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Research Article

Approximation by α-Bernstein-Schurer operator

Nursel Çetin

Ankara Hacı Bayram Veli University, Polatlı Faculty of Science and Letters, Department of Mathematics, 06900, Ankara, Turkey

Abstract

In this paper, we introduce a new family of generalized Bernstein-Schurer operators and investigate some approximation properties of these operators. We obtain a uniform approximation result using the well-known Korovkin theorem and give the degree of approximation via second modulus of smoothness. Also, we present Voronovskaya and Grüss-Voronovskaya type results for these operators.

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1. Introduction

In 1912, the classical Bernstein polynomials

$$B_n(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}, \qquad x\in[0,1] \tag{1.1}$$

for any $n\in\mathbb{N}$ and $f\in C[0,1]$, the space of all real valued continuous functions on $[0,1]$, were proposed by Bernstein [6] as one of the simplest way to prove Weierstrass Approximation Theorem and then, considering their simple structure and many useful approximation properties, discovery of their various generalizations and modifications in different ways has been an intensive research area. Some works on generalizations and modifications of the Bernstein polynomials can be found in [8, 9, 13, 18, 19, 21–24].

In 1932, Voronovskaya [25] proved the first asymptotic formula describing the degree of pointwise convergence for Bernstein operators. Later on, many researchers have studied widely some generalizations of Voronovskaya's result. In 1935, Grüss [16] gave an inequality which estimates the difference between the integral of product of two functions and the product of their integrals. Recently, Acu et al. [3] initially studied the Grüss-type inequality for linear positive operators by using the least concave majorant of the modulus of continuity. In [15], Gonska and Tachev proved Grüss-type inequalities in terms of the second order modulus of continuity and the second order Ditizian-Totik modulus of smoothness. Very recently, Gal and Gonska [14] obtained a Voronovskaya-type theorem with the help of Grüss inequality for Bernstein operators in both the real and the complex case and termed it as Grüss-Voronovskaya-type theorem.

Email address: nurselcetin07@gmail.com
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In 1962, considering a given non-negative integer p, Schurer [21] introduced and studied the following generalization of Bernstein operators $B_{n,p} : C[0,1+p] \to C[0,1]$ defined as

$$B_{n,p}(f,x) = \sum_{k=0}^{n+p} f\left(\frac{k}{n}\right)\binom{n+p}{k} x^k (1-x)^{n+p-k}, \; x \in [0,1] \tag{1.2}$$

for any $n \in \mathbb{N}$ and $f \in C[0,1+p]$. Note that the special case $p = 0$ gives the classical Bernstein polynomials.

In 2017, Chen et al. [9] constructed a new family of generalized Bernstein operators which is called as $\alpha-$Bernstein operator, depending on a non-negative real parameter, is given by

$$T_{n,\alpha}(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) p_{n,i}^{(\alpha)}(x), \tag{1.3}$$

for any function $f(x)$ defined on $[0,1]$, each positive integer n and any fixed real α. Here, for $i = 0,1,...,n$, the $\alpha-$Bernstein polynomial $p_{n,i}^{(\alpha)}(x)$ of degree n is defined by $p_{1,0}^{(\alpha)}(x) = 1-x$, $p_{1,1}^{(\alpha)}(x) = x$ and

$$\begin{aligned} p_{n,i}^{(\alpha)}(x) \;\; &= \;\; \left[\binom{n-2}{i}(1-\alpha)x + \binom{n-2}{i-2}(1-\alpha)(1-x) + \binom{n}{i}\alpha x(1-x)\right] \times \\ & \quad\;\; x^{i-1}(1-x)^{n-i-1}, \end{aligned}$$

where $n \geq 2$, $x \in [0,1]$ and the binomial coefficients $\binom{k}{l}$ are given by

$$\binom{k}{l} = \begin{cases} \dfrac{k!}{(k-l)! \cdot l!}, & \text{if} \quad 0 \leq l \leq k, \\ 0, & \text{else.} \end{cases}$$

For $\alpha = 1$, the α-Bernstein operator becomes the classical Bernstein polynomial. Also, the α-Bernstein operators are linear positive operators for $0 \leq \alpha \leq 1$. In [9], the authors gave some elementary properties and proved the uniform convergency of the sequence of the α-Bernstein operators to $f \in C[0,1]$ with the help of the well known Korovkin theorem. They obtained the rate of convergence and Voronovskaya-type theorem for the α-Bernstein operators. Also, they gave an upper bound for the approximation error by means of the modulus of continuity and proved that the α-Bernstein operators satisfy some shape preserving results. Then, many researchers have studied intensively $\alpha-$Bernstein operators and their generalizations for the last two years (see, e.g., [1,2,4,7,10,11,17,20]).

In this paper, inspired by the above works, we introduce a generalization of Bernstein-Schurer operators given in (1.2) as follows:

$$T_{n,\alpha,p}(f;x) = \sum_{i=0}^{n+p} f\left(\frac{i}{n}\right) \widetilde{p}_{n,i}^{(\alpha)}(x), \tag{1.4}$$

for any function $f(x)$ defined on $[0,1+p]$, $x \in [0,1]$, each positive integer n, fixed $p \in \mathbb{N} \cup \{0\}$ and any fixed real α. Here, for $i = 0,1,...,n$, the α-Bernstein-Schurer polynomial $\widetilde{p}_{n,i}^{(\alpha)}(x)$ of degree n is defined by $\widetilde{p}_{1,0}^{(\alpha)}(x) = 1-x$, $\widetilde{p}_{1,1}^{(\alpha)}(x) = x$ and

