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Convex and Starshaped Sets in Manifolds Without Conjugate Points

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ABSTRACT

Let $\mathcal{W}^n$ be the class of C^∞ complete simply connected $n-$dimensional manifolds without conjugate points. The hyperbolic space as well as Euclidean space are good examples of such manifolds. Let $W \in \mathcal{W}^n$ and let A be a subset of W. This article aims at characterization and building convex and starshaped sets in this class from inside. For example, it is proven that, for a compact starshaped set, the convex kernel is the intersection of stars of extreme points only. Also, if a closed unbounded convex set A does not contain a totally geodesic hypersurface and its boundary has no geodesic ray, then A is the convex hull of its extreme points. This result is a refinement of the well-known Karein-Millman theorem.

Keywords: *Manifolds without conjugate points; hyperbolic space; convex sets; stars; starshaped sets; extreme points; kernel.*

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1. An Introduction

Let M be a C^∞ complete Riemannian manifold. A vector field J along a geodesic α is called a Jacobi vector field if

$$D_T^2 J + \Re(\alpha', J)\,\alpha' = 0,$$

where D is the covariant derivative and $\Re$ is the curvature tensor. Two points on a geodesic α are said to be conjugate to each other if there is a non-trivial Jacobi vector field along α that vanishes at both of them. A geodesic α has no conjugate points if every Jacobi field along α vanishes at most once. A C^∞ complete Riemannian manifold M is called a manifold without conjugate points if every geodesic of M has no conjugate points. In this case, the exponential map is a covering map at every point of M. Moreover, if M is simply connected, then $\exp_p$ is a diffeomorphism and M has the property that for every two distinct points p and q in M, there is a unique geodesic joining them. Let $\mathcal{W}^n$ be the class of C^∞ complete simply connected $n-$dimensional Riemannian manifolds without conjugate points. The hyperbolic space H^n, the $n-$dimensional Euclidean space E^n and all manifolds with non-positive curvature are good examples of such manifolds. We refer to [6, 7, 8, 9, 10, 11, 12, 17, 18, 20, 21] and references therein for more details and examples of these manifolds.

It is very nice to study the boundary of a closed set A in $W \in \mathcal{W}^n$ and get global properties of A. For instance, the Krein-Milman theorem [2, 14, 15, 16, 19, 22] in the $n-$dimensional Euclidean space E^n asserts that every compact convex set is the convex hull of its extreme points i.e. given a compact convex set $A \subset E^n$, one only needs its extreme points $E(A)$ to recover the set shape.

The aim of this paper is to characterize convex and starshaped sets in manifolds without conjugate points using their extreme points. Sufficient conditions for a set A in $W \in \mathcal{W}^n$ to be convex, totally geodesic, and starshaped are considered. A generalization of Krein-Milman theorem to the setting of closed unbounded convex sets is given. It is clear that the convex kernel of a starshaped set $A \subset W$, $W \in \mathcal{W}^2$, is the intersection of stars of all points of A. In this work, it is proven that, for a compact starshaped set, the convex kernel is the intersection of stars of extreme points only. Moreover, the original starshaped condition is replaced by a more general condition where the intersection of the stars of certain extreme points is not empty. Thus we get a characterization of starshaped sets in $\mathcal{W}^2$.

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2. Results

Let $W \in \mathcal{W}^n$ and let A be a non-empty subset of W. The geodesic segment joining two points p and q is denoted by $[pq]$. If p is removed we write $(pq]$. The geodesic ray with vertex at p and passing through q is denoted by $R(pq)$ while the geodesic passing through p and q is denoted by $G(pq)$. We say that p sees q via A if $[pq] \subset A$. The set of all points of A that p sees via A is called the star of A at p and is denoted by A_p. A is a starshaped set if there is a point $p \in A$ that sees every point in A i.e. $A_p = A$. The set of all such points p is called the kernel of A and is denoted by $\ker A$. A is convex if $\ker A = A$. A point $p \in A$ is called an extreme point of A if p is not a relative interior point of any segment in A. The set of all extreme points of A is called the profile of A and is denoted by $E(A)$. Note that, the definition of extreme points is introduced here to a non-convex set so it is somewhat different form the classical one. The convex hull, $C(A)$, of A is the intersection of all convex subsets of E^n that contain A. Three concepts of convex sets were introduced to complete Riemannian manifolds in [1]. The three concepts coincide in complete simply connected Riemannian manifolds without conjugate points since geodesics of these manifolds are global minimizers.[3, 4, 5, 13, 21].

We begin with the following important lemmas.

