{n\in\mathbb{N}} |a_n| + \sum_{n=1}^{\infty} |a_n| \ ,$$

for all $(a_n)_n \in \ell^1$.

2.1. The space and its fixed point properties behind the notion of ai copy of $\ell^{1\boxplus 0}$

In this section, we introduce some renormings of ℓ^1 and we notice that the resulting spaces are some degenerate Lorentz-Marinkiewicz spaces.

As we stated in the introduction section, we have recently constructed an equivalent renorming of ℓ^1 which turns out to produce a degenerate ℓ^1-analog Lorentz-Marcinkiewicz space $\ell_{\delta,1}$. In the same work [14], we obtained its isometrically isomorphic predual $\ell^0_{\delta,\infty}$ and dual $\ell_{\delta,\infty}$, corresponding degenerate c_0-analog and ℓ^∞-analog Lorentz-Marcinkiewicz spaces, respectively. Then, we investigated some types of fixed point properties such as weak and regular fixed point properties.

Furthermore, very recently in [15], generalizing our work [14] by constructing another equivalent norm on ℓ^1 and obtaining our generalized degenerate ℓ^1-analog Lorentz-Marcinkiewicz space $\ell_{\delta,1}$, where the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}} = (\alpha+\beta, \beta, \beta, \cdots)$ for $\beta \ge \alpha > 0$, we have showed that $\ell_{\delta,1}$ has the weak fixed point property but fails to have the fixed point property for nonexpansive mappings.

Now, we will consider these two new spaces after reminding definitions of Lorentz-Marcinkiewicz spaces because our results derive from these renormings of ℓ^1 space.

We should note that in the author's Ph.D. thesis [13], written under supervisor of Chris Lennard, the usual Lorentz-Marcienkiewicz spaces and their fixed point properties were studied; hence, we can give definitions of the usual Lorentz-Marcienkiewicz spaces below to understand how different the degenerate ones are.

Let $w \in (c_0 \backslash \ell^1)^+$, $w_1 = 1$, and $(w_n)_{n\in\mathbb{N}}$ be decreasing; that is, consider a scalar sequence given by $w = (w_n)_{n\in\mathbb{N}}$, $w_n > 0, \forall n \in \mathbb{N}$ such that $1 = w_1 \geq w_2 \geq w_3 \geq \cdots \geq w_n \geq w_{n+1} \geq \ldots, \forall n \in \mathbb{N}$ with $w_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $\sum\limits_{n=1}^{\infty} w_n = \infty$. This sequence is called a weight sequence. For example, $w_n = \frac{1}{n}, \forall n \in \mathbb{N}$.

Definition 2.7. $l_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \;\middle|\; \|x\|_{w,\infty} := \sup\limits_{n\in\mathbb{N}} \dfrac{\sum\limits_{j=1}^{n} {x_j}^\star}{\sum\limits_{j=1}^{n} w_j} < \infty \right\}$.

Here, $x^\star$ represents the decreasing rearrangement of the sequence x, which is the sequence of $|x| = (|x_j|)_{j\in\mathbb{N}}$, arranged in a non-increasing order, followed by infinitely many zeros when $|x|$ has only finitely many non-zero terms. This space is non-separable and an analogue of ℓ_∞ space.

Definition 2.8. $l^0_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \;\middle|\; \limsup\limits_{n\longrightarrow\infty} \dfrac{\sum\limits_{j=1}^{n} {x_j}^\star}{\sum\limits_{j=1}^{n} w_j} = 0 \right\}$.

This is a separable subspace of $\ell_{w,\infty}$ and an analogue of c_0 space.

Definition 2.9. $l_{w,1} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \;\middle|\; \|x\|_{w,1} := \sum\limits_{j=1}^{\infty} w_j \, {x_j}^\star < \infty \right\}$.

This is a separable subspace of $\ell_{w,\infty}$ and an analogue of ℓ_1 space with following facts: $(\ell^0_{w,\infty})^\star \cong l_{w,1}$ and $(\ell_{w,1})^\star \cong l_{w,\infty}$ where the star denotes the dual of a space while $\cong$ denotes isometrically isomorphic.

More information about these spaces can be seen in [10, 12].

Now, we will introduce our construction given in [14]. For all $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$, we define $|||x||| := \|x\|_1 + \|x\|_\infty = \sum\limits_{n=1}^{\infty} |x_n| + \sup\limits_{n\in\mathbb{N}} |x_n|$. Clearly $|||\cdot|||$ is an equivalent norm on ℓ^1 with $\|x\|_1 \leq |||x||| \leq 2\|x\|_1$, $\forall x \in \ell^1$. Note that $\forall x \in \ell^1$, $|||x||| = 2x_1^* + x_2^* + x_3^* + x_4^* + \cdots$ where z^* is the decreasing rearrangement of $|z| = (|z_n|)_{n\in\mathbb{N}}$, $\forall z \in c_0$. Let $\delta_1 := 2, \delta_2 := 1, \delta_3 := 1, \cdots, \delta_n := 1$, $\forall n \geq 4$. We see that $(\ell^1, |||\cdot|||)$ is a (degenerate) Lorentz space $\ell_{\delta,1}$, where the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}}$ is a decreasing positive sequence in $\ell^\infty \backslash c_0$, rather than in $c_0 \backslash \ell^1$ (the usual Lorentz case).

Generalizing this construction, in [15], we constructed another equivalent norm as the following: let $\beta \geq \alpha > 0$. For all $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$, we define $|||x||| := \beta\|x\|_1 + \alpha\|x\|_\infty = \beta \sum\limits_{n=1}^{\infty} |x_n| + \alpha \sup\limits_{n\in\mathbb{N}} |x_n|$. Clearly $|||\cdot|||$ is an equivalent norm on ℓ^1 with $\beta\|x\|_1 \leq |||x||| \leq (\alpha+\beta)\|x\|_1$, $\forall x \in \ell^1$. Note that $\forall x \in \ell^1$, $|||x||| = \beta\left(\frac{\alpha+\beta}{\beta} x_1^* + x_2^* + x_3^* + x_4^* + \cdots\right)$ where z^* is the decreasing rearrangement of $|z| = (|z_n|)_{n\in\mathbb{N}}$, $\forall z \in c_0$. Let $\delta_1 := (\alpha+\beta), \delta_2 := \beta, \delta_3 := \beta, \cdots, \delta_n := \beta$, $\forall n \geq 4$. Then, we see that $(\ell^1, |||\cdot|||)$ is a (degenerate) Lorentz space $\ell_{\delta,1}$.