$$\begin{aligned} \widetilde{p}_{n,i}^{(\alpha)}(x) \;\; &= \;\; \left[\binom{n+p-2}{i}(1-\alpha)x + \binom{n+p-2}{i-2}(1-\alpha)(1-x) + \binom{n+p}{i}\alpha x(1-x)\right] \\ & \quad\;\; \times (x^{i-1}(1-x)^{n+p-i-1}), \end{aligned}$$

where $n+p \geq 2$, $x \in [0,1]$. The $\alpha-$Bernstein-Schurer operators are a family of linear positive operators for $0 \leq \alpha \leq 1$. We should note that if $\alpha = 1$, then $T_{n,\alpha,p}$ gives the Bernstein-Schurer operators. If $\alpha = 1$ and $p = 0$, the operator $T_{n,\alpha,p}$ reduces to the classical Bernstein polynomials. Also, the case $p = 0$ gives the α-Bernstein operator. Here, we obtain an important recurrence formula to compute higher order moments and give an estimate, related to the degree of approximation, via second modulus of smoothness. Also, we prove a Voronovskaya and Grüss-Voronovskaya-type results for α-Bernstein-Schurer operator $T_{n,\alpha,p}$.

2. Auxiliary results

In this section, we give some useful results which are necessary for the proof of the main results. Let us denote by $e_k(x) = x^k$, $k \in \mathbb{N} \cup \{0\}$ the test functions and $\varphi_x^j(t) := (t-x)^j$, $j \in \mathbb{N}$.

The α-Bernstein-Schurer operator defined by (1.4) has the following another representation.

Theorem 2.1. *The α-Bernstein-Schurer operator can be stated as*

$$\begin{aligned} & T_{n,\alpha}(f;x) \qquad (2.1) \\ = \ & (1-\alpha)\sum_{i=0}^{n+p-1} g_i \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} + \alpha \sum_{i=0}^{n+p} f_i \binom{n+p}{i} x^i (1-x)^{n+p-i}, \end{aligned}$$

where $f \in C[0,1+p]$, $x \in [0,1]$, $f_i = f\left(\frac{i}{n}\right)$ and

$$g_i = \left(1 - \frac{i}{n+p-1}\right) f_i + \frac{i}{n+p-1} f_{i+1}, \ n+p \geq 2.$$

Proof. It can be easily obtained by the method used for the α-Bernstein operator (see p. 247-248 in [9]). So we omit the details. □

For the Bernstein-Schurer polynomials given in (1.2), we have the following properties

$$\begin{aligned} B_{n,p}(e_0;x) &= 1, \qquad (2.2) \\ B_{n,p}(e_1;x) &= \left(1+\frac{p}{n}\right)x, \qquad (2.3) \\ B_{n,p}(e_2;x) &= \left(1+\frac{p}{n}\right)^2 x^2 + \frac{(n+p)}{n^2} x(1-x). \qquad (2.4) \end{aligned}$$

(see e.g., p. 320-321 in [5]).

Furthermore, by direct computation, we have the following moments.

Lemma 2.2. *For the operators $B_{n,p}$ given by (1.2) we have*

$$\begin{aligned} & B_{n,p}(e_3;x) \\ = \ & \frac{(n+p)x}{n^3}\left[(n+p-1)(n+p-2)x^2 + 3(n+p-1)x + 1\right] \qquad (2.5) \end{aligned}$$

and

$$\begin{aligned} B_{n,p}(e_4;x) = \ & \frac{(n+p)x}{n^4}\Big[(n+p-1)(n+p-2)(n+p-3)x^3 \\ & +6(n+p-1)(n+p-2)x^2 + 7(n+p-1)x + 1\Big]. \qquad (2.6) \end{aligned}$$

In what follows, we will give a recurrence formula to calculate higher order moments of the new operator $T_{n,\alpha,p}$.

Theorem 2.3. *For all $j, p \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ and $x \in [0,1]$, we have the recurrence formula*

$$\begin{aligned} & T_{n,\alpha,p}(e_{j+1};x) \\ = \ & \frac{x(1-x)}{n}\left(T_{n,\alpha,p}(e_j;x)\right)' + \frac{[1+(n+p-1)x]}{n} T_{n,\alpha,p}(e_j;x) \end{aligned}$$

$$+\frac{(1-\alpha)}{n+p-1}\left(\frac{n-1}{n}\right)^{j+1} B_{n-1,p}\left(e_{j+1};x\right)-\frac{(1-\alpha)}{n}\left(\frac{n-1}{n}\right)^{j} B_{n-1,p}\left(e_{j};x\right)$$
$$-\frac{\alpha\left(1-x\right)}{n}B_{n,p}\left(e_j;x\right), \tag{2.7}$$

where $\alpha \in [0,1]$ and $B_{n,p}$ is the n th Bernstein-Schurer operator.

