Tubular Surface Having Pointwise 1-type Gauss Map in Euclidean 4-Space

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(Communicated by Cihan Özgür)

ABSTRACT

In this paper, we deal with a tubular surface in Euclidean 4-space $\mathbb{E}^4$. We study this surface with respect to its Gauss map. We show that there is not any tubular surface having harmonic Gauss map and we give the complete classification of tubular surface having pointwise 1-type Gauss map in Euclidean 4-space $\mathbb{E}^4$.

1. Introduction

The term of finite type immersions is presented by Chen, and then the same author writes some papers related to this topic [11, 12]. If a submanifold M is given in Euclidean $m-$space $\mathbb{E}^m$, and if an isometric immersion $x : M \to \mathbb{E}^m$, also known as the position vector field of M, is written as a finite sum of eigenvectors of the Laplacian Δ of M for a constant map x_0, and non-constant maps $x_1, x_2, ..., x_k$, i.e.,

$$x = x_0 + \sum_{i=1}^{k} x_i,$$

then x is called as a finite type. Here, $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \le i \le k$. If the numbers λ_is are different from each other, the submanifold is called as $k-$type [10].

This term is extended to the Gauss map of M as

$$\Delta G = a(G + C) \tag{1.1}$$

for a real number a and a constant vector C by Chen and Piccinni in [13]. In this respect, a submanifold satisfying (1.1) is said to have 1-type Gauss map G. Then many papers have been written about submanifolds having 1-type Gauss map G [4, 5, 6, 23].

Afterwards, in (1.1), the real number a is replaced with a non-constant function f. That is, the equation (1.1) becomes

$$\Delta G = f(G + C). \tag{1.2}$$

A submanifold satisfying (1.2) is said to have pointwise 1-type Gauss map G. If the function f is non-constant, the pointwise 1-type Gauss map is called as proper. Also, if the vector C is zero, the pointwise 1-type Gauss map is called as the first kind. Otherwise, second kind [14].

Surfaces satisfying (1.2) have been the subject of many studies such as [1, 15, 16, 17, 21, 22, 26, 27]. In recent years, authors deal with the meridian surfaces with pointwise 1-type Gauss map in some spaces in [2, 3].

When a space curve $\gamma(u)$, spine curve, is given, we can define a canal surface as the envelope of a one parameter family of spheres whose centers are the points of the spine curve $\gamma(u)$, and whose radii $r(u)$ are varying. If the radius function $r(u)$ is constant, the canal surface is called a tubular (tube) or a pipe surface. Actually, the notion of the canal surface is a generalization of an offset of a plane curve. In [18] and [19], some

features of offset curves are discussed thoroughly. In [8], authors consider canal surfaces in $\mathbb{E}^4$. Also in [24, 25], the authors studied canal surfaces with parallel transport frame in $\mathbb{E}^4$.

In the present study, we consider tubular surface with respect to its Gauss map in $\mathbb{E}^4$. We find that there is no tubular surface having harmonic Gauss map and we give the complete classification of tubular surface with pointwise 1-type Gauss map in Euclidean 4-space $\mathbb{E}^4$.

2. Basic Concepts

Given a regular surface M in $\mathbb{E}^4$ with the parametrization $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$, at any point $p = X(u,v)$, the vectors X_u and X_v span the tangent space of M. Then the first fundamental form's coefficients are computed as

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v. \rangle \tag{2.1}$$

Here, $\langle , \rangle$ is the Euclidean inner product. For the regularity of the surface patch $X(u,v)$, $W^2 = EG - F^2 \neq 0$.

At any point p in M, there is a decomposition $T_p\mathbb{E}^4 = T_pM \oplus T_p^\perp M$, where $T_p^\perp M$ is the orthogonal component of T_pM in $\mathbb{E}^4$. Let $\widetilde{\nabla}$ be the Riemannian connection of $\mathbb{E}^4$. Then the induced Riemannian connection on M for any given local vector fields X, Y tangent to M is defined as

$$\nabla_X Y = (\widetilde{\nabla}_X Y)^T,$$

where T represents the tangential component.