Lemma 2.1. *Let $W \in \mathcal{W}^n$ and let A be a closed subset of W. If a and b are points of A and $[ab] \nsubseteq A$, then there are two points $x, y \in \partial A \cap [ab]$ such that $(xy) \cap A = \varphi$.*

Lemma 2.2. *Let $W \in \mathcal{W}^n$ and let A be a compact subset of W. Then A has at least one extreme point.*

Proof. Let p be in $W \backslash A$. Define the real-valued continuous function f on A by $f(x) = d(p, x)\,, x \in A$. Since A is compact, f attains its maximum value at a point $y \in A$. Thus A is a subset of the closed disc $\overline{B}(p, r)$ with centre at p and radius $r = d(p, y)$ defined by

$$\overline{B}(p, r) = \{x \in W : d(p, x) \leq r\}$$

The point y is an extreme point of A since any geodesic segment containing p in its relative interior cuts the exterior of A. □

Theorem 2.1. *Let $W \in \mathcal{W}^2$ and let A be a compact starshaped subset of W. Then*

$$\ker A = \bigcap_{x \in E(A)} A_x$$

Proof. Let $B = \bigcap_{x \in E(A)} A_x$. By the definition of the kernel of a starshaped set we have

$$\ker A = \bigcap_{x \in A} A_x \subset \bigcap_{x \in E(A)} A_x = B$$

So, we need only to show that $B \subset A$. Let $x \in B \backslash \ker A$. Then there is a point $y \in A$ such that $[xy] \nsubseteq A$. By Lemma 2.1, we find two points $\overline{x}$, $\overline{y}$ in $\partial A \cap [xy]$ such that $(\overline{x}\,\overline{y}) \cap A = \phi$. Let $z \in \ker A$, then z sees $\overline{y}$ via A and hence $R(z\overline{y}) \cap A$ is a closed geodesic segment. Let $q \in \partial A$ such that $R(z\overline{y}) \cap A = [zq]$. Suppose that $q \neq \overline{y}$. Since $x \in E(A)$, q sees x via A. Then z sees the geodesic segment $[xq]$ via A and consequently z sees $[x\overline{y}]$ via A which is a contradiction and q is not an extreme point i.e. there is a geodesic segment $[ab] \subset A$ such that $p \in (ab)$. It is clear from the choice of q that $(ab) \nsubseteq R(z\overline{y})$. Since $z \in \ker A$, z sees (ab) via A. Thus we get two points $\overline{a} = [za] \cap [xy]$ and $\overline{b} = [zb] \cap [xy]$ such that $\overline{y} \in \left[\overline{a}\overline{b}\right]$ which contradicts the choice of $\overline{y}$. So, $q = \overline{y}$. $\overline{y}$ is not an extreme point otherwise $\overline{y}$ sees x. Therefore, we get a geodesic segment $[rs]$ such that $\overline{y} \in (rs) \subset A$. The geodesic $G(rs)$ separates the points x and z otherwise, as we do above, z sees (rs) via A and we get a point $\overline{r} = [zr] \cap [xy] \in A$ that contradicts the choice of $\overline{y}$. Let H_1 be the closed half space generated by $G(x\overline{y})$ that does not contain z and let H_2 be the half space generated by $G(z\overline{y})$ that does not contain x. Let $D = A \cap H_1 \cap H_2$. D has a non-empty intersection with the geodesic segment (rs) i.e. D has points close to $\overline{y}$. Since D is compact, D has an extreme point $p \in \partial D$ by Lemma 2.2. The boundary points of D are either boundary points of A or points of $G(x\overline{y})$. Thus p is an extreme point of A i.e. p sees x via A. Since $z \in \ker A$, z sees the geodesic segment $[px]$ via A and consequently $[x\overline{y}] \subset A$ which is a contradiction and the point x does not exist. □

Theorem 2.2. *Let $W \in \mathcal{W}^2$ and let A be a compact subset of W. Suppose that $B = \bigcap_{x \in E(A)} A_x \neq \varphi$. Then $\ker A = B$ if and only if for every $x \notin A$, there is a geodesic ray with vertex at x having a non-empty intersection with A.*

Proof. Suppose that $\ker A \neq B$ i.e. $B \nsubseteq \ker A$. Let $y \in B \backslash \ker A$. Thus there is a point $z \in A$ such that $[yz] \nsubseteq A$. Then by Lemma 2.1, there are two pints $\overline{y}$ and $\overline{z}$ in $\partial A \cap [yz]$ such that $(\overline{y}\overline{z}) \cap A = \phi$. Let $p \in (\overline{y}\overline{z})$, then we get a point $\overline{p} \notin A$ such that the geodesic ray $R(p\overline{p})$ has a non-empty intersection with A. Rotate the ray $R(p\overline{p})$ to touch ∂A such that p is fixed and the angle between $[p\overline{p}]$ and $[pz]$ decreases. The intersection of the new geodesic ray and A has an extreme point x of A. Thus y sees x via A and $[xy]$ cuts the geodesic ray $R(p\overline{p})$ in a point a which is a contradiction otherwise $a \in [yx]$ which is also a contradiction. Thus $\ker A = B$.