Proof. Using (2.1), we can write

$$T_{n,\alpha,p}\left(e_j;x\right)=$$
$$(1-\alpha)\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^i\left(1-x\right)^{n+p-i-1}\left[\left(1-\frac{i}{n+p-1}\right)\left(\frac{i}{n}\right)^j+\frac{i}{n+p-1}\left(\frac{i+1}{n}\right)^j\right]$$
$$+\alpha\sum_{i=0}^{n+p}\binom{n+p}{i}x^i\left(1-x\right)^{n+p-i}\left(\frac{i}{n}\right)^j.$$

By some calculations, we get

$$\begin{aligned}
&\left(T_{n,\alpha,p}\left(e_j;x\right)\right)' \\
=\;& (1-\alpha)\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}\left[ix^{i-1}\left(1-x\right)^{n+p-i-1}-\left(n+p-i-1\right)x^i\left(1-x\right)^{n+p-i-2}\right] \\
&\times\left[\left(1-\frac{i}{n+p-1}\right)\left(\frac{i}{n}\right)^j+\frac{i}{n+p-1}\left(\frac{i+1}{n}\right)^j\right] \\
&+\alpha\sum_{i=0}^{n+p}\binom{n+p}{i}\left[ix^{i-1}\left(1-x\right)^{n+p-i}-\left(n+p-i\right)x^i\left(1-x\right)^{n+p-i-1}\right]\left(\frac{i}{n}\right)^j \\
=\;& \frac{(1-\alpha)}{x\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^i\left(1-x\right)^{n+p-i-1}i\left[\left(1-\frac{i}{n+p-1}\right)\left(\frac{i}{n}\right)^j\right. \\
&\left.+\frac{i}{n+p-1}\left(\frac{i+1}{n}\right)^j\right]-\frac{\left(n+p-1\right)\left(1-\alpha\right)}{\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^i\left(1-x\right)^{n+p-i-1} \\
&\times\left[\left(1-\frac{i}{n+p-1}\right)\left(\frac{i}{n}\right)^j+\frac{i}{n+p-1}\left(\frac{i+1}{n}\right)^j\right] \\
&+\frac{\alpha}{x\left(1-x\right)}\sum_{i=0}^{n+p}\binom{n+p}{i}x^i\left(1-x\right)^{n+p-i}i\left(\frac{i}{n}\right)^j \\
&-\frac{\alpha\left(n+p\right)}{\left(1-x\right)}\sum_{i=0}^{n+p}\binom{n+p}{i}x^i\left(1-x\right)^{n+p-i}\left(\frac{i}{n}\right)^j \\
=\;& \frac{(1-\alpha)}{x\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^i\left(1-x\right)^{n+p-i-1}i\left[\left(1-\frac{i}{n+p-1}\right)\left(\frac{i}{n}\right)^j\right] \\
&+\frac{(1-\alpha)}{x\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^i\left(1-x\right)^{n+p-i-1}i\left[\frac{i}{n+p-1}\left(\frac{i+1}{n}\right)^j\right] \\
&-\frac{\left(n+p-1\right)\left(1-\alpha\right)}{\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^i\left(1-x\right)^{n+p-i-1}\left[\left(1-\frac{i}{n+p-1}\right)\left(\frac{i}{n}\right)^j\right. \\
&\left.+\frac{i}{n+p-1}\left(\frac{i+1}{n}\right)^j\right]+\frac{\alpha}{x\left(1-x\right)}\sum_{i=0}^{n+p}\binom{n+p}{i}x^i\left(1-x\right)^{n+p-i}i\left(\frac{i}{n}\right)^j
\end{aligned}$$