Let $\chi(M)$ and $\chi^\perp(M)$ be the spaces of the smooth vector fields tangent and normal to M, respectively. The second fundamental map is defined as follows:

$$h : \chi(M) \times \chi(M) \to \chi^\perp(M), \ \ h(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y. \tag{2.2}$$

This map is well-defined, symmetric and bilinear. The equation (2.2) is known as the Gauss equation.

For each $X \in \chi(M)$ and $\xi \in \chi^\perp(M)$, the shape operator of M is defined as

$$\begin{aligned} A &: \chi^\perp(M) \times \chi(M) \to \chi(M) \\ A_\xi X &= -(\widetilde{\nabla}_X \xi)^T = -\widetilde{\nabla}_X \xi + \nabla_X^\perp \xi, \end{aligned} \tag{2.3}$$

where A_ξ is the shape operator tensor, $\nabla^\perp$ is the normal connection belongs to $\chi^\perp(M)$. For any $X, Y \in \chi(M)$,

$$\langle A_\xi X, Y \rangle = \langle h(X,Y), \xi \rangle \tag{2.4}$$

holds. The operator A_ξ is self-adjoint and bilinear. The equation (2.3) is known as the Weingarten equation [9]. Thus, the coefficients of the second fundamental forms of M can be defined as follows:

$$h_{ij}^k = \langle h(X_i, X_j), N_k \rangle, \ 1 \leq k \leq 2, \tag{2.5}$$

where X_i and X_j are the orthonormal vectors of T_pM [9].

The shape operator matrix corresponded to normal vector N_k of M is given

$$A_{N_k} = \begin{pmatrix} h_{11}^k & h_{12}^k \\ h_{12}^k & h_{22}^k \end{pmatrix} \tag{2.6}$$

[7]. The Gaussian curvature and the mean curvature vector of M are given as

$$K = \det(A_{N_1}) + \det(A_{N_2}), \tag{2.7}$$

and

$$\overrightarrow{H} = \frac{1}{2} \{iz(A_{N_1}) N_1 + iz(A_{N_2}) N_2\}, \tag{2.8}$$

respectively [9].

Now, let us recall some basic concepts of the Gauss map of a submanifold M in $\mathbb{E}^m$. The Grassmannian manifold, $G(n,m)$, consists of all oriented n-planes through the origin of $\mathbb{E}^m$ and the vector space $\wedge^n \mathbb{E}^m$,

obtained by the exterior product of n-vectors in $\mathbb{E}^m$, can be defined as an Euclidean space $\mathbb{E}^N$, where $N = \begin{pmatrix} m \\ n \end{pmatrix}$. In the light of this information, we can define the Gauss map. Let $\{e_1, e_2, ..., e_n, e_{n+1}, ..., e_m\}$ be an adapted orthonormal frame field in $\mathbb{E}^m$ such that $e_1, e_2, ..., e_n$ are tangent and $e_{n+1}, ..., e_m$ are normal to M, respectively. The map $G : M \to G(n,m) \subset \mathbb{E}^N$ which is defined as $G(p) = (e_1 \wedge e_2 \wedge ... \wedge e_n)(p)$ is called the Gauss map of M. This is a smooth map and assigns a point p into an oriented n-plane in $\mathbb{E}^m$ by the parallel translation of the tangent space of M at p in $\mathbb{E}^m$.

For any real function f on M, the Laplacian Δf of f is defined as

$$\Delta f = -\sum_i \left(\widetilde{\nabla}_{e_i} \widetilde{\nabla}_{e_i} f - \widetilde{\nabla}_{\nabla_{e_i} e_i} f \right).$$

