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De-Moivre and Euler Formulae for Dual-Complex Numbers

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Abstract

In this study, we generalize the well-known formulae of De-Moivre and Euler of complex numbers to dual-complex numbers. Furthermore, we investigate the roots and powers of a dual-complex number by using these formulae. Consequently, we give some examples to illustrate the main results in this paper.

1. Introduction

The complex numbers have emerged from the need to solve cubic equations. First studies on complex numbers were produced by G. Cardan (1501-1576) and B. Bombelli (1526-1572). Later, Euler used the formula

$$x+iy=r(\cos\theta+i\sin\theta)$$

and he studied the root of the equation $z^n=1$. Also, he proved that a complex number can be written in the form of

$$e^{i\theta}=\cos\theta+i\sin\theta$$

[1]. Abraham de Moivre found the formula

$$(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$$

and gave his own name to this formula. The developments in the number theory present us new number systems including the dual numbers which are expressed by the real and dual parts like complex numbers. This idea was first introduced by W. K. Clifford to solve some algebraic problems, [2]. Afterwards, E. Study presented different theorems with his studies on kinematics and line geometry, [3]. A dual number is a pair of real numbers which consists of the real unit $+1$ and dual unit ε satisfying $\varepsilon^2=0$ for $\varepsilon\neq 0$. Therefore, dual numbers are elements of two-dimensional real algebra $\mathbb{D}=\left\{z=x+\varepsilon y | x,y\in\mathbb{R},\varepsilon^2=0,\varepsilon\neq 0\right\}$ which is generated by 1 and ε. Similar to the complex numbers, the module of a dual number z is defined by $|z|=|x+\varepsilon y|=|x|=r$, its argument is $\theta=\frac{y}{x}$ and represented by $\arg(z)$. The set of all points which satisfy the equation $|z|=|x|=r>0$ and which are on the dual plane are $x=\pm r$ lines, [4].

This circle is called Galilean circle on a dual plane. Let S be a circle centered with O and M be a point on S. If d is OM line, and α is the angle δ_{Od}, a Galilean circle is represented by

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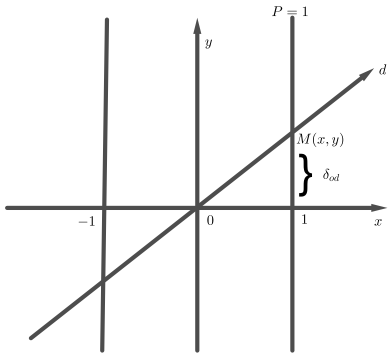

Figure 1.1: Galilean unit circle

So, one can easily see that

$$\cos g\alpha = \frac{|OP|}{|OM|} = 1 \quad , \quad \sin g\alpha = \frac{|MP|}{|OM|} = \frac{\delta_{Od}}{1} = \alpha.$$

On the other hand, exponential representation of a dual number $z = x + \varepsilon y$ is in the form of $z = xe^{\varepsilon\theta}$ where $\frac{y}{x}$ is dual angle and it is shown as $\arg(z) = \frac{y}{x} = \theta$, [5]. In addition, from the definitions of Galilean cosine and sine, we realize

$$\cos g(\theta) = 1 \text{ and } \sin g(\theta) = \tfrac{y}{x} = \theta.$$

By considering the exponential rules, we write

$$\begin{aligned}
&\cos g(x+y) = \cos g(x)\cos g(y) - \varepsilon^2 \sin g(x)\sin g(y)\\
&\sin g(x+y) = \sin g(x)\cos g(y) + \cos g(x)\sin g(y)\\
&\cos g^2(x) + \varepsilon^2 \sin g^2(x) = 1.
\end{aligned}$$

[6].

E. Cho proved that De-Moivre formula for the complex numbers is admissible for quaternions, [7]. Yaylı and Kabadayı gave De-Moivre formula for dual quaternions, [8]. This formula is also investigated for the case of hyperbolic quaternions in [9]. In this study, we first introduce dual-complex numbers and algebraic expressions on dual complex numbers. We also generalize De-Moivre and Euler formulae which are given for complex and dual numbers to dual-complex numbers. Then we have found the roots and forces of the dual-complex numbers. Finally, the obtained results are supported by examples.