$$
\begin{aligned}
& -\frac{\alpha\left(n+p\right)}{\left(1-x\right)}\sum_{i=0}^{n+p}\binom{n+p}{i}x^{i}\left(1-x\right)^{n+p-i}\left(\frac{i}{n}\right)^{j} \\
= \quad & \frac{n\left(1-\alpha\right)}{x\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^{i}\left(1-x\right)^{n+p-i-1}\left[\left(1-\frac{i}{n+p-1}\right)\left(\frac{i}{n}\right)^{j+1}\right] \\
& +\frac{n\left(1-\alpha\right)}{x\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^{i}\left(1-x\right)^{n+p-i-1}\left[\frac{i}{n+p-1}\left(\frac{i+1}{n}\right)^{j+1}\right] \\
& -\frac{\left(1-\alpha\right)}{x\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^{i}\left(1-x\right)^{n+p-i-1}\left[\frac{i}{n+p-1}\left(\frac{i+1}{n}\right)^{j}\right] \\
& -\frac{\left(n+p-1\right)\left(1-\alpha\right)}{\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^{i}\left(1-x\right)^{n+p-i-1}\left[\left(1-\frac{i}{n+p-1}\right)\left(\frac{i}{n}\right)^{j}\right. \\
& \left.+\frac{i}{n+p-1}\left(\frac{i+1}{n}\right)^{j}\right]+\frac{\alpha n}{x\left(1-x\right)}\sum_{i=0}^{n+p}\binom{n+p}{i}x^{i}\left(1-x\right)^{n+p-i}\left(\frac{i}{n}\right)^{j+1} \\
& -\frac{\alpha\left(n+p\right)}{\left(1-x\right)}\sum_{i=0}^{n+p}\binom{n+p}{i}x^{i}\left(1-x\right)^{n+p-i}\left(\frac{i}{n}\right)^{j} \\
= \quad & \frac{n\left(1-\alpha\right)}{x\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^{i}\left(1-x\right)^{n+p-i-1}\left[\left(1-\frac{i}{n+p-1}\right)\left(\frac{i}{n}\right)^{j+1}+\frac{i}{n+p-1}\right. \\
& \left.\times\left(\frac{i+1}{n}\right)^{j+1}\right]-\frac{\left(1-\alpha\right)}{x\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^{i}\left(1-x\right)^{n+p-i-1}\left[\frac{i}{n+p-1}\left(\frac{i+1}{n}\right)^{j}\right] \\
& -\frac{\left(n+p-1\right)\left(1-\alpha\right)}{\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^{i}\left(1-x\right)^{n+p-i-1}\left[\left(1-\frac{i}{n+p-1}\right)\left(\frac{i}{n}\right)^{j}\right. \\
& \left.+\frac{i}{n+p-1}\left(\frac{i+1}{n}\right)^{j}\right]+\frac{\alpha n}{x\left(1-x\right)}\sum_{i=0}^{n+p}\binom{n+p}{i}x^{i}\left(1-x\right)^{n+p-i}\left(\frac{i}{n}\right)^{j+1} \\
& -\frac{\alpha\left(n+p\right)}{\left(1-x\right)}\sum_{i=0}^{n+p}\binom{n+p}{i}x^{i}\left(1-x\right)^{n+p-i}\left(\frac{i}{n}\right)^{j} \\
= \quad & \frac{n}{x\left(1-x\right)}T_{n,\alpha,p}\left(e_{j+1};x\right) \\
& -\frac{\left(1-\alpha\right)}{x\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^{i}\left(1-x\right)^{n+p-i-1}\left[\frac{i}{n+p-1}\left(\frac{i+1}{n}\right)^{j}\right] \\
& -\frac{\left(n+p-1\right)}{\left(1-x\right)}T_{n,\alpha,p}\left(e_{j};x\right)-\frac{\alpha}{\left(1-x\right)}\sum_{i=0}^{n+p}\binom{n+p}{i}x^{i}\left(1-x\right)^{n+p-i}\left(\frac{i}{n}\right)^{j} \\
= \quad & \frac{n}{x\left(1-x\right)}T_{n,\alpha,p}\left(e_{j+1};x\right)-\frac{1}{x\left(1-x\right)}T_{n,\alpha,p}\left(e_{j};x\right) \\
& +\frac{\left(1-\alpha\right)}{x\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^{i}\left(1-x\right)^{n+p-i-1}\left(\frac{i}{n}\right)^{j} \\
& -\frac{\left(1-\alpha\right)}{x\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^{i}\left(1-x\right)^{n+p-i-1}\frac{i}{n+p-1}\left(\frac{i}{n}\right)^{j}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{\alpha}{x\left(1-x\right)}\sum_{i=0}^{n+p}\binom{n+p}{i}x^{i}\left(1-x\right)^{n+p-i}\left(\frac{i}{n}\right)^{j} \\
& -\frac{\left(n+p-1\right)}{\left(1-x\right)}T_{n,\alpha,p}\left(e_{j};x\right)-\frac{\alpha}{\left(1-x\right)}\sum_{i=0}^{n+p}\binom{n+p}{i}x^{i}\left(1-x\right)^{n+p-i}\left(\frac{i}{n}\right)^{j} \\
= & \frac{n}{x\left(1-x\right)}T_{n,\alpha,p}\left(e_{j+1};x\right)-\frac{\left[1+\left(n+p-1\right)x\right]}{x\left(1-x\right)}T_{n,\alpha,p}\left(e_{j};x\right) \\
& +\frac{\left(1-\alpha\right)}{x\left(1-x\right)}\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^{i}\left(1-x\right)^{n+p-i-1}\left(\frac{i}{n}\right)^{j} \\
& -\frac{\left(1-\alpha\right)}{x\left(1-x\right)}\left(\frac{n}{n+p-1}\right)\sum_{i=0}^{n+p-1}\binom{n+p-1}{i}x^{i}\left(1-x\right)^{n+p-i-1}\left(\frac{i}{n}\right)^{j+1} \\
& +\frac{\alpha-\alpha x}{x\left(1-x\right)}\sum_{i=0}^{n+p}\binom{n+p}{i}x^{i}\left(1-x\right)^{n+p-i}\left(\frac{i}{n}\right)^{j} \\
= & \frac{n}{x\left(1-x\right)}T_{n,\alpha,p}\left(e_{j+1};x\right)-\frac{\left[1+\left(n+p-1\right)x\right]}{x\left(1-x\right)}T_{n,\alpha,p}\left(e_{j};x\right) \\
& +\frac{\left(1-\alpha\right)}{x\left(1-x\right)}\left(\frac{n-1}{n}\right)^{j}B_{n-1,p}\left(e_{j};x\right) \\
& -\frac{\left(1-\alpha\right)}{x\left(1-x\right)}\left(\frac{n}{n+p-1}\right)\left(\frac{n-1}{n}\right)^{j+1}B_{n-1,p}\left(e_{j+1};x\right)+\frac{\alpha}{x}B_{n,p}\left(e_{j};x\right),
\end{aligned}
$$

which completes the proof. □

Now, we give moments and central moments of the α-Bernstein-Schurer operators, below.

Lemma 2.4. *For the operators $T_{n,\alpha,p}$ given in (2.1), one has*

$$T_{n,\alpha,p}(e_0;x)=1, \tag{2.8}$$

$$T_{n,\alpha,p}(e_1;x)=\left(1+\frac{p}{n}\right)x, \tag{2.9}$$

$$T_{n,\alpha,p}(e_2;x)=\left(1+\frac{p}{n}\right)^{2}x^{2}+\frac{\left[n+p+2(1-\alpha)\right]}{n^{2}}x\left(1-x\right), \tag{2.10}$$

$$
\begin{aligned}
& T_{n,\alpha,p}(e_3;x) \\
= & \frac{1}{n^{3}}\Big\{\Big[(n+p)^{3}-3\left(n+p\right)^{2}+2\left(3\alpha-2\right)\left(n+p\right)+12(1-\alpha)\Big]x^{3} \\
& +3\Big[(n+p)^{2}+\left(1-2\alpha\right)\left(n+p\right)+6\left(\alpha-1\right)\Big]x^{2} \\
& +\left[n+p+6\left(1-\alpha\right)\right]x\Big\},
\end{aligned} \tag{2.11}
$$