This suggests that $\ell^0_{\delta,\infty} = (c_0, \|\cdot\|)$ is an isometric predual of $(\ell^1, |||\cdot|||)$ where for all $z \in c_0$,

$$\|z\| := \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^{n} z_j^*}{\sum_{j=1}^{n} \delta_j} = \sup_{n\in\mathbb{N}} \frac{1}{\alpha + n\beta} \sum_{j=1}^{n} z_j^*.$$

But there is a way to re-write $\|z\|$ without using decreasing rearrangements of $|z|$. This may help with calculations involving this norm.

Fix $z \in c_0$, arbitrary. Note that $\forall n \in \mathbb{N}$, $\sum_{j=1}^{n} z_j^* = \sup_{\substack{K\subseteq\mathbb{N}\\ \#(K)=n}} \sum_{i\in K} |z_i|$, where $\#(K)$ is the number of elements in K for all finite subsets $K \subseteq \mathbb{N}$.

Thus, $\|z\| = \sup_{n\in\mathbb{N}} \frac{1}{\alpha+n\beta} \sup_{\substack{K\subseteq\mathbb{N}\\ \#(K)=n}} \sum_{i\in K} |z_i| = \sup_{n\in\mathbb{N}} \sup_{\substack{K\subseteq\mathbb{N}\\ \#(K)=n}} \frac{1}{\alpha+\#(K)\beta} \sum_{i\in K} |z_i|$.

Hence, for all $z \in c_0$,

$$\|z\| = \sup_{\substack{\emptyset\neq K\subseteq\mathbb{N}\\ \#(K)<\infty}} \frac{1}{\alpha + \#(K)\beta} \sum_{i\in K} |z_i|. \tag{2.1}$$

Also, note that the formula (2.1) can be extended to ℓ^∞: $\forall w = (w_i)_{i\in\mathbb{N}} \in \ell^\infty$, we define

$$\|w\| := \sup_{\substack{\emptyset\neq K\subseteq\mathbb{N}\\ \#(K)<\infty}} \frac{1}{\alpha + \#(K)\beta} \sum_{i\in K} |w_i|. \tag{2.2}$$

It is easy to see that dual space of $(\ell^1, |||\cdot|||)$ is isometrically isomorphic to $(\ell^\infty, \|\cdot\|)$; i.e., $(\ell^1, |||\cdot|||)^* \cong (\ell^\infty, \|\cdot\|)$.

In the following sections, we will see that our new notion is related with these spaces.

3. Main results

In this section, we will give our main results. Recall that we introduced our property in the Definition 2.6. We will show that our property is an alternative property for a Banach space to contain an ai copy of ℓ^1 and to do that, we will be working on its generalized version introduced below.

Definition 3.1. We will say that a Banach space $(X, \|\cdot\|)$ contains an ai copy of $\ell^{1(\beta)\boxplus 0(\alpha)}$ if there exist $\beta \geq \alpha > 0$, a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that

$$\alpha \sup_{n\in\mathbb{N}} (1-\varepsilon_n)\,|a_n| + \beta \sum_{n=1}^{\infty} (1-\varepsilon_n)\,|a_n| \leq \left\| \sum_{n=1}^{\infty} a_n\, x_n \right\| \leq \alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n|\,,$$

for all $(a_n)_n \in \ell^1$.

Note that indeed the Definition 2.6 implies Definition 3.1. Now using the Definition 2.6, the next theorem conclude that If a Banach Space $(X, \|\cdot\|)$ contains an ai copy of $\ell^{1\boxplus 0}$ then it fails to have the fixed point property for $\|\cdot\|$-nonexpansive mappings, but we will also prove the same conclusion by a direct way.

Theorem 3.2. *If a Banach Space $(X, \|\cdot\|)$ contains an ai copy of $\ell^{1(\beta)\boxplus 0(\alpha)}$ for some $\beta \geq \alpha > 0$ then it contains an ai copy of ℓ^1 and it fails to have the fixed point property for $\|\cdot\|$-nonexpansive mappings.*

Proof. Indeed, suppose there exist $\beta \geq \alpha > 0$ and a null sequence $(\varepsilon_n)_n$ in $(0,1)$, and a sequence $(x_n)_n$ in X such that for all $(a_n)_n \in \ell^1$,

$$\alpha \sup_{n\in\mathbb{N}} (1-\varepsilon_n)\,|a_n| + \beta \sum_{n=1}^{\infty} (1-\varepsilon_n)\,|a_n| \leq \left\| \sum_{n=1}^{\infty} a_n\, x_n \right\| \leq \alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n| \ . \quad (3.1)$$

Fix an increasing sequence $(m_i)_i$ in $\mathbb{N}$ and define for each $k \in \mathbb{N}$, $M_k := \sum_{i=1}^{k} m_i$ with $M_0 := 0$ and $y_k = \frac{1}{\beta(1+m_k)} \sum_{n=M_{k-1}+1}^{M_k} x_n$.

Then by (3.1), we get for each $k \in \mathbb{N}$,

$$\begin{aligned} \|y_k\| &\leq \frac{1}{\beta}\frac{1}{m_k+1}(\alpha + \beta(M_k - M_{k-1})) = \frac{1}{m_k+1}\left(\frac{\alpha}{\beta} + m_k\right) \\ &\leq \frac{1}{m_k+1}(1+m_k) = 1, \end{aligned}$$

and thus, for all $(a_n)_n \in \ell^1$, $\left\| \sum_{n=1}^{\infty} a_n\, y_n \right\| \leq \sum_{n=1}^{\infty} |a_n|$.