2. Dual-Complex Numbers

A dual-complex number w can be written in the form of complex pair (z,t) such that $+1$ is the real unit and ε is the dual unit. Thus, we denote dual-complex numbers set by $\mathbb{DC} = \{ w = z + \varepsilon t \mid z,t \in \mathbb{C}, \varepsilon^2 = 0, \varepsilon \neq 0 \}$. If we consider complex numbers $z = x_1 + ix_2$ and $t = x_3 + ix_4$, we represent a dual-complex number $w = x_1 + x_2 i + x_3\varepsilon + x_4\varepsilon i$. Here i, ε and εi are unit vectors in three-dimensional vectors space such that i is a complex unit, ε is a dual unit, and εi is a dual-complex unit, [10]. So, the multiplication table of dual–complex numbers' base elements is given below.

x	1	i	ε	$i\varepsilon$
1	1	i	ε	$i\varepsilon$
i	i	1	$i\varepsilon$	ε
ε	ε	$i\varepsilon$	0	0
$i\varepsilon$	$i\varepsilon$	ε	0	0

Table 1: Multiplication Table of Dual-Complex Numbers

We define addition and multiplication on dual-complex numbers as follows

$$\begin{aligned}
&w_1 + w_2 = (z_1 \pm \varepsilon z_2) + (z_3 \pm \varepsilon z_4) = (z_1 \pm z_3) + \varepsilon(z_2 \pm z_4)\\
&w_1 \times w_2 = (z_1 + \varepsilon z_2) \times (z_3 + \varepsilon z_4) = z_1 z_3 + \varepsilon(z_1 z_4 + z_2 z_3)
\end{aligned}$$

where w_1 and w_2 are dual-complex numbers and $z_1, z_2, z_3, z_4 \in \mathbb{C}$. On the other hand, the division of two dual-complex numbers is $\frac{w_1}{w_2} = \frac{z_1+\varepsilon z_2}{z_3+\varepsilon z_4} = \frac{z_1}{z_3} + \varepsilon \frac{z_2 z_3 - z_1 z_4}{{z_3}^2}$ where $\mathscr{R}\mathrm{e}(w_2) \neq 0$. Thus, dual-complex numbers yield a commutative ring whose characteristic is 0. If we consider both algebraic and geometric properties of dual-complex numbers, we define five possible conjugations of dual-complex numbers.

These are

$$w^{\dagger_1}=\bar{z}+\varepsilon\bar{t}\ \text{(complex conjugation)}$$
$$w^{\dagger_2}=z-\varepsilon t\ \text{(dual conjugation)}$$
$$w^{\dagger_3}=\bar{z}-\varepsilon\bar{t}\ \text{(coupled conjugation)}$$
$$w^{\dagger_4}=\bar{z}\left(1-\varepsilon\frac{t}{z}\right)\ \text{(dual-complex conjugation)}$$
$$w^{\dagger_5}=t-\varepsilon\, z\ \text{(anti-dual conjugation)}$$

such that $w=z+\varepsilon t\in\mathbb{DC}$ is a dual-complex number, [11]. In regards to these definitions, we give the following proposition for modules of dual-complex numbers.

Proposition 2.1. *Let be a dual-complex number. Then we write*

$$|w|^2_{\dagger_1}=w\times w^{\dagger_1}=(z+t\varepsilon)(\bar{z}+\bar{t}\varepsilon)=z\bar{z}+z\bar{t}\varepsilon+\bar{z}t\varepsilon=z\bar{z}+(z\bar{t}+\bar{z}t)\,\varepsilon=|z|^2+2\varepsilon\mathscr{R}e\,(z\bar{t})\ \in\mathbb{D}$$

$$|w|^2_{\dagger_2}=w\times w^{\dagger_2}=(z+t\varepsilon)(z-t\varepsilon)=zz-zt\varepsilon+zt\varepsilon=z^2\in\mathbb{C}$$

$$|w|^2_{\dagger_3}=w\times w^{\dagger_3}=(z+t\varepsilon)(\bar{z}-\bar{t}\varepsilon)=z\bar{z}-z\bar{t}\varepsilon+\bar{z}t\varepsilon=|z|^2-(z\bar{t}-\bar{z}t)\,\varepsilon=|z|^2-2i\varepsilon\,\mathrm{Im}\,(z\bar{t})\in\mathbb{D}$$

$$|w|^4_{\dagger_4}=w\times w^{\dagger_4}=(z+t\varepsilon)\left(\bar{z}\left(1-\frac{t}{z}\varepsilon\right)\right)=z\bar{z}-z\bar{z}\frac{t}{z}\varepsilon+\bar{z}t\varepsilon=z\bar{z}-\bar{z}t\varepsilon+\bar{z}t\varepsilon=z\bar{z}=|z|^2\in\mathbb{DC}\ (\mathscr{R}e\,(w)\neq 0)$$

$$|w|^2_{\dagger_5}=w\times w^{\dagger_5}=(z+t\varepsilon)(t-z\varepsilon)=zt+t^2\varepsilon-z^2\varepsilon=zt+\varepsilon\left(t^2-z^2\right)\in\mathbb{DC}$$

[11].