and

$$
\begin{aligned}
& T_{n,\alpha,p}(e_4;x) \\
= & \frac{1}{n^{4}}\Big\{\Big[(n+p)^{4}-6\left(n+p\right)^{3}+\left(12\alpha-1\right)\left(n+p\right)^{2}+6\left(9-10\alpha\right)\left(n+p\right)-72(1-\alpha)\Big]x^{4} \\
& +6\Big[(n+p)^{3}-\left(1+2\alpha\right)\left(n+p\right)^{2}+2\left(8\alpha-7\right)\left(n+p\right)+24(1-\alpha)\Big]x^{3} \\
& +\Big[7\left(n+p\right)^{2}+\left(29-36\alpha\right)\left(n+p\right)-86\left(1-\alpha\right)\Big]x^{2}+\left[n+p+14\left(1-\alpha\right)\right]x\Big\}.
\end{aligned} \tag{2.12}
$$

Proof. By the recurrence formula obtained in Theorem 2.3, for $j=0$ we have

$$\begin{aligned}T_{n,\alpha,p}(e_1;x) =& \frac{x\left(1-x\right)}{n}\left(T_{n,\alpha,p}(e_0;x)\right)' + \frac{\left[1+\left(n+p-1\right)x\right]}{n} T_{n,\alpha,p}(e_0;x)\\ &+\frac{\left(1-\alpha\right)}{n+p-1}\left(\frac{n-1}{n}\right)B_{n-1,p}\left(e_1;x\right)-\frac{\left(1-\alpha\right)}{n}B_{n-1,p}\left(e_0;x\right)\\ &-\frac{\alpha\left(1-x\right)}{n}B_{n,p}\left(e_0;x\right).\end{aligned}$$

Using (2.2), (2.3) and (2.8), it follows that

$$\begin{aligned}T_{n,\alpha,p}(e_1;x) &= \frac{1+\left(n+p-1\right)x}{n}+\frac{\left(1-\alpha\right)}{n+p-1}\left(\frac{n-1}{n}\right)\left(1+\frac{p}{n-1}\right)x\\ &\quad -\frac{\left(1-\alpha\right)}{n}-\frac{\alpha\left(1-x\right)}{n}\\ &= \left(1+\frac{p}{n}\right)x. \end{aligned}\tag{2.13}$$

Then, from (2.7), for $j=1$ we get

$$T_{n,\alpha,p}(e_2;x) = \frac{x\left(1-x\right)}{n}\left(T_{n,\alpha,p}(e_1;x)\right)' + \frac{\left[1+\left(n+p-1\right)x\right]}{n} T_{n,\alpha,p}(e_1;x)$$

$$+\frac{\left(1-\alpha\right)}{n+p-1}\left(\frac{n-1}{n}\right)^2B_{n-1,p}\left(e_2;x\right)-\frac{\left(1-\alpha\right)}{n}\left(\frac{n-1}{n}\right)B_{n-1,p}\left(e_1;x\right)-\frac{\alpha\left(1-x\right)}{n}B_{n,p}\left(e_1;x\right).$$

Using the moments expressions (2.13), (2.3) and (2.4), we obtain the followings

$$\begin{aligned}T_{n,\alpha,p}(e_2;x) &= \frac{x\left(1-x\right)}{n}\left(1+\frac{p}{n}\right)+\frac{\left[1+\left(n+p-1\right)x\right]}{n}\left(1+\frac{p}{n}\right)x\\ &\quad +\frac{\left(1-\alpha\right)}{n+p-1}\left(\frac{n-1}{n}\right)^2\left[\left(1+\frac{p}{n-1}\right)^2x^2+\frac{\left(n+p-1\right)}{\left(n-1\right)^2}x\left(1-x\right)\right]\\ &\quad -\frac{\left(1-\alpha\right)}{n}\left(\frac{n-1}{n}\right)\left(1+\frac{p}{n-1}\right)x-\frac{\alpha\left(1-x\right)}{n}\left(1+\frac{p}{n}\right)x\\ &= \frac{[\left(n+p\right)^2-\left(n+p\right)-2\left(1-\alpha\right)]x^2+[n+p+2\left(1-\alpha\right)]x}{n^2}\\ &= \frac{\left(n+p\right)^2}{n^2}x^2+\frac{\left[n+p+2\left(1-\alpha\right)\right]}{n^2}x\left(1-x\right).\end{aligned}\tag{2.14}$$

Writing for $j=2$ and $j=3$ in (2.7), by making use of (2.14), (2.4), (2.5) and (2.11), (2.5), (2.6) respectively, the remain of the proof can be easily shown. $\square$

Lemma 2.5. *For the operators $T_{n,\alpha,p}$, one has*

$$T_{n,\alpha,p}\left(\varphi_x^1;x\right) = \frac{p}{n}x, \tag{2.15}$$

$$T_{n,\alpha,p}\left(\varphi_x^2;x\right) = \frac{p^2}{n^2}x^2+\frac{\left[n+p+2\left(1-\alpha\right)\right]}{n^2}x\left(1-x\right), \tag{2.16}$$

$$\begin{aligned}&T_{n,\alpha,p}(\varphi_x^4;x)\\ =& \frac{1}{n^4}\left\{\left[3n^2+2n\left(3-6\alpha+7p-3p^2\right)+p^4-6p^3+\left(12\alpha-1\right)p^2+6p\left(9-10\alpha\right)\right.\right.\\ &\left.-72(1-\alpha)\right]x^4+6\left[-n^2+n\left(p^2-4p+4\alpha-2\right)+p^3-\left(1+2\alpha\right)p^2+2\left(8\alpha-7\right)p\right.\\ &\left.+24\left(1-\alpha\right)\right]x^3+\left[3n^2+n\left(10p+5-12\alpha\right)+7p^2+\left(29-36\alpha\right)p-86\left(1-\alpha\right)\right]x^2\\ &\left.+\left[n+p+14\left(1-\alpha\right)\right]x\right\}.\end{aligned}\tag{2.17}$$