3. Tubular Surfaces Having Pointwise 1-Type Gauss Map in $\mathbb{E}^4$

Let $\gamma(u) = (f_1(u), f_2(u), f_3(u), 0)$ be a curve parametrized by arclength. The corresponding Frenet formulas have the following form:

$$
\begin{aligned}
\gamma'(u) &= t(u), \\
t'(u) &= \kappa(u)n(u), \\
n'(u) &= -\kappa(u)t(u) + \tau(u)b_1(u), \\
b_1'(u) &= -\tau(u)n(u), \\
b_2'(u) &= 0,
\end{aligned}
\tag{3.1}
$$

where $\{t(u), n(u), b_1(u), b_2(u)\}$ is the Frenet orthonormal basis of γ. The canal surface in $\mathbb{E}^4$ has the following parametrization (see [20]):

$$M : X(u,v) = \gamma(u) + r(u)(b_1(u)\cos v + b_2(u)\sin v). \tag{3.2}$$

Now, we consider the surface M in (3.2) as a tubular surface. Then its parametrization is given by

$$M : X(u,v) = \gamma(u) + r(b_1(u)\cos v + b_2(u)\sin v), \tag{3.3}$$

where r is a real constant. For the vector fields X_1, X_2 are tangent, and N_1, N_2 are normal to M, we can choose an orthonormal frame X_1, X_2, N_1, N_2 given as in the following:

$$
\begin{aligned}
X_1 &= \frac{X_u}{\|X_u\|} = \frac{t - (r\tau \cos v)n}{\sqrt{E}}, \\
X_2 &= \frac{X_v}{\|X_v\|} = -(\sin v)b_1 + (\cos v)b_2, \\
N_1 &= (\cos v)b_1 + (\sin v)b_2, \\
N_2 &= \frac{(r\tau \cos v)t + n}{\sqrt{E}}.
\end{aligned}
\tag{3.4}
$$

Here, $E = 1 + r^2\tau^2 \cos^2 v$ is the coefficient of the first fundamental form. Moreover, by differentiating (3.4) covariantly with respect to X_1 and X_2, we obtain the following derivative formulas

$$
\begin{aligned}
\widetilde{\nabla}_{X_1} X_1 &= a_1 X_2 + a_2 N_1 + a_3 N_2, \\
\widetilde{\nabla}_{X_1} X_2 &= -a_1 X_1 + a_4 N_2, \\
\widetilde{\nabla}_{X_1} N_1 &= -a_2 X_1 + a_5 N_2, \\
\widetilde{\nabla}_{X_1} N_2 &= -a_3 X_1 - a_4 X_2 - a_5 N_1, \\
\widetilde{\nabla}_{X_2} X_1 &= a_4 N_2, \\
\widetilde{\nabla}_{X_2} X_2 &= -\frac{1}{r} N_1, \\
\widetilde{\nabla}_{X_2} N_1 &= \frac{1}{r} X_2, \\
\widetilde{\nabla}_{X_2} N_2 &= -a_4 X_1,
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
a_1(u,v) &= \frac{r\tau^2 \cos v \sin v}{E}, \\
a_2(u,v) &= \frac{-r\tau^2 \cos^2 v}{E}, \\
a_3(u,v) &= \frac{\kappa E - r\tau' \cos v}{E^{\frac{3}{2}}}, \\
a_4(u,v) &= \frac{\tau \sin v}{E}, \\
a_5(u,v) &= \frac{-\tau \cos v}{E}
\end{aligned}
\tag{3.6}
$$

are differentiable functions.

From the equation (2.4), the coefficients of the second fundamental form become

$$
h_{ij}^k = \langle h(X_i, X_j), N_k \rangle = \langle A_{N_k} X_i, X_j \rangle, \ 1 \le i, j \le 2, \ k = 3, 4.
\tag{3.7}
$$

By considering the equations (2.2), (3.4), and (3.7), we obtain the coefficients

$$
\begin{aligned}
h_{11}^1 &= a_2, \ h_{12}^1 = 0, \ h_{22}^1 = -\frac{1}{r}, \\
h_{11}^2 &= a_3, \ h_{12}^2 = a_4, \ h_{22}^2 = 0.
\end{aligned}
\tag{3.8}
$$

Lemma 3.1. *Let M be a tubular surface given with the parametrization (3.3) in $\mathbb{E}^4$. Then the shape operator matrices are given as follows:*

$$
A_{N_1} = \begin{bmatrix} a_2 & 0 \\ 0 & -\frac{1}{r} \end{bmatrix}, \quad A_{N_2} = \begin{bmatrix} a_3 & a_4 \\ a_4 & 0 \end{bmatrix}.
$$

In the following corollary, we obtain the special cases of the results which have been given in [8] with a different calculation method.