To prove the second implication, let $p \notin A$ and $q \in \ker A$. Consider the geodesic ray $R(qp)$ passing through p. The geodesic ray $R(qp) \setminus [qp)$ has a non-empty intersection with A otherwise $q \notin \ker A$. □

Corollary 2.1. *Let $W \in \mathcal{W}^2$ and let A be a compact subset of W. Then A is starshaped if and only if $\bigcap_{x \in E(A)} A_x \neq \phi$ and for every $x \notin A$, there is a geodesic ray with vertex at x having a non-empty intersection with A. Moreover, $\ker A = \bigcap_{x \in E(A)} A_x$.*

Theorem 2.3. *Let $W \in \mathcal{W}^n$ and let A be a non-empty closed subset of W. If ∂A is convex, then A is a convex set. Moreover, if A has a non-empty interior, then A is unbounded, ∂A is totally geodesic and A^c is also convex.*

Proof. Suppose that A is not convex i.e. we get two points p and q in A such that (pq) is not contained in A. Since A is closed, we find two points r, s in ∂A such that $(rs) \cap A = \phi$ which is a contradiction and so A is convex.

To show that A is unbounded, let $p \in int(A)$. Suppose that A is bounded and so we find a real number ε such that A is contained in the closed ball $\overline{B}(p, \varepsilon)$ of radius ε and center at p. Let $[ab]$ be any chord of $\overline{B}(p, \varepsilon)$ that runs through p. Since A and $[ab]$ are both closed and convex sets, we find a' and b' in ∂A such that $A \cap [ab] = [a'b']$ which is a contradiction since $[a'b']$ cuts the interior of A. Therefore A is unbounded.

Assume that ∂A is not totally geodesic i.e. there are two points a and b in ∂A such that the line $G(ab)$ passing through a and b is not contained in ∂A. Since ∂A and $G(ab)$ are closed convex sets, there are a' and b' in ∂A such that

$$[ab] \subset \partial A \cap G(ab) = [a'b']$$

Let $p \in G(ab) \setminus [a'b']$ (i.e. $p \in int(A) \cap G(ab)$ or $p \in A^c \cap G(ab)$) and assume that $p \in R(b'a')$. If $p \in int(A) \cap G(ab)$, then the geodesic convex cone $C\left(b, \overline{B}(p, \varepsilon)\right)$ with vertex b and base $\overline{B}(p, \varepsilon)$ for a sufficiently small ε shows that a is an interior point which is a contradiction see Figure 1.

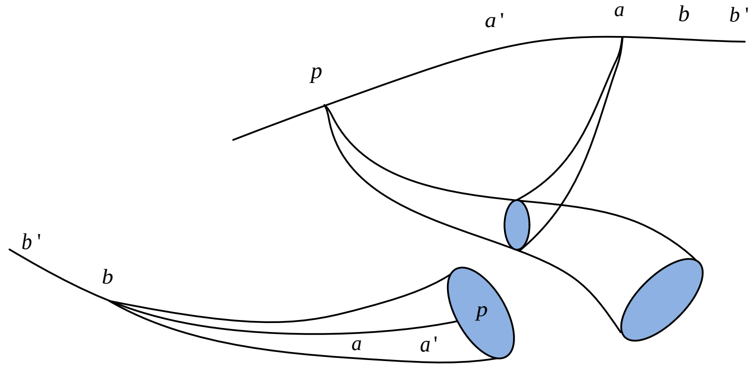

Figure 1. Two cases for the point p

Now we take $p \in A^c \cap G(ab)$. Let q be a point of $int(A)$. The sets $[pq]$ and A are closed convex sets and so there is a point $q' \in \partial A$ such that $[pq] \cap A = [q'q]$. This implies that the intersection $B = \partial A \cap C\left(p, \overline{B}(q, \epsilon)\right)$, for a small ϵ, is a non-empty closed convex set since ∂A is convex. Therefore, B is a convex cross section of $C\left(p, \overline{B}(q, \epsilon)\right)$ that determines a hypersurface H whose intersection with $C\left(p, \overline{B}(q, \epsilon)\right)$ is B. At least one of the points a and b(say a) does not lie in H otherwise the line $G(ab)$ lies in H which contradicts the fact that p is the vertex of the convex cone $C\left(p, \overline{B}(q, \epsilon)\right)$. Now, the convex cone $C(a, B)$ has dimension n i.e. $C(a, B)$ has interior points which is a contradiction since both a and B are in ∂A see Figure 1. This contradiction completes the proof. □