Lemma 2.6. *For the operators $T_{n,\alpha,p}$, the following expressions hold*

$$\lim_{n\to\infty} nT_{n,\alpha,p}\left(\varphi_x^1;x\right) = px, \tag{2.18}$$

$$\lim_{n\to\infty} nT_{n,\alpha,p}\left(\varphi_x^2;x\right) = x\left(1-x\right), \tag{2.19}$$

$$\lim_{n\to\infty} nT_{n,\alpha,p}(\varphi_x^4;x) = 0 \tag{2.20}$$

3. Main results

Applying the classical Korovkin Theorem to the sequence of linear positive operators $T_{n,\alpha,p}$, from (2.8)-(2.10) we have the convergence theorem as follows.

Theorem 3.1. *For any $f\in C\left[0,1+p\right]$ and $\alpha\in\left[0,1\right]$, the sequence $\left\{T_{n,\alpha,p}\left(f;x\right)\right\}$ converges to f uniformly on $\left[0,1\right]$.*

Now, we will give an upper bound for the approximation error in terms of $K-$functional.

For $f\in C\left[0,1+p\right]$ and $\delta>0$, the first modulus of smoothness, modulus of continuity, of f is defined by

$$\omega\left(f,\delta\right) = \sup_{|h|\leq\delta}\ \sup_{x,x+h\in[0,1+p]} \left|f\left(x+h\right)-f\left(x\right)\right|$$

and second modulus of smoothness of f is defined by

$$\omega_2\left(f,\delta\right) = \sup_{|h|\leq\delta}\ \sup_{x,x+2h\in[0,1+p]} \left|f\left(x+2h\right)-2f\left(x+h\right)+f\left(x\right)\right|.$$

For convenience, we need the following Peetre's K-functional defined by

$$K\left(f,\delta\right) = \inf_{g\in C^2[0,1+p]} \left\{\left\|f-g\right\|+\delta\left\|g''\right\|\right\},$$

where $\delta>0$ and

$$C^2\left[0,1+p\right] = \left\{g\in C\left[0,1+p\right] : g',g''\in C\left[0,1+p\right]\right\}.$$

Note that the modulus of smoothness and the K-functional of an $f\in C\left[0,1+p\right]$ are related to each other as in the following sense: *there exist positive constants C_1 and C_2 such that*

$$C_1\omega_2\left(f,\delta\right) \leq K\left(f,\delta^2\right) \leq C_2\omega_2\left(f,\delta\right) \tag{3.1}$$

(see, e.g., Theorem 2.4 in p. 177 of [12]).

Theorem 3.2. *Let $p\in\mathbb{N}_0$ be fixed, $\alpha\in\left[0,1\right]$ and $f\in C\left[0,1+p\right]$. Then, for each $x\in\left[0,1\right]$, one has*

$$\left|T_{n,\alpha,p}\left(f;x\right)-f\left(x\right)\right| \leq C\omega_2\left(f,\delta_{n,\alpha,p}\left(x\right)\right)+\omega\left(f,\frac{p}{n}x\right),$$

where $C>0$ is an absolute constant and

$$\delta_{n,\alpha,p}\left(x\right) = \frac{1}{2}\sqrt{\frac{p^2}{n^2}x^2+\frac{\left[n+p+2\left(1-\alpha\right)\right]}{2n^2}x\left(1-x\right)}. \tag{3.2}$$

Proof. Consider an auxiliary operator

$$\overline{T}_{n,\alpha,p}\left(f;x\right) := T_{n,\alpha,p}\left(f;x\right)+f\left(x\right)-f\left(\left(1+\frac{p}{n}\right)x\right) \tag{3.3}$$

for $f\in C\left[0,1+p\right]$ and $x\in\left[0,1\right]$. In this case, from (2.8) and(2.9), $\overline{T}_{n,\alpha,p}$ are linear and positive and each operator preserves the linear functions:

$$\overline{T}_{n,\alpha,p}\left(t-x;x\right) = 0. \tag{3.4}$$

Now, let $g\in C^{2}\left[0,1+p\right].$ For every $x\in\left[0,1\right],$ from Taylor's formula one has

$$g\left(t\right)=g\left(x\right)+g'\left(x\right)\left(t-x\right)+\int\limits_{x}^{t}\left(t-y\right)g''\left(y\right)dy. \tag{3.5}$$

Application of the operators $\overline{T}_{n,\alpha,p}$ on both sides of (3.5) gives that

$$\begin{aligned}&\overline{T}_{n,\alpha,p}\left(g;x\right)-g(x)\\ =\;&g'\left(x\right)\overline{T}_{n,\alpha,p}\left(t-x;x\right)+\overline{T}_{n,\alpha,p}\left(\int\limits_{x}^{t}\left(t-y\right)g''\left(y\right)dy;x\right).\end{aligned} \tag{3.6}$$