Corollary 3.1. *[8] Let M be a tubular surface given with the parametrization (3.3) in $\mathbb{E}^4$. The Gaussian curvature, and the mean curvature vector of M are respectively given by*

$$
K = \frac{2\tau^2 \cos^2 v + r^2\tau^4 \cos^4 v - \tau^2}{E^2},
$$

and

$$
\overrightarrow{H} = \frac{1}{2} \left\{ \left(\frac{-2r^2\tau^2 \cos^2 v - 1}{Er} \right) N_1 + \left(\frac{\kappa E - r\tau' \cos v}{E^{\frac{3}{2}}} \right) N_2 \right\}.
$$

Proof. From the Lemma 3.1, using the equations (2.7), and (2.8), we get

$$
\begin{aligned}
K &= \det(A_{N_1}) + \det(A_{N_2}) \\
&= \frac{\tau^2 \cos^2 v}{E} - \frac{\tau^2 \sin^2 v}{E^2} \\
&= \frac{2\tau^2 \cos^2 v + r^2\tau^4 \cos^4 v - \tau^2}{E^2},
\end{aligned}
$$

and

$$
\begin{aligned}
\overrightarrow{H} &= \frac{1}{2} \left\{ iz(A_{N_1}) N_1 + iz(A_{N_2}) N_2 \right\} \\
&= \frac{1}{2} \left\{ \left(\frac{-r\tau^2 \cos^2 v}{E} - \frac{1}{r} \right) N_1 + \left(\frac{\kappa E - r\tau' \cos v}{E^{\frac{3}{2}}} \right) N_2 \right\} \\
&= \frac{1}{2} \left\{ \left(\frac{-2r^2\tau^2 \cos^2 v - 1}{Er} \right) N_1 + \left(\frac{\kappa E - r\tau' \cos v}{E^{\frac{3}{2}}} \right) N_2 \right\}.
\end{aligned}
$$

$\square$

The Gauss map of a given surface M in $\mathbb{E}^4$ is defined by $G = X_1 \wedge X_2$. Using the equations (3.5), and a straight-forward computation, we get the Laplacian of the Gauss map as follows:

$$
\begin{aligned}
\Delta G \quad = \quad & \left(a_2^2 + a_3^2 + 2a_4^2 + \frac{1}{r^2} \right) X_1 \wedge X_2 \\
& + \left(-a_1 a_2 + a_4 a_5 - \frac{a_1}{r} \right) X_1 \wedge N_1 \\
& + \left(-X_1\left[a_4 \right] - a_1 a_3 \right) X_1 \wedge N_2 \\
& + \left(X_1\left[a_2 \right] - a_3 a_5 \right) X_2 \wedge N_1 \\
& + \left(X_1\left[a_3 \right] + a_2 a_5 - 2a_1 a_4 + X_2\left[a_4 \right] \right) X_2 \wedge N_2 \\
& + \left(-2a_2 a_4 - \frac{2a_4}{r} \right) N_1 \wedge N_2,
\end{aligned}
\tag{3.9}
$$

where $X_i\left[a_j \right]$ $(i = 1, 2; j = 2, 3, 4)$ is the directional derivative with respect to X_i.