On the other hand, we have

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} a_n\, y_n \right\| &= \frac{\alpha}{\beta} \left\| \sum_{n=1}^{\infty} a_n \frac{1}{m_n+1} \sum_{j=M_{n-1}+1}^{M_n} \frac{x_j}{\alpha} \right\| \\ &\geq \frac{\alpha}{\beta} \left[\begin{array}{l} \alpha \sup\limits_{n\in\mathbb{N}} |a_n| \left(\frac{1}{m_n+1} \sum\limits_{j=M_{n-1}+1}^{M_n} \frac{(1-\varepsilon_j)}{\alpha} \right) \\ +\beta \sum\limits_{n=1}^{\infty} |a_n| \left(\frac{1}{m_n+1} \sum\limits_{j=M_{n-1}+1}^{M_n} \frac{(1-\varepsilon_j)}{\alpha} \right) \end{array} \right] \\ &\geq \sum_{n=1}^{\infty} |a_n| \left(\frac{1}{m_n+1} \sum_{j=M_{n-1}+1}^{M_n} (1-\varepsilon_j) \right) \\ &\geq \sum_{n=1}^{\infty} |a_n| \left(\frac{m_n}{m_n+1} (1-\varepsilon_{M_{n-1}+1}) \right). \end{aligned}$$

Hence, since $\lim_{k\to\infty} m_k = \infty$, there exists a null sequence $(\varepsilon_n^{'})_n$ in $(0,1)$ such that for all $k \in \mathbb{N}$, $1 - \varepsilon_k^{'} = \frac{m_k}{m_k+1}(1-\varepsilon_{M_{k-1}+1})$ and therefore $(y_k)_k$ is an ai ℓ^1 sequence in X. ☐

Remark 3.3. One can also prove that if a Banach Space $(X, \|\cdot\|)$ contains an ai copy of $\ell^{1\boxplus 0}$ then it fails to have the fixed point property for $\|\cdot\|$-nonexpansive mappings without using the fact given in previous theorem by a similar method to Theorem 1.2 in [3].

The following result gives alternative methods of recognizing ai copies of $\ell^{1\boxplus 0}$.

Theorem 3.4. *A Banach space $(X, \|\cdot\|)$ contains an ai copy of $\ell^{1(\beta)\boxplus 0(\alpha)}$ for some $\beta \geq \alpha > 0$ if and only if there is a sequence $(x_n)_n$ in X such that*

(1) *there are constants $A \geq \frac{\alpha+\beta}{\beta}$ and $0 < r \leq R \leq \frac{r(2\beta A - (\alpha+\beta)) + \sqrt{(\alpha+\beta-2\beta A)^2 r^2 + 8(\alpha+\beta)^2 r^2}}{4(\alpha+\beta)}$ so that for all $(a_n)_n \in \ell^1$,*

$$r\left(\alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n|\right) \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq R\left(\alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n|\right),$$

and

(2) $\lim_{n\to\infty} \|x_n\| = \frac{(R+r)(\alpha+\beta)}{2A}$.

Proof. Suppose that $(X, \|\cdot\|)$ contains an ai copy of $\ell^{1(\beta)\boxplus 0(\alpha)}$ for some $\beta \geq \alpha > 0$. Then there are a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(y_n)_n$ in X such that for all $(a_n)_n \in \ell^1$,

$$\alpha \sup_{n\in\mathbb{N}} (1-\varepsilon_n)|a_n| + \beta \sum_{n=1}^{\infty} (1-\varepsilon_n)|a_n| \leq \left\| \sum_{n=1}^{\infty} a_n y_n \right\| \leq \alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n| \ .$$

Let $x_n = (1-\varepsilon_n)^{-1} y_n$ for each $n \in \mathbb{N}$.

Then for all $(a_n)_n \in \ell^1$,

$$\begin{aligned} \alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n| &\leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \\ &\leq \alpha \sup_{n\in\mathbb{N}} (1-\varepsilon_n)^{-1}|a_n| + \beta \sum_{n=1}^{\infty} (1-\varepsilon_n)^{-1}|a_n| \\ &\leq \frac{1}{1-\varepsilon_1} \left(\alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n| \right) \ . \end{aligned}$$

Also, since $(\alpha+\beta) \leq \|x_n\| \leq \frac{(\alpha+\beta)}{1-\varepsilon_n}$ by the inequality above, $\lim_{n\to\infty} \|x_n\| = (\alpha+\beta)$.

Hence, conditions (1) and (2) hold for $r = 1$, $R = \frac{1}{1-\varepsilon_1}$ and $A = \frac{1+\frac{1}{1-\varepsilon_1}}{2}$.

Conversely, assume that conditions (1) and (2) hold.

Then, there are constants $A \geq \frac{\alpha+\beta}{\beta}$ and

$0 < r \leq R \leq \frac{r(2\beta A - (\alpha+\beta)) + \sqrt{(\alpha+\beta-2\beta A)^2 r^2 + 8(\alpha+\beta)^2 r^2}}{4(\alpha+\beta)}$ so that for all $(a_n)_n \in \ell^1$,

$$r\left(\alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n| \right) \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq R\left(\alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n| \right) ,$$

and $\lim_{n\to\infty} \|x_n\| = \frac{(R+r)(\alpha+\beta)}{2A}$.

Then, $\lim_{n\to\infty} \|\frac{2rA}{r+R} x_n\| = (\alpha+\beta)r$ and for all $(a_n)_n \in \ell^1$,

since $R \leq \frac{r(2\beta A - (\alpha+\beta)) + \sqrt{(\alpha+\beta-2\beta A)^2 r^2 + 8(\alpha+\beta)^2 r^2}}{4(\alpha+\beta)} \leq (2A-1)r$,

$$\begin{aligned} r\left(\alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n| \right) &\leq \frac{2r^2 A}{r+R} \left(\alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n| \right) \\ &\leq \left\| \sum_{n=1}^{\infty} a_n \frac{2rA}{r+R} x_n \right\| \\ &\leq \frac{2rRA}{r+R} \left(\alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n| \right) \ . \end{aligned}$$

Now, define $y_n := \frac{2rA}{r+R} x_n, \ \forall n \in \mathbb{N}$.