Using (3.4) and taking (3.3) into account for $f\left(t\right)=\int\limits_{x}^{t}\left(t-y\right)g''\left(y\right)dy$, then (3.6) reduces to

$$\begin{aligned}\overline{T}_{n,\alpha,p}\left(g;x\right)-g(x)&=T_{n,\alpha,p}\left(\int\limits_{x}^{t}\left(t-y\right)g''\left(y\right)dy;x\right)\\&\quad-\int\limits_{x}^{\left(1+\frac{p}{n}\right)x}\left[\left(1+\frac{p}{n}\right)x-y\right]g''\left(y\right)dy.\end{aligned} \tag{3.7}$$

Using the fact

$$\left|\int\limits_{x}^{\left(1+\frac{p}{n}\right)x}\left[\left(1+\frac{p}{n}\right)x-y\right]g''\left(y\right)dy\right|\leq\frac{1}{2}\left(T_{n,\alpha,p}\left(\varphi_{x}^{1};x\right)\right)^{2}\left\|g''\right\|$$

and Lemma 2.5, we obtain

$$\begin{aligned}&\left|\overline{T}_{n,\alpha,p}\left(g;x\right)-g(x)\right|\\ \leq\;&T_{n,\alpha,p}\left(\left|\int\limits_{x}^{t}\left(t-y\right)g''\left(y\right)dy\right|;x\right)+\left|\int\limits_{x}^{\left(1+\frac{p}{n}\right)x}\left[\left(1+\frac{p}{n}\right)x-y\right]\left|g''\left(y\right)\right|dy\right|\\ \leq\;&\frac{\left\|g''\right\|}{2}\left[T_{n,\alpha,p}\left(\varphi_{x}^{2};x\right)+\left(T_{n,\alpha,p}\left(\varphi_{x}^{1};x\right)\right)^{2}\right]\\ \leq\;&\frac{\left\|g''\right\|}{2}\left[\frac{2p^{2}}{n^{2}}x^{2}+\frac{\left[n+p+2\left(1-\alpha\right)\right]}{n^{2}}x\left(1-x\right)\right].\end{aligned} \tag{3.8}$$

On the other hand, from (3.3), (3.4) and (2.8), it can be easily obtained that

$$\left|\overline{T}_{n,\alpha,p}\left(f;x\right)\right|\leq\left\|f\right\|T_{n,\alpha,p}\left(1;x\right)+2\left\|f\right\|\leq3\left\|f\right\|. \tag{3.9}$$

Thus, taking (3.3), (3.8) and (3.9) into consideration, for $f,g\in C\left[0,1+p\right]$ one has

$$\begin{aligned}&\left|T_{n,\alpha,p}\left(f;x\right)-f\left(x\right)\right|\\&\quad\leq\left|\overline{T}_{n,\alpha,p}\left(f-g;x\right)-\left(f-g\right)\left(x\right)\right|+\left|\overline{T}_{n,\alpha,p}\left(g;x\right)-g(x)\right|\\&\quad+\left|f\left(\left(1+\frac{p}{n}\right)x\right)-f(x)\right|\\&\quad\leq4\left\{\left\|f-g\right\|+\frac{\left\|g''\right\|}{8}\left[\frac{2p^{2}}{n^{2}}x^{2}+\frac{\left[n+p+2\left(1-\alpha\right)\right]}{n^{2}}x\left(1-x\right)\right]\right\}+\omega\left(f,\frac{p}{n}x\right).\end{aligned}$$

Finally, taking infimum over all $g \in C^2[0,1+p]$ on the right hand-side of the last inequality and application of (3.1) gives rise to

$$\begin{aligned}\left|T_{n,\alpha,p}(f;x)-f(x)\right| &\leq 4K\left(f,\delta^2_{n,\alpha,p}(x)\right)+\omega\left(f,\frac{p}{n}x\right)\\ &\leq C\omega_2\left(f,\delta_{n,\alpha,p}(x)\right)+\omega\left(f,\frac{p}{n}x\right),\end{aligned}$$

where $\delta_{n,\alpha,p}(x)$ is given by (3.2). □

In the next, we investigate a Voronovskaya-type result for the $\alpha-$Bernstein-Schurer operator $T_{n,\alpha,p}$.

Theorem 3.3. *Suppose that $f \in C[0,1+p]$ and f has the second order derivative at $x \in [0,1]$. Then we have*

$$\lim_{n\to\infty} n\left[T_{n,\alpha,p}(f;x)-f(x)\right]=pxf'(x)+\frac{x(1-x)}{2}f''(x),$$

where $0 \leq \alpha \leq 1$.

Proof. From Taylor's formula, one has

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+h(t-x)(t-x)^2, \tag{3.10}$$

at the fixed point $x \in [0,1]$, where $h(t-x)$ is a continuous function on $[0,1+p]$ and $\lim_{t\to x} h(t-x)=0$. Application of the operators $T_{n,\alpha,p}$ to (3.10) implies

$$\begin{aligned}n\left[T_{n,\alpha,p}(f;x)-f(x)\right] &= f'(x)\,nT_{n,\alpha,p}(t-x;x)+\frac{f''(x)}{2}nT_{n,\alpha,p}\left((t-x)^2;x\right)\\ &\quad +nT_{n,\alpha,p}\left(h(t-x)(t-x)^2;x\right).\end{aligned}$$

Using (2.18) and (2.19), we can write

$$\lim_{n\to\infty} n\left[T_{n,\alpha,p}(f;x)-f(x)\right]=pxf'(x)+\frac{x(1-x)}{2}f''(x)+\lim_{n\to\infty} nT_{n,\alpha,p}\left(h(t-x)(t-x)^2;x\right).$$