Corollary 2.2. *Let $W \in \mathcal{W}^n$ and let A be a non-empty open subset of W and $int(\overline{A}) = A$. If ∂A is convex, then A is unbounded convex set and ∂A is totally geodesic.*

Proof. It is clear that $\overline{A}$ satisfies the hypothesis of Theorem 2.3. Therefore $\partial\overline{A} = \partial A$ is affine and $\overline{A}$ is convex and unbounded. Note that if A is bounded, then $\overline{A}$ is also bounded and equivalently, $\overline{A}$ is unbounded implies that A is unbounded. Since the interior of a closed convex set is also convex, the convexity of $\overline{A}$ implies that A is convex. □

Theorem 2.4. *Let $W \in \mathcal{W}^n$ and let A be a non-empty closed subset of W. If $(pq) \subset int\,(A)$ for every pair of boundary points p, q of A, then A is strictly convex.*

Proof. It is enough to prove that A is convex since the strict convexity of A is direct. Now, we assume that A is not convex i.e. there are p, q in A such that $[pq]$ is not contained in A. Since A is closed, there are p', q' in ∂A such that $(p'q') \cap A = \phi$ which is a contradiction and A is convex. □

Corollary 2.3. *Let $W \in \mathcal{W}^n$ and let A be a non-empty closed subset of W. A is convex if and only if $(pq) \subset \partial A$ or $(pq) \subset int\,(A)$ for each pair of boundary points p, q.*

Since the interior of a closed convex set is again convex, this result is still true for open sets such that $int\left(\overline{A}\right) = A$. The following example shows that the closeness is important. Let A be a subset of $E^2 \in \mathcal{W}^2$ defined by $A = \{(x, y) : 0 \prec x \prec 1, 0 \prec y \prec 1\} \cup \{(0, 0), (1, 1), (1, 0), (0, 1)\}$. A is neither closed nor open and $(pq) \subset \partial A$ or $(pq) \subset int\,(A)$ for each pair of boundary points p, q but A is not convex.

Proposition 2.1. *Let $W \in \mathcal{W}^n$ and let A be a non-empty closed subset of W. If there is a point $p \in A$ that sees ∂A via A, then A is starshaped.*

Proof. We claim that $p \in \ker A$. Suppose that p is not in $\ker A$ i.e. there is a point $q \in A$ such that $[pq]$ is not contained in A. Since A is closed, there are two points p' and q' in $\partial A \cap [pq]$ such that $(p'q') \cap A = \phi$. Thus p does not see neither p' nor q'. This contradicts the fact that p sees ∂A via A and the proof is complete. □

It is clear that the converse of this result is also true. Thus we can say that this proposition is a characterization of the kernel of the closed starshaped sets. This means that the kernel of a closed starshaped set A is only the points of A that see ∂A. The following corollary is direct.

Corollary 2.4. *Let $W \in \mathcal{W}^n$ and let A be a non-empty closed convex subset of W. If ∂A is starshaped, then $\ker(\partial A) \subset \ker A$.*

In the light of the above results, one can test the convexity and starshapedness of a closed set A using its boundary points. In the next part a minimal subset of these boundary points will build A up from inside.

Theorem 2.5. *Let $W \in \mathcal{W}^n$ and let A be a non-empty closed convex subset of W. If A has no hyperplane, then $A = C\,(\partial A)$.*

Proof. Since A is convex, A is connected. We will prove that $C\,(\partial A)$ is open and closed in the relative topology on A and hence $A = C\,(\partial A)$.

First, we prove that $C\,(\partial A)$ is open in A. Let $p \in C\,(\partial A) \subset A$. We have the following cases:

1. $p \in C\,(\partial A) \cap int\,(A)$: Let $B_\delta = B\,(p, \delta) \cap A$. In this case there exists a real number δ such that $B\,(p, \delta) \subset A$ and so $B_\delta = B\,(p, \delta)$. Suppose that p is not an interior point of $C(\partial A)$ i.e. for any δ, B_δ is not contained in $C\,(\partial A)$ and so p is a boundary point of $C(\partial A)$. Therefore, there is a supporting hyperplane H_1 of $\overline{C(\partial A)}$(the closure of $C(\partial A)$ is a closed convex subset of A) at p and $\overline{C(\partial A)}$ is contained in a closed half-space with boundary H_1. Let x be any point of $B\,(p, \delta)$ that lies on the other side of H_1 and let H_2 be a parallel hyperplane to H_1 at x. Since A does not contain a hyperplane, we find a point $y \in H_2 \setminus A$. The line segment $[xy]$ cuts ∂A at a point $z \in H_2$ which contradicts the fact that H_1 supports $\overline{C\,(\partial A)}$. This contradiction implies that p is an interior point of $C\,(\partial A)$ in the relative topology of A.