Note that $\lim_{n\to\infty} \|y_n\| = (\alpha+\beta)r$ and for all $n \in \mathbb{N}$, $(\alpha+\beta)r \leq \|y_n\| \leq \frac{2(\alpha+\beta)rRA}{r+R}$.

Fix a null sequence $(\varepsilon_n)_n$ in $(0,1)$.

Then, since for all $n \in \mathbb{N}$, $2r \leq \|y_n\| \leq \frac{2(\alpha+\beta)rRA}{r+R}$ and $\lim_{n\to\infty} \|y_n\| = (\alpha+\beta)r$, passing to a subsequence of $(\varepsilon_n)_n$ as well, we can suppose that there exists a subsequence $(y_{n_k})_k$ such that $(\alpha+\beta)r \leq \|y_{n_k}\| \leq (\alpha+\beta)r(1+\varepsilon_k) \leq \frac{2r(\alpha+\beta)RA}{r+R}$ for all $k \in \mathbb{N}$.

Now, define $z_k := \frac{y_{n_k}}{(\alpha+\beta)r(1+\varepsilon_k)} = \frac{2Ax_{n_k}}{(r+R)(\alpha+\beta)(1+\varepsilon_k)}$ for all $k \in \mathbb{N}$.

Then, since $\|z_k\| \leq 1$ for all $k \in \mathbb{N}$ (from the inequality above), we have

$$\left\|\sum_{n=1}^{\infty} a_n z_n\right\| \leq \sum_{n=1}^{\infty} |a_n| \text{ for all } (a_n)_n \in \ell^1 .$$

Now, consider the sequence $(u_k)_k$ given by $u_k := \frac{x_{n_k}}{2R} + \frac{\beta z_k}{2}$, $\forall k \in \mathbb{N}$.
Then, for all $(a_n)_n \in \ell^1$,

$$\alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n| \ \geq \left\|\sum_{n=1}^{\infty} a_n u_n\right\|. \tag{3.2}$$

Also, for all $(a_n)_n \in \ell^1$,

$$\begin{aligned}
\left\|\sum_{k=1}^{\infty} a_n u_n\right\| &= \left\|\sum_{k=1}^{\infty} a_k \left(\frac{x_{n_k}}{2R} + \frac{\beta z_k}{2}\right)\right\| \\
&= \left\|\sum_{k=1}^{\infty} a_k \left(\frac{1}{2R} + \frac{\beta A}{(\alpha+\beta)(1+\varepsilon_k)(r+R)}\right) x_{n_k}\right\| \\
&\geq r\alpha \sup_{n\in\mathbb{N}} |a_n| \left(\frac{(1+\varepsilon_n)}{2R} + \frac{\beta A}{(\alpha+\beta)(r+R)}\right)\frac{1}{(1+\varepsilon_n)} \\
&\quad + r\beta \sum_{n=1}^{\infty} |a_n| \ \left(\frac{(1+\varepsilon_n)}{2R} + \frac{\beta A}{(\alpha+\beta)(r+R)}\right)\frac{1}{(1+\varepsilon_n)} \\
&\geq \left(\frac{r}{2R} + \frac{\beta A r}{(\alpha+\beta)(r+R)}\right)\left(\alpha \sup_{n\in\mathbb{N}} \frac{|a_n|}{(1+\varepsilon_n)} + \beta \sum_{n=1}^{\infty} \frac{|a_n|}{(1+\varepsilon_n)}\right) \\
&\geq \alpha \sup_{n\in\mathbb{N}} \frac{|a_n|}{(1+\varepsilon_n)} + \beta \sum_{n=1}^{\infty} \frac{|a_n|}{(1+\varepsilon_n)} \\
&\geq \alpha \sup_{n\in\mathbb{N}} |a_n|(1-\varepsilon_n) + \beta \sum_{n=1}^{\infty} |a_n|(1-\varepsilon_n)
\end{aligned}$$

since $R \leq \frac{r(2\beta A-(\alpha+\beta))+\sqrt{(\alpha+\beta-2\beta A)^2 r^2+8(\alpha+\beta)^2 r^2}}{4(\alpha+\beta)}$.

Thus, from two inequalities above, we have

$$\begin{aligned}
\alpha \sup_{n\in\mathbb{N}} |a_n|(1-\varepsilon_n) + \beta \sum_{n=1}^{\infty} |a_n|(1-\varepsilon_n) &\leq \left\|\sum_{n=1}^{\infty} a_n y_n\right\| \\
&\leq \alpha \sup_{n\in\mathbb{N}} |a_n| + \beta \sum_{n=1}^{\infty} |a_n|
\end{aligned}$$

for all $(a_n)_n \in \ell^1$.

Hence, $(X, \|\cdot\|)$ contains an ai copy of $\ell^{1(\beta)\boxplus 0(\alpha)}$ for some $\beta \geq \alpha > 0$. □

By taking $\alpha = \beta = 1$ in Theorem 3.4, we arrive at the following corollary.

Corollary 3.5. *A Banach space $(X, \|\cdot\|)$ contains an ai copy of $\ell^{1\boxplus 0}$ if and only if there is a sequence $(x_n)_n$ in X such that*

(1) *there are constants $A \geq 2$ and $0 < r \leq R \leq \frac{r(A-1)+\sqrt{(A-1)^2 r^2+8r^2}}{4}$ such that for all $(a_n)_n \in \ell^1$,*

$$r\left(\sup_{n\in\mathbb{N}} |a_n| + \sum_{n=1}^{\infty} |a_n|\right) \ \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq R\left(\sup_{n\in\mathbb{N}} |a_n| + \sum_{n=1}^{\infty} |a_n|\right),$$

and

(2) $\lim_{n\to\infty} \|x_n\| = \frac{R+r}{A}$.