Then, it suffices to prove that $\lim_{n\to\infty} nT_{n,\alpha,p}\left(h(t-x)(t-x)^2;x\right)=0$. Since $\lim_{t\to x} h(t-x)=0$, for each $\varepsilon>0$ there is a $\delta>0$ such that $|h(t-x)|<\varepsilon$ for all t satisfying $|t-x|<\delta$. On the other hand, since $h(t-x)$ is bounded on $[0,1+p]$, there is an $M>0$ such that $|h(t-x)|\leq M$ for all t. Therefore, we may write

$$|h(t-x)|\leq M\frac{(t-x)\,,^2}{\delta^2}$$

when $|t-x|\geq\delta$. So, these arguments enable us to write $|h(t-x)|\leq\varepsilon+M\frac{(t-x)^2}{\delta^2}$ for all t.

Thus, we have

$$\begin{aligned}nT_{n,\alpha,p}\left(h(t-x)(t-x)^2;x\right) &\leq \varepsilon nT_{n,\alpha,p}\left((t-x)^2;x\right)+\frac{M}{\delta^2}nT_{n,\alpha,p}\left((t-x)^4;x\right)\\ &= \varepsilon nT_{n,\alpha,p}\left(\varphi_x^2;x\right)+\frac{M}{\delta^2}nT_{n,\alpha,p}\left(\varphi_x^4;x\right).\end{aligned}$$

Making use of (2.19) and (2.20), we arrive at the desired result. □

In what follows, by using the approach in [14] we will give Grüss-Voronovskaya type theorem for the α-Bernstein-Schurer operator $T_{n,\alpha,p}$.

Theorem 3.4. *Let $f,g \in C^2[0,1+p]$ and $\alpha \in [0,1]$. Then, for each $x \in [0,1]$ we have*

$$\lim_{n\to\infty} n\left[T_{n,\alpha,p}(fg;x)-T_{n,\alpha,p}(f;x)\,T_{n,\alpha,p}(g;x)\right]=x(1-x)\,f'(x)\,g'(x).$$

Proof. Since

$$(fg)(x) = f(x)\,g(x), \quad (fg)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$$

and

$$(fg)''(x) = f''(x)\,g(x) + 2f'(x)\,g'(x) + f(x)\,g''(x),$$

we can easily write

$$\begin{aligned}
& T_{n,\alpha,p}(fg;x) - T_{n,\alpha,p}(f;x)\,T_{n,\alpha,p}(g;x) \\
= & \left[T_{n,\alpha,p}(fg;x) - f(x)\,g(x) - (fg)'(x)\,T_{n,\alpha,p}\left(\varphi_x^1;x\right) - \frac{(fg)''(x)}{2}T_{n,\alpha,p}\left(\varphi_x^2;x\right)\right] \\
& -g(x)\left[T_{n,\alpha,p}(f;x) - f(x) - f'(x)\,T_{n,\alpha,p}\left(\varphi_x^1;x\right) - \frac{f''(x)}{2}T_{n,\alpha,p}\left(\varphi_x^2;x\right)\right] \\
& -T_{n,\alpha,p}(f;x)\left[T_{n,\alpha,p}(g;x) - g(x) - g'(x)\,T_{n,\alpha,p}\left(\varphi_x^1;x\right) - \frac{g''(x)}{2}T_{n,\alpha,p}\left(\varphi_x^2;x\right)\right] \\
& +\frac{1}{2}T_{n,\alpha,p}\left(\varphi_x^2;x\right)\left[f(x)\,g''(x) + 2f'(x)\,g'(x) - g''(x)\,T_{n,\alpha,p}(f;x)\right] \\
& +g'(x)\,T_{n,\alpha,p}\left(\varphi_x^1;x\right)\left[f(x) - T_{n,\alpha,p}(f;x)\right].
\end{aligned}$$

By using (2.18) and (2.19), we get

$$\begin{aligned}
& \lim_{n\to\infty} n\left[T_{n,\alpha,p}(fg;x) - T_{n,\alpha,p}(f;x)\,T_{n,\alpha,p}(g;x)\right] \\
= & \lim_{n\to\infty} n\left[T_{n,\alpha,p}(fg;x) - f(x)\,g(x)\right] - (fg)'(x)\,px - \frac{(fg)''(x)}{2}x(1-x) \\
& -g(x)\left\{\lim_{n\to\infty} n\left[T_{n,\alpha,p}(f;x) - f(x)\right] - f'(x)\,px - \frac{f''(x)}{2}x(1-x)\right\} \\
& -\lim_{n\to\infty} T_{n,\alpha,p}(f;x)\left\{\lim_{n\to\infty} n\left[T_{n,\alpha,p}(g;x) - g(x)\right] - g'(x)\,px - \frac{g''(x)}{2}x(1-x)\right\} \\
& +\frac{x(1-x)}{2}\left\{g''(x)\lim_{n\to\infty}\left[f(x) - T_{n,\alpha,p}(f;x)\right] + 2f'(x)\,g'(x)\right\} \\
& +g'(x)\,px\lim_{n\to\infty}\left[f(x) - T_{n,\alpha,p}(f;x)\right].
\end{aligned}$$

Considering Theorem 3.1 and Theorem 3.3, we obtain

$$\lim_{n\to\infty} n\left[T_{n,\alpha,p}(fg;x) - T_{n,\alpha,p}(f;x)\,T_{n,\alpha,p}(g;x)\right] = x(1-x)\,f'(x)\,g'(x),$$

which is the desired result. □

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