Corollary 3.2. *There is no tubular surface having harmonic Gauss map.*

Proof. Assume that the surface M has harmonic Gauss map, i.e., $\Delta G = 0$. From (3.9), we have $a_2^2 + a_3^2 + 2a_4^2 + \frac{1}{r^2} = 0$, a contradiction. Hence, there is no tubular surface having harmonic Gauss map. $\square$

Now, we assume that the tubular surface M given by the parametrization (3.3) has pointwise 1-type Gauss map. From (1.2), and (3.9), the equations

$$
\begin{aligned}
f + f \left\langle C, X_1 \wedge X_2 \right\rangle \quad &= \quad a_2^2 + a_3^2 + 2a_4^2 + \frac{1}{r^2}, \\
f \left\langle C, X_1 \wedge N_1 \right\rangle \quad &= \quad -a_1 a_2 + a_4 a_5 - \frac{a_1}{r}, \\
f \left\langle C, X_1 \wedge N_2 \right\rangle \quad &= \quad -X_1\left[a_4 \right] - a_1 a_3, \\
f \left\langle C, X_2 \wedge N_1 \right\rangle \quad &= \quad X_1\left[a_2 \right] - a_3 a_5, \\
f \left\langle C, X_2 \wedge N_2 \right\rangle \quad &= \quad X_1\left[a_3 \right] + a_2 a_5 - 2a_1 a_4 + X_2\left[a_4 \right], \\
f \left\langle C, N_1 \wedge N_2 \right\rangle \quad &= \quad -2a_2 a_4 - \frac{2a_4}{r}
\end{aligned}
$$

hold. Here, f is a non-zero smooth function. Using the equation (1.2), we can write

$$
\begin{aligned}
C \quad = \quad & A_1 X_1 \wedge X_2 + A_2 X_1 \wedge N_1 + A_3 X_1 \wedge N_2 \\
& + \quad A_4 X_2 \wedge N_1 + A_5 X_2 \wedge N_2 + A_6 N_1 \wedge N_2,
\end{aligned}
\tag{3.10}
$$

where

$$
\begin{aligned}
A_1(u, v) \quad &= \quad \frac{a_2^2 + a_3^2 + 2a_4^2 + \frac{1}{r^2}}{f} - 1, \\
A_2(u, v) \quad &= \quad \frac{-a_1 a_2 + a_4 a_5}{f} - \frac{a_1}{fr}, \\
A_3(u, v) \quad &= \quad \frac{-X_1\left[a_4 \right] - a_1 a_3}{f}, \\
A_4(u, v) \quad &= \quad \frac{X_1\left[a_2 \right] - a_3 a_5}{f}, \\
A_5(u, v) \quad &= \quad \frac{X_1\left[a_3 \right] + a_2 a_5 - 2a_1 a_4 + X_2\left[a_4 \right]}{f}, \\
A_6(u, v) \quad &= \quad \frac{-2a_2 a_4}{f} - \frac{2a_4}{fr}
\end{aligned}
\tag{3.11}
$$

are differentiable functions.

In the following theorems, we give some results about tubular surfaces having pointwise 1-type Gauss map of the first and the second kinds.

Theorem 3.1. *Let M be a tubular surface given with the parametrization (3.3) in $\mathbb{E}^4$. If M has pointwise 1-type Gauss map of the first kind, then M is one of the following surfaces in $\mathbb{E}^4$:*

(i) *circular cylinder,*

(ii) *torus with the parametrization,*

$$M : X(u, v) = (\cos u, \sin u, r \cos v, r \sin v). \tag{3.12}$$

Proof. Let M be a tubular surface given with the parametrization (3.3) in $\mathbb{E}^4$. We suppose that M has pointwise 1-type Gauss map of the first kind, i.e. C is zero. From the equations (3.10), and (3.11), we have $-2a_2a_4r - 2a_4 = 0$, so $a_4 = 0$, which implies $\tau = 0$. Writing $\tau = 0$ in the equations (3.6), we obtain $a_1 = a_2 = a_5 = 0$, and $a_3 = \kappa$. Since the coefficient of $X_2 \wedge N_2$ is zero, $X_1[a_3] = \kappa' = 0$.