4. Some examples and remarks

In this section, we present important examples such that the final example points our vital result: if a Banach space contains an ai copy of $\ell^{1\boxplus 0}$, then it contains an ai copy of ℓ^1 but the converse does not hold.

But firstly, we would like to give some interesting remarks noting a confusion may occur as follows. From our definition of ai copy of $\ell^{1\boxplus 0}$, one could ask what would be the isomorphic copy of $\ell^{1\boxplus 0}$ which could be confused with the concept of isomorphic copy of $\ell^1 \oplus c_0$.

As a result of valuable discussions with Lennard [8], we will provide the following remarks with examples. First, we can define a Banach space containing an isomorphic copy of $\ell^1 \oplus c_0$ as below but we leave further applications open.

Definition 4.1. We say that a Banach space $(X, \|\cdot\|)$ contains an isomorphic copy Y of $\ell^1 \oplus c_0$ if $\exists (x_n)_{n\in\mathbb{N}} \in X$ and $(z_n)_{n\in\mathbb{N}} \in X$ with $[x_n]_{n\in\mathbb{N}} \cap [z_n]_{n\in\mathbb{N}} = \{0\}$ (where $[x_n]_{n\in\mathbb{N}}$ and $[z_n]_{n\in\mathbb{N}}$ are closed linear spans of $\{x_n \; n \in \mathbb{N}\}$ and $\{z_n \; n \in \mathbb{N}\}$, respectively) and $0 < A \le B < \infty$ such that $\forall s = (s_n)_{n\in\mathbb{N}} \in \ell^1$ and $\forall t = (t_n)_{n\in\mathbb{N}} \in c_0$,

$$A\left(\sum_{n=1}^{\infty}|s_n| + \sup_n |t_n|\right) \le \left\|\sum_{n=1}^{\infty} s_n x_n + \sum_{n=1}^{\infty} t_n z_n\right\| \le B\left(\sum_{n=1}^{\infty}|s_n| + \sup_n |t_n|\right).$$

Note that $V := [x_n]_{n\in\mathbb{N}}$ is an isomorphic copy of ℓ^1 inside $(X, \|\cdot\|)$ and $W := [z_n]_{n\in\mathbb{N}}$ is an isomorphic copy of c_0 inside $(X, \|\cdot\|)$ such that $V \cap W = \{0\}$.

Remark 4.2. From the Definition 4.1, one can obtain that if a Banach space $(X, \|\cdot\|)$ contains an isomorphic copy Y of $\ell^1 \oplus c_0$, by letting $s_n = t_n$ and by re-labelling $x_n + z_n$, $\forall n \in \mathbb{N}$, then

$$\left[\begin{array}{l} \text{there exist constants } 0 < r \le R < \infty \text{ and there exists a sequence} \\ (x_n)_n \text{ in } Y \text{ so that} \\ \frac{1}{2}r\left(\sum_{n=1}^{\infty}|a_n| + \sup_n |a_n|\right) \le \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \\ \qquad\qquad\qquad\qquad \le \frac{1}{2}R\left(\sum_{n=1}^{\infty}|a_n| + \sup_n |a_n|\right) \end{array}\right] \tag{4.1}$$

where $A = \frac{r}{2}$ and $B = \frac{R}{2}$.

Remark 4.3. Let $(X, \|\cdot\|)$ be a Banach space. Then, we cannot obtain conclusions similarly to those of James' Theorem from statement (4.1).

That is, generally statement (4.1) does not imply the following statement: if $\varepsilon > 0$, then there exists a sequence $(x_n)_n$ in X so that

$$\frac{1}{2}(1-\varepsilon)\left(\sum_{n=1}^{\infty}|a_n| + \sup_n |a_n|\right) \le \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \le \frac{1}{2}\left(\sum_{n=1}^{\infty}|a_n| + \sup_n |a_n|\right),$$

for all $(a_n)_n \in \ell$.

Indeed, two statements above are very different.

Now consider $(X, \|\cdot\|) = (\ell^1, \|\cdot\|_1)$. Let $x_n = e_n$, $\forall n \in \mathbb{N}$.

Then, X satisfies (4.1):

$$\begin{aligned} \frac{1}{2}\left(\sum_{n=1}^{\infty}|a_n| + \sup_n |a_n|\right) &\le \left\|\sum_{n=1}^{\infty} a_n x_n\right\|_1 \\ &\le \sum_{n=1}^{\infty}|a_n| + \sup_n |a_n|, \end{aligned}$$

for all $(a_n)_n \in \ell^1$.

Then, fix $\varepsilon \in \left(0,\ \frac{1}{6}\right)$ and assume $\exists (z_n = x_n{}^{\sim})_n$ in $(\ell^1, \|\cdot\|_1)$ such that

$$\begin{aligned} \forall (a_n)_n \in \ell^1,\ \frac{1}{2}(1-\varepsilon)\left(\sum_{n=1}^{\infty}|a_n| + \sup_n |a_n|\right) &\leq \left\|\sum_{n=1}^{\infty} a_n z_n\right\|_1 \\ &\leq \frac{1}{2}\left(\sum_{n=1}^{\infty}|a_n| + \sup_n |a_n|\right). \end{aligned} \tag{4.2}$$

Then, easily it can be seen that

$$1-\varepsilon \leq \|z_n\|_1 \leq 1,\ \forall n \in \mathbb{N}. \tag{4.3}$$

We note that $(e_n)_n$ is a Schauder basis for $(\ell^1, \|\cdot\|_1)$.

By the Bessaga-Pełczyński Selection Principle [1, p.46] with $x_n = e_n$, $x_n^* \in \ell^\infty \cong (\ell^1)^*$ & $y_n = z_n$ and using also (4.3), there exists a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ of $(z_n)_{n\in\mathbb{N}}$ that is equivalent to a block basic sequence $(q_k)_{k\in\mathbb{N}}$ of $(e_k)_{k\in\mathbb{N}}$ which can be written as $q_k = \sum\limits_{m_k+1}^{m_{k+1}} \alpha_k e_k$ where $m_k \in \mathbb{N}$, $1 \leq m_1 < m_2 < m_3 < \cdots$ and each $\alpha_k \in \mathbb{R}$.