3. De-Moivre and Euler Formulae for Dual-Complex Number

Definition 3.1. *Exponential representation of a dual-complex number is $e^w=ze^{\frac{t}{z}\varepsilon}$ where $w=z+t\varepsilon\in\mathbb{DC}$ is a dual-complex number and $(z\neq 0)$, [11].*

Definition 3.2. *Let $w=z+t\varepsilon$ be a dual-complex number with the exponential representation $e^w=ze^{\frac{t}{z}\varepsilon}$. The dual-complex angle $\frac{t}{z}$ is called argument of dual-complex number and it is denoted by $\arg w=\frac{t}{z}=\varphi$, [11].*

Definition 3.3. *Let $w=z+t\varepsilon$ be a dual-complex number and φ be its principal argument. Every dual-complex number can be written in the form of $w=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))$ such that $\cos g(\varphi)=1$ and $\sin g(\varphi)=\varphi$, [11].*

Theorem 3.4. *(Euler Formula) Let $w=z+t\varepsilon$ be a dual-complex number and φ be the principal argument of w. Then* $w=ze^{\varepsilon\varphi}=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))$ *.*

Proof. As it is aforementioned in Definition 3.2, the exponential representation of a dual-complex number $w=z+t\varepsilon\in\mathbb{DC}$ is $e^w=ze^{\frac{t}{z}\varepsilon}$, where dual-complex number $\frac{t}{z}$ is the principal argument φ. Thus, if we write w in the form of $w=ze^{\varepsilon\varphi}=z\left(1+\varepsilon\varphi+\frac{(\varepsilon\varphi)^2}{2!}+\frac{(\varepsilon\varphi)^3}{3!}+...\right)$, from properties of the dual unit, we see that $w=ze^{\varepsilon\varphi}=z(1+\varepsilon\varphi)=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))$. □

Theorem 3.5. *Let $w=z+t\varepsilon$ be a dual-complex number and $\varphi=\frac{t}{z}$. Then $\frac{1}{e^{\varepsilon\varphi}}=e^{\varepsilon(-\varphi)}$.*

Proof. If we use Euler formula for $\frac{1}{e^{\varepsilon\varphi}}$, we have

$$\frac{1}{e^{\varepsilon\varphi}}=\frac{1}{\left(1+\varepsilon\varphi+\frac{(\varepsilon\varphi)^2}{2!}+\frac{(\varepsilon\varphi)^3}{3!}+....\right)}=\frac{1}{\cos g(\varphi)+\varepsilon\sin g(\varphi)}.$$

If we multiplicate both the numerator and the denominator with $\cos g(\varphi)-\varepsilon\sin g(\varphi)$ in the last expression, we get

$$\frac{1}{e^{\varepsilon\varphi}}=\frac{1}{\cos g(\varphi)+\varepsilon\sin g(\varphi)}\frac{(\cos g(\varphi)-\varepsilon\sin g(\varphi))}{(\cos g(\varphi)-\varepsilon\sin g(\varphi))}=\frac{\cos g(\varphi)-\varepsilon\sin g(\varphi)}{\cos g^2(\varphi)}.$$

If we consider the equality $\cos g^2(\varphi)=1$, we have $\frac{1}{e^{\varepsilon\varphi}}=\cos g(\varphi)-\varepsilon\sin g(\varphi)$. Considering the last equation, we write $\frac{1}{e^{\varepsilon\varphi}}=\cos g(\varphi)-\varepsilon\sin g(\varphi)=\cos g(-\varphi)+\varepsilon\sin g(-\varphi)$. As a consequence, we get $\frac{1}{e^{\varepsilon\varphi}}=e^{\varepsilon(-\varphi)}$. □

Theorem 3.6. *(De-Moivre Formula) Let $w=z+t\varepsilon$ be a dual-complex number and $w=ze^{\varepsilon\varphi}=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))$ be its polar representation. Then, the equation $w^n=(ze^{\varepsilon\varphi})^n=(z(\cos g(\varphi)+\varepsilon\sin g(\varphi))^n=z^n(\cos g(n\varphi)+\varepsilon\sin g(n\varphi))$ yields for all non-negative integers.*

Proof. Considering Galelian trigonometric identities for dual-complex number $w=z+t\varepsilon$, we will prove that $w^n=(ze^{\varepsilon\varphi})^n=(z(\cos g(\varphi)+\varepsilon\sin g(\varphi))^n=z^n(\cos g(n\varphi)+\varepsilon\sin g(n\varphi))$ is admissible by the help of induction. For $n=2$, we have

$$\begin{aligned}\left(ze^{\varepsilon\varphi}\right)^2&=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))\,z(\cos g(\varphi)+\varepsilon\sin g(\varphi))\\&=z^2\left(\cos^2 g(\varphi)+\varepsilon(\cos g(\varphi)\sin g(\varphi)+\sin g(\varphi)\cos g(\varphi))\right)\\&=z^2(\cos g(2\varphi)+\varepsilon\sin g(2\varphi))\end{aligned}$$