2. $p \in C\,(\partial A) \cap \partial A$: in this case, $B\,(p, \delta)$ has a non-empty intersection with A for any real number δ. Let $B_\delta = B\,(p, \delta) \cap A$. Suppose that p is not an interior point of $C\,(\partial A)$. Then, for any δ, the set B_δ has a point x which is not in $C\,(\partial A)$. But $\overline{C\,(\partial A)}$ is closed convex set and $x \notin \overline{C\,(\partial A)}$, and so we get a hyperplane H passing through x that separates x and $\overline{C\,(\partial A)}$. Since A does not have a hyperplane, there is a point y in $H \setminus A$ where $[xy]$ cuts ∂A. Thus H cuts ∂A and so H cuts $C\,(\partial A)$ which is a contradiction and so p is an interior point of $C\,(\partial A)$ in the relative topology on A.

This discussion above implies that $C(\partial A)$ is an open set in A. Now, we want to prove that $C(\partial A)$ is closed in A. Let p be a boundary point of $C(\partial A)$. If $p \in \partial A$, then $p \in C(\partial A)$. Let $p \in intA$, then there is a small positive real number δ such that $B(p,\delta) \subset A$. Since p is a boundary point of $\overline{C(\partial A)}$, $B(p,\delta) \neq B(p,\delta) \cap C(\partial A) \neq \phi$. Therefore, we find a point x in $B(p,\delta)$ which is not in $C(\partial A)$. Since $\overline{C(\partial A)}$ is a closed convex set, we get a hyperplane H passing through x and does not intersect $C(\partial A)$. But A does not have a hyperplane and so H cuts ∂A which is a contradiction and $p \in C(\partial A)$ i.e. $C(\partial A)$ is closed in the relative topology on A and the proof is complete. □

In general, sets need not have extreme points. The following proposition gives a sufficient condition for the existence of extreme points.

Proposition 2.2. *Let $W \in \mathcal{W}^n$ and let A be a non-empty closed convex subset of W. A contains at least one extreme point if and only if A has no geodesic.*

Proof. Let us assume that A has a geodesic l. Suppose that A has an extreme point p. It is clear that $p \notin l$. Let B be the closed convex hull of p and l. B is a subset of A since A is a closed convex set containing both p and l. It is clear that B contains a line passing through p and parallel to l i.e. either p is not an extreme point or the line l does not exist. □

Lemma 2.3. *Let $W \in \mathcal{W}^n$ and let A be a non-empty closed convex subset of W. If H is a supporting totally geodesic hypersurface of A, then $E(H \cap A) \subset E(A)$*

Proof. Let p be an extreme point of $H \cap A$. Suppose that $p \notin E(A)$ i.e. there are x, y in ∂A such that $p \in (xy)$. The hypersurface H supports A at p and so $[xy] \subset H$. This implies that $p \in [xy] \subset H \cap A$ which contradicts the fact that p is an extreme point of $H \cap A$. This contradiction completes the proof. □

The minimal subset of a compact convex set A which generates A is is its extreme points. Our next main theorem shows that this property is more general.

Theorem 2.6. *Let $W \in \mathcal{W}^n$ and let A be a non-empty closed convex subset of W. If A has no hyperplane and its boundary has no ray, then $A = C(E(A))$.*

Proof. To prove that $A = C(E(A))$, it suffices to prove that $\partial A \subset C(E(A))$ and by Theorem 2.5, we get that $A = C(\partial A) \subset C(E(A)) \subset A$ and hence $A = C(E(A))$. Let $p \in \partial A$. If p is an extreme point, then $p \in E(A) \subset C(E(A))$. Now suppose that p is not an extreme point i.e. there are x, y in ∂A such that $p \in (xy)$. Since A is a closed convex set, there is a supporting totally geodesic hypersurface H of A at p. It is clear that the set $H \cap A$ is a non-empty closed convex subset of ∂A. Since ∂A has no ray, the set $H \cap A$ is bounded i.e. $H \cap A$ is a compact convex set. Therefore, $H \cap A = C(E(H \cap A))$. But $E(H \cap A) \subset E(A)$ and so $p \in C(E(H \cap A)) \subset C(E(A))$ and the proof is complete. □

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