Moreover, one can show that in $(\ell^1, \|\cdot\|_1)$, by passing to a subsequence if necessary, we can choose $(q_k)_{k\in\mathbb{N}}$ so that $\varepsilon_k := \|z_{n_k} - q_k\|_1 \longrightarrow 0$ as $k \longrightarrow \infty$ and (♣) $\varepsilon_n \in (0, \varepsilon)$, $\forall n \in \mathbb{N}$.

Fix an arbitrary $(b_j)_{j\in\mathbb{N}} \in \ell^1$. From (4.2),

$$\left\|\sum_{k=1}^{\infty} b_k z_{n_k}\right\|_1 \leq \frac{1}{2}\left(\sum_{k=1}^{\infty}|b_k| + \sup_k |b_k|\right). \tag{4.4}$$

Also,

$$\left\|\sum_{k=1}^{\infty} b_k q_k\right\|_1 = \sum_{k=1}^{\infty}|b_k|\|q_k\|_1 \geq \sum_{k=1}^{\infty}|b_k|\left(\|z_{n_k}\|_1 - \varepsilon_k\right) \geq \sum_{k=1}^{\infty}|b_k|\left(1-\varepsilon-\varepsilon\right)$$

by (♣) and (4.3).

Furthermore, from the last inequality and (4.4),

$$\begin{aligned} (1-2\varepsilon)\sum_{k=1}^{\infty}|b_k| &\leq \left\|\sum_{k=1}^{\infty} b_k q_k\right\|_1 \\ &\leq \left\|\sum_{k=1}^{\infty} b_k z_{n_k}\right\|_1 + \left\|\sum_{k=1}^{\infty} b_k (q_k - z_{n_k})\right\|_1 \\ &\leq \frac{1}{2}\left(\sum_{k=1}^{\infty}|b_k| + \sup_k |b_k|\right) + \sum_{k=1}^{\infty}|b_k|\varepsilon_k \\ &\leq \frac{1}{2}\sum_{k=1}^{\infty}|b_k| + \frac{1}{2}\sup_k |b_k| + \varepsilon\sum_{k=1}^{\infty}|b_k|. \end{aligned}$$

Thus,

$$\sum_{k=1}^{\infty}|b_k| \leq \frac{1}{1-6\varepsilon}\sup_k |b_k|,\ \forall b = (b_k)_{k\in\mathbb{N}} \in \ell^1.$$

Now, fix $N \in \mathbb{N}$ arbitrary and let $b := \left(\underbrace{\frac{1}{N}, \frac{1}{N}, \cdots, \frac{1}{N}}_{N \text{ times}}, 0, 0, \cdots, 0, \cdots\right)$.

Then, $\|b\|_1 = \sum\limits_{k=1}^{\infty}|b_k| = 1$ and $\|b\|_\infty = \sup\limits_k |b_k| = \frac{1}{N}$.

Therefore, $1 \leq \frac{1}{1-6\varepsilon}\frac{1}{N}$, $\forall N \in \mathbb{N}$ but this is clearly a contradiction.

Now we will consider examples of spaces containing ai copy of $\ell^{1\boxplus 0}$ and those of not containing an ai copy of $\ell^{1\boxplus 0}$.

Example 4.4. Clearly, the degenerate Lorentz-Marcinkiewicz space $(\ell^1, |||\cdot|||) = (\ell_{\delta,1}, |||\cdot|||)$ given in section 2.1 with the weight sequence $\delta := (2, 1, 1, 1, ...)$ contains an ai copy of $\ell^{1\boxplus 0}$ and in fact is an isometric copy.

Now, in the next example, we will consider Lin's norm in generalized form firstly constructed by Dowling, Johnson, Lennard, and Turett [6].

Example 4.5. Dowling, Johnson, Lennard, and Turett [6] constructed the following equivalent norm $|||\cdot|||^{\sim}$ on ℓ^1 and showed that $(\ell^1, |||\cdot|||^{\sim})$ does not contain an ai copy of ℓ^1 and later Lin [9] showed that ℓ^1 can be renormed to have the fixed point property for nonexpansive mappings with a special version of the norm $|||\cdot|||^{\sim}$.

Now consider the norm $|||\cdot|||^{\sim}$ as follows: for $x = (\xi_k)_k \in \ell^1$, write $|||x|||^{\sim} := \sup_{k\in\mathbb{N}} \gamma_k \sum_{j=k}^{\infty} |\xi_j|$ where $\gamma_k \uparrow_k 1$, γ_k is strictly increasing. Then, we can show that $(\ell^1, |||\cdot|||^{\sim})$ does not contain an asymptotically isometric copy of $\ell^{1\boxplus 0}$ either.

Proof. We use the similar ideas expressed in [6] and by contradiction, assume $(\ell^1, |||\cdot|||^{\sim})$ does contain an ai copy of $\ell^{1\boxplus 0}$.

That is, there exists a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in ℓ^1 such that

$$\begin{aligned} \frac{1}{2}\sup_{n\in\mathbb{N}}(1-\varepsilon_n)\,|t_n| + \frac{1}{2}\sum_{n=1}^{\infty}(1-\varepsilon_n)\,|t_n| &\leq \left|\left|\left|\sum_{n=1}^{\infty} t_n\, x_n\right|\right|\right|^{\sim} \\ &\leq \frac{1}{2}\sup_{n\in\mathbb{N}}|t_n| + \frac{1}{2}\sum_{n=1}^{\infty}|t_n|, \end{aligned} \tag{4.5}$$

for every $(t_n)_{n\in\mathbb{N}} \in \ell^1$.

Without loss of generality we suppose that $(x_n)_n$ is disjointly supported and that by passing to a subsequence, we can assume that (x_n) converges weak* (and so it is pointwise) to some $y \in \ell^1$.