For $n=k$ non-negative integer, let $(z(\cos g(\varphi)+\varepsilon\sin g(\varphi))^k=z^k(\cos g(k\varphi)+\varepsilon\sin g(k\varphi))$ be true. For $n=k+1$, we get

$$\begin{aligned}(z(\cos g(\varphi)+\varepsilon\sin g(\varphi))^{k+1}&=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))^k(z(\cos g(\varphi)+\varepsilon\sin g(\varphi))\\&=z^k(\cos g(k\varphi)+\varepsilon\sin g(k\varphi))\,z(\cos g(k\varphi)+\varepsilon\sin g(k\varphi))\\&=z^k(\cos g(k\varphi)\cos g(\varphi)+\varepsilon(\cos g(k\varphi)\sin g(\varphi)+\sin g(k\varphi)\cos g(\varphi)))\\&=z^{k+1}(\cos g((k+1)\varphi)+\varepsilon\sin g((k+1)\varphi))\end{aligned}$$

So, the desired equality holds for $n=k+1$. This completes the proof. □

Theorem 3.7. *For the dual-complex number $w=z+t\varepsilon\in\mathbb{DC}$, the following equality yields for any integer n.*

Proof. We give the proof for non-negative integers in Theorem 3.6. Let $-n$ be a negative integer considering Theorem 3.5., we get

$$\begin{aligned}(w)^{-1}&=z^{-1}(\cos g(\varphi)-\varepsilon\sin g(\varphi))\\w^{-n}&=z^{-n}(\cos g(n\varphi)-\varepsilon\sin g(n\varphi))\\&=z^{-n}(\cos g(-n\varphi)+\varepsilon\sin g(-n\varphi))\end{aligned}$$

Thus, we see that for any integer $w^n=(ze^{\varepsilon\varphi})^n=(z(\cos g(\varphi)+\varepsilon\sin g(\varphi))^n=z^n(\cos g(n\varphi)+\varepsilon\sin g(n\varphi))$. □

Example 3.8. *Let $w=1+i+\varepsilon+\varepsilon i$ be a dual-complex number, we investigate (w^4), 4th- degree power of w where w is written in the form of $w=z+t\varepsilon$ and $z=1+i$, $t=1+i$ are complex numbers. Seeing that argument of w is $\frac{t}{z}=\varphi$, polar representation of w is given by $w=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))$. From Theorem 3.7, we have $w^4=z^4(\cos g(4\varphi)+\varepsilon\sin g(4\varphi))$ We gave equivalence for these Galilean trigonometric functions. So we find,*

$$w^4=(1+i)^4(1+\varepsilon 4)=-4(1+\varepsilon 4)=-4-16\varepsilon$$

Example 3.9. *We find values of w^2 and w^{10} for the dual-complex number $w=1-i+\varepsilon+3\varepsilon i\in\mathbb{DC}$. If we write w in the form of $w=z+t\varepsilon$, then its argument is $\frac{t}{z}=\frac{1-i}{1+3i}=\varphi$ where $z,t\in\mathbb{C}$ and $z=1-i, t=1+3i$. Thus, the polar representation of w is $w=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))$. So, we find*

$$w^2=z^2(\cos g(2\varphi)+\varepsilon\sin g(2\varphi))=(1-i)^2\left(1+\varepsilon\frac{2(1+3i)}{(1-i)}\right)=-2i+8\varepsilon+4\varepsilon i$$

and

$$w^{10}=z^{10}(\cos g(10\varphi)+\varepsilon\sin g(10\varphi))=(1-i)^{10}\left(1+\varepsilon 10\frac{(1+3i)}{(1-i)}\right)=-32i+640\varepsilon+320i$$

Theorem 3.10. *n-th degree root of w is $\sqrt[n]{w}=\sqrt[n]{z}\left(\cos g\left(\frac{\varphi}{n}\right)+\varepsilon\sin g\left(\frac{\varphi}{n}\right)\right)$ where $w=z+t\varepsilon\in\mathbb{DC}$ is a dual-complex number.*

Proof. Polar representation of $w=z+t\varepsilon\in\mathbb{DC}$ is $w=z(\cos g(\varphi)+\varepsilon\sin g(\varphi))$. From Theorem 3.7, we know that $w^n=(z.(\cos g(\varphi)+\varepsilon\sin g(\varphi))^n=z^n(\cos g(n\varphi)+\varepsilon\sin g(n\varphi))$. So, we get

$$\sqrt[n]{w}=w^{\frac{1}{n}}=z^{\frac{1}{n}}\left(\cos g\left(\frac{1}{n}\varphi\right)+\varepsilon\sin g\left(\frac{1}{n}\varphi\right)\right)=\sqrt[n]{z}\left(\cos g\left(\frac{\varphi}{n}\right)+\varepsilon\sin g\left(\frac{\varphi}{n}\right)\right)$$

This completes the proof. □

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