Then there are two cases: $\kappa = 0$ or κ is constant. If $\kappa = 0$, γ is a straight line and M is circular cylinder. If κ is constant, γ is a circle and M is a torus with the parametrization (3.12). $\qquad\square$

Corollary 3.3. *If M has pointwise 1-type Gauss map of the first kind, then the function f is constant, and expressed as $f = \kappa^2 + \frac{1}{r^2}$. In this case, the Gauss map is not proper.*

Proof. Let M has pointwise 1-type Gauss map of the first kind. Since A_1 is zero, from the equations (3.11),

$$\frac{a_2^2 + a_3^2 + 2a_4^2 + \frac{1}{r^2}}{f} - 1 = 0,$$

i.e., $\kappa^2 + \frac{1}{r^2} = f$. Since κ and r is constant, f is constant and so the Gauss map is not proper. $\qquad\square$

Theorem 3.2. *Let M be a tubular surface given with the parametrization (3.3) in $\mathbb{E}^4$. If M has pointwise 1-type Gauss map of the second kind,*

$$
\begin{aligned}
X_1[A_1] - a_4A_3 + a_2A_4 + a_3A_5 &= 0, \\
X_1[A_2] - a_5A_3 - a_1A_4 + a_3A_6 &= 0, \\
X_1[A_3] + a_4A_1 + a_5A_2 - a_1A_5 - a_2A_6 &= 0, \\
X_1[A_4] - a_2A_1 + a_1A_2 - a_5A_5 + a_4A_6 &= 0, \\
X_1[A_5] - a_3A_1 + a_1A_3 + a_5A_4 &= 0, \\
X_1[A_6] - a_3A_2 + a_2A_3 - a_4A_4 &= 0,
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
X_2[A_1] + a_4A_5 + \frac{A_2}{r} &= 0, \\
X_2[A_2] + a_4A_6 - \frac{A_1}{r} &= 0, \\
X_2[A_3] &= 0, \\
X_2[A_4] &= 0, \\
X_2[A_5] - a_4A_1 + \frac{A_6}{r} &= 0, \\
X_2[A_6] - a_4A_2 - \frac{A_5}{r} &= 0
\end{aligned}
\tag{3.14}
$$

hold, where $X_i[A_j]$ $(i = 1, 2; j = 1, ..., 6)$ is the directional derivative with respect to X_i.

Proof. Let M has pointwise 1-type Gauss map of the second kind, i.e., C is constant. Using (3.5), and (3.10), we get

$$
\begin{aligned}
\widetilde{\nabla}_{X_1} C &= (X_1[A_1] - a_4A_3 + a_2A_4 + a_3A_5) X_1 \wedge X_2 \\
&\quad + (X_1[A_2] - a_5A_3 - a_1A_4 + a_3A_6) X_1 \wedge N_1 \\
&\quad + (X_1[A_3] + a_4A_1 + a_5A_2 - a_1A_5 - a_2A_6) X_1 \wedge N_2 \\
&\quad + (X_1[A_4] - a_2A_1 + a_1A_2 - a_5A_5 + a_4A_6) X_2 \wedge N_1 \\
&\quad + (X_1[A_5] - a_3A_1 + a_1A_3 + a_5A_4) X_2 \wedge N_2 \\
&\quad + (X_1[A_6] - a_3A_2 + a_2A_3 - a_4A_4) N_1 \wedge N_2,
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\nabla}_{X_2} C \;=\; & \left(X_2\left[A_1\right] + a_4 A_5 + \frac{A_2}{r} \right) X_1 \wedge X_2 \\
& + \left(X_2\left[A_2\right] + a_4 A_6 - \frac{A_1}{r} \right) X_1 \wedge N_1 \\
& + X_2\left[A_3\right] X_1 \wedge N_2 \\
& + X_2\left[A_4\right] X_2 \wedge N_1 \\
& + \left(X_2\left[A_5\right] - a_4 A_1 + \frac{A_6}{r} \right) X_2 \wedge N_2 \\
& + \left(X_2\left[A_6\right] - a_4 A_2 - \frac{A_5}{r} \right) N_1 \wedge N_2.
\end{aligned}
$$

Since C is constant, then the coefficient of the vectors $\widetilde{\nabla}_{X_1} C$ and $\widetilde{\nabla}_{X_2} C$ must vanish. Thus, we have the equations (3.13) and (3.14) which completes the proof. $\qquad\square$

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