Next, replacing x_n by the $|||\cdot|||^{\sim}$-normalization of $\left(\frac{x_{2n}-x_{2n-1}}{2}\right)_n$ satisfying (4.5), we can suppose that $y = 0$.

By the proof of the Bessaga-Pełczyński Theorem [1], we may pass to an essentially disjointly supported subsequence of x_n. Hence, when it is normalized and truncated this subsequence appropriately, we get a disjointly supported sequence satisfying (4.5). Also, by passing to subsequences if necessary, we may suppose that $\varepsilon_n < \frac{1}{3n}$ for all $n \in \mathbb{N}$.

Let $(m(k))_{k\in\mathbb{N}_0}$ with $m(0) = 0$ and $(\xi_k)_{k\in\mathbb{N}}$ a sequence of scalars such that for each $k \in \mathbb{N}$, $y_k = \sum_{j=m(k-1)+1}^{m(k)} \xi_j e_j$. Using the triangular inequality of the norm, for each $K \in \mathbb{N}$, we get

$$\frac{K-K\varepsilon_K}{2}+\frac{K+1-\varepsilon_1-K\varepsilon_K}{2}\le |||x_1+Kx_K|||^{\sim}$$

$$\le \sup_{\substack{1\le j\le m(1)\\ m(K-1)+1\le i\le m(K)}}\left\{\begin{array}{l}\gamma_j\left(\begin{array}{c}\sum\limits_{k=j}^{m(1)}|\xi_k|\\ +K\sum\limits_{k=m(K-1)+1}^{m(K)}|\xi_k|\end{array}\right),\\ K\gamma_i\sum\limits_{k=i}^{m(K)}|\xi_k|\end{array}\right\}$$

$$\le \sup_{\substack{1\le j\le m(1)\\ m(K-1)+1\le i\le m(K)}}\left\{\begin{array}{l}\gamma_j\sum\limits_{k=j}^{m(1)}|\xi_k|\\ +K\gamma_{m(1)}\sum\limits_{k=m(K-1)+1}^{m(K)}|\xi_k|,\\ K\gamma_i\sum\limits_{k=i}^{m(K)}|\xi_k|\end{array}\right\}.$$

Thus,

$$\frac{K-K\varepsilon_K}{2}+\frac{K+1-\varepsilon_1-K\varepsilon_K}{2}$$

$$\le \sup_{\substack{1\le j\le m(1)\\ m(K-1)+1\le i\le m(K)}}\left\{\begin{array}{l}K\frac{\gamma_{m(1)}}{\gamma_{m(K-1)+1}}\\ \times\gamma_{m(K-1)+1}\sum\limits_{k=m(K-1)+1}^{m(K)}|\xi_k|\\ +\gamma_j\sum\limits_{k=j}^{m(1)}|\xi_k|,\ K\gamma_i\sum\limits_{k=i}^{m(K)}|\xi_k|\end{array}\right\}.$$

Therefore, $K+\frac{1-\varepsilon_1}{2}-K\varepsilon_K\le\max\left\{1+K\frac{\gamma_{m(1)}}{\gamma_{m(K-1)+1}},K\right\}$ for all $K\in\mathbb{N}$.

But since $\varepsilon_1<\frac{1}{3}$ and $K\varepsilon_K<\frac{1}{3}$, we have $K+\frac{1-\varepsilon_1}{2}-K\varepsilon_K>K$, and so $1+\frac{1}{2K}-\frac{\varepsilon_1}{2K}-\varepsilon_K\le\frac{1}{K}+\frac{\gamma_{m(1)}}{\gamma_{m(K-1)+1}}$, for all $K\in\mathbb{N}$.

Thus, we get a contradiction by letting $K\to\infty$ since we would have $1\le\gamma_m(1)$. □

Our final example shows that there exists a Banach space that contains an ai copy of ℓ^1 but it does not contain any ai copy of $\ell^{1\boxplus 0}$.

Example 4.6. We can show that if a Banach space $(X,\|.\|)$ contains an ai copy of $\ell^{1\boxplus 0}$, then it contains a sequence $(x_n)_{n\in\mathbb{N}}$ such that there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,1)$ satisfying the condition

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)\frac{|a_n|}{4^n}+\sum_{n=1}^{\infty}(1-\varepsilon_n)\frac{|a_n|}{2^n}\le\left\|\sum_{n=1}^{\infty}a_nx_n\right\|\le\sum_{n=1}^{\infty}\frac{|a_n|}{4^n}+\sum_{n=1}^{\infty}\frac{|a_n|}{2^n}$$

for all $(a_n)_{n\in\mathbb{N}}\in\ell^1$. It can be noticed that behind this notion, there is a-nother degenerate Lorentz-Marcinkiewicz space $\ell_{\delta,1}$ with the weight sequence $\delta:=\left(\frac{1}{4^n}+\frac{1}{2^n}\right)_{n\in\mathbb{N}}$. Let's call the Banach space containing this type of sequence the Banach space that contains an ai copy of $\ell_{\delta,1}$ for $\delta=\left(\frac{1}{4^n}+\frac{1}{2^n}\right)_{n\in\mathbb{N}}$. Obviously, if a Banach space $(X,\|.\|)$ contains an ai copy of $\ell_{\delta,1}$ for $\delta=\left(\frac{1}{4^n}+\frac{1}{2^n}\right)_{n\in\mathbb{N}}$, it contains an ai copy of ℓ^1.

Considering degenerate Lorentz-Marcinkiewicz space $\ell_{\delta,1}$; or say, $(\ell^1,\|.\|)$ with the norm given by $\|x\|=\sum\limits_{n=1}^{\infty}\frac{|\xi_n|}{4^n}+\sum\limits_{n=1}^{\infty}\frac{|\xi_n|}{2^n}$, $\forall x=(\xi_n)_{n\in\mathbb{N}}\in\ell^1$, we obtain an important result which shows our notion is different from ai ℓ^1 sequence. Indeed, we can see that $(\ell^1, \|.\|)$ does not contain an ai copy of $\ell^{1 \boxplus 0}$ while clearly that it contains an ai copy of ℓ^1.

Proof. First of all, indeed, if a Banach space $(X, \|.\|)$ contains an ai copy of $\ell^{1 \boxplus 0}$, then using the proof method in Theorem 3.2, but considering $\alpha = \beta = 1$ and $y_k = \left(\frac{1}{4^n} + \frac{1}{2^n}\right) \frac{1}{\beta(1+m_k)} \sum_{n=M_{k-1}+1}^{M_k} x_n$ there, we obtain that $(X, \|.\|)$ contains an ai copy of $\ell_{\delta,1}$ for $\delta = \left(\frac{1}{4^n} + \frac{1}{2^n}\right)_{n\in\mathbb{N}}$.

Next, as the most important part of our example which shows the difference of our notion from ai ℓ^1 sequences, we consider the degenerate Lorentz-Marcinkiewicz space $\ell_{\delta,1}$ with the weight sequence $\delta = \left(\frac{1}{4^n} + \frac{1}{2^n}\right)_{n\in\mathbb{N}}$. Then, we will apply the same proof method as the proof of previous example to see $(\ell^1, \|.\|)$ does not contain any ai copy of $\ell^{1 \boxplus 0}$.

We assume for the contradiction that $(\ell^1, \|.\|)$ does contain any ai copy of $\ell^{1 \boxplus 0}$ and we can skip all the details in the previous proof until the inequality part where the norm is essentially used. So considering the difference of our norm, we get the following inequality: for each $K \in \mathbb{N}$, we get

$$\begin{aligned}
&\frac{K - K\varepsilon_K}{2} + \frac{K + 1 - \varepsilon_1 - K\varepsilon_K}{2} \leq \|x_1 + Kx_K\| \\
&\leq \sum_{k=1}^{m(1)} \left(\frac{1}{4^n} + \frac{1}{2^n}\right) |\xi_k| + K \sum_{k=m(K-1)+1}^{m(K)} \left(\frac{1}{4^n} + \frac{1}{2^n}\right) |\xi_k| \\
&\leq \frac{3}{4} \sum_{k=1}^{m(1)} |\xi_k| + K \left(\frac{1}{4^{m(K-1)+1}} + \frac{1}{2^{m(K-1)+1}}\right) \sum_{k=m(K-1)+1}^{m(K)} |\xi_k| \; .
\end{aligned}$$

Therefore, $K + \frac{1-\varepsilon_1}{2} - K\varepsilon_K \leq \frac{3}{4} + K\left(\frac{1}{4^{m(K-1)+1}} + \frac{1}{2^{m(K-1)+1}}\right)$ for all $K \in \mathbb{N}$.

But since $\varepsilon_1 < \frac{1}{3}$ and $K\varepsilon_K < \frac{1}{3}$, we have $K + \frac{1-\varepsilon_1}{2} - K\varepsilon_K > K$ and so

$$1 + \frac{1}{2K} - \frac{\varepsilon_1}{2K} - \varepsilon_K \leq \frac{3}{4K} + \left(\frac{1}{4^{m(K-1)+1}} + \frac{1}{2^{m(K-1)+1}}\right), \text{ for all } K \in \mathbb{N}.$$

Thus, we get a contradiction by letting $K \to \infty$ since we would have $1 \leq \frac{3}{4}$. □

Acknowledgment. The author is grateful to Chris Lennard for his valuable comments and helpful discussions on the subject. We would also like to express our sincere thanks to the referees for their valuable suggestions to improve this manuscript.

References

[1] J. Diestel, *Sequences and series in Banach spaces*, Springer Science & Business Media, 2012.

[2] S.J. Dilworth, M. Girardi and J. Hagler, *Dual Banach spaces which contain an isometric copy of L_1*, Bull. Polish Acad. Sci. Math. **48**, 1–12, 2000.

[3] P.N. Dowling and C.J. Lennard, *Every nonreflexive subspace of $L_1[0, 1]$ fails the fixed point property*, Proc. Amer. Math. Soc. **125**, 443–446, 1997.

[4] P.N. Dowling, C.J. Lennard and B. Turett, *Asymptotically isometric copies of c_0 in Banach Spaces*, J. Math. Anal. Appl. **219**, 377–391, 1998.

[5] P.N. Dowling, C.J. Lennard and B. Turett, *Renormings of ℓ^1 and c_0 and fixed point properties*, in: Handbook of Metric Fixed Point Theory, Springer, Netherlands, 269–297, 2001.

[6] P.N. Dowling, W.B. Johnson, C.J. Lennard and B. Turett, *The optimality of James's distortion theorems*, Proc. Amer. Math. Soc. **125**, 167–174, 1997.

[7] R.C. James, *Uniformly non-square Banach spaces*, Ann. of Math. **80** (3), 542–550, 1964.

[8] C.J. Lennard, Personal communication, 2017.

[9] P.K. Lin, *There is an equivalent norm on* ℓ_1 *that has the fixed point property*, Nonlinear Anal. **68**, 2303–2308, 2008.

[10] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I: Sequence Spaces*, in: Ergebnisse der Mathematik und ihrer Grenzgebiete **92**, Springer-Verlag, 1977.

[11] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II: Function Spaces*, in: Ergebnisse der Mathematik und ihrer Grenzgebiete **97**, Springer-Verlag, 1979.

[12] G.G. Lorentz, *Some new functional spaces*, Ann. Math. **51** (1), 37–55, 1950.

[13] V. Nezir, *Fixed point properties for* c_0*-like spaces*, Ph.D., University of Pittsburgh, Pittsburgh, PA, USA, 2012.

[14] V. Nezir, *Fixed point properties for a degenerate Lorentz-Marcinkiewicz space*, Turkish J. Math. **43**, 1919-1939, 2019.

[15] V. Nezir and N. Mustafa, *On the fixed point property for a degenerate Lorentz-Marcinkiewicz space*, in: Proceedings of the 5th International Conference on Recent Advances in Pure and Applied Mathematics (icrapam 2018), Karadeniz Technical University, Trabzon, 23–27 July 2018.