INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 12 NO. 2 PAGE 241–249 (2019)

Notes About a New Metric on the Cotangent Bundle

Filiz Ocak*

(Communicated by Arif Salimov)

ABSTRACT

In this article, we construct a new metric $\breve{G} = {}^{R}\nabla + \sum\limits_{i,j=1}^{m} a^{ji}\delta p_j \delta p_i$ in the cotangent bundle, where ${}^{R}\nabla$ is the Riemannian extension and a^{ji} is a symmetric (2,0)-tensor field on a differentiable manifold.

Keywords: *Cotangent bundle; Riemannian extension; geodesic; para-Nordenian metric.*
AMS Subject Classification (2010): *Primary: 53C07; Secondary: 53C15.*

1. Introduction

Cotangent bundle of differentiable manifold with a Riemannian extension, which was introduced by Patterson and Walker [11], was investigated by many authors [2, 3, 4, 7, 12, 17]. Riemannian extension has been developed in several ways. Calviño-Louzao et.al [5] introduced the modified Riemannian extension using a symmetric tensor field of type (0, 2) and studied some geometric applications. Gezer and his collaborators studied the curvature properties and the Kähler- Norden structure with respect to the modified Riemannian extension [8]. Aslanci and Cakan [1] discussed the curvature properties of the deformed Riemannian extension in the cotangent bundle by means of musical isomorphism between tangent and cotangent bundle. Then Salimov and Cakan [13] investigated the deformed Riemannian extension using twin Norden metric.

In this paper, after the introduction and preliminaries, in section 3, we construct a new metric on the cotangent bundle using the Riemannian extension and quadratic differential form $\sum\limits_{i,j=1}^{m} a^{ji}\delta p_j \delta p_i$, where $\delta p_j = dp_j - p_h \Gamma_{ij}^{h} dx^j$. Then we calculate Levi-Civita connection and components of the curvature tensor for this metric. In section 4, we get the necessary condition for the horizontal lift of any connection on the cotangent bundle to be a metric connection. In section 5, we investigate the geodesics on the cotangent bundle with respect to the new metric. Then we obtain the horizontal lift of a geodesic on (M, g) that does not need to be a geodesic on $(T^*M, \breve{G})$. In section 6, we investigate the almost para-Nordenian, the para- Kählerian and the para-Nordenian properties of the new metric in the cotangent bundle.

2. Preliminaries

Let M be an m-dimensional C^∞-manifold with torsion-free connection ∇ , T^*M be the cotangent bundle of M and $\pi : T^*M \to M$ be the natural projection. For any local coordinates $\left(U, x^i\right), i = 1, ..., m$ on M, we denote by $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i\right), \bar{i} = m+1, ..., 2m$ the corresponding local coordinates on T^*M, where $x^{\bar{i}} = p_i$ are the components of the covector p in each cotangent space $T^*_x M, x \in U$ with respect to the natural coframe $\left\{dx^i\right\}$. Let $F(M)\,(F(T^*M))$ be the ring of real-valued C^∞ functions on $M(T^*M)$ and $\Im_s^r(M)\,(\Im_s^r(T^*M))$ be the module over $F(M)\,(F(T^*M))$ of C^∞ tensor fields of type (r,s).

The local expression of a vector and covector field is given by $Z = Z^i \frac{\partial}{\partial x^i}$ and $\theta = \theta_i dx^i$ in $U \subset M$, respectively. With respect to the natural frame $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\right\}$, then the vertical lift ${}^{V}\theta \in \Im_0^1(T^*M)$ of $\theta \in \Im_1^0(M)$, the horizontal

Received : *21–March–2019,* ***Accepted :*** *18–May–2019*
*** Corresponding author***

and complete lifts ${}^{H}Z, {}^{C}Z \in \Im_0^1(T^*M)$ of $Z \in \Im_0^1(M)$ are given by

$$ {}^{V}\theta = \sum_i \theta_i \frac{\partial}{\partial x^{\bar{i}}}, \tag{2.1} $$

$$ {}^{H}Z = Z^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma_{ij}^h Z^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{2.2} $$

$$ {}^{C}Z = Z^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i Z^h \frac{\partial}{\partial x^{\bar{i}}}, \tag{2.3} $$

where the coefficients Γ_{ij}^h are the Christoffel symbols of the Levi-Civita connection ∇ on M (for details, see [17]).

In [17], the adapted frame $\{\tilde{e}_{(\alpha)}\} = \{\tilde{e}_{(j)}, \tilde{e}_{(\bar{j})}\}$ is given by

$$ \tilde{e}_{(j)} = {}^{H}Z_{(j)} = \frac{\partial}{\partial x^j} + \sum_h p_a \Gamma_{hj}^a \frac{\partial}{\partial x^{\bar{h}}}, $$
$$ \tilde{e}_{(\bar{j})} = {}^{V}\omega^{(j)} = \frac{\partial}{\partial x^{\bar{j}}}. \tag{2.4} $$

From (2.1), (2.2), (2.3) and (2.4), in the the adapted frame $\{\tilde{e}_{(\alpha)}\}$, we see that ${}^{V}\theta$, ${}^{H}Z$ and ${}^{C}Z$ have the following components

$$ {}^{V}\theta = \sum_i \theta_i \tilde{e}_{(\bar{i})}, \quad {}^{V}\theta = ({}^{V}\theta^\alpha) = \begin{pmatrix} 0 \\ \theta_i \end{pmatrix} \tag{2.5} $$

$$ {}^{H}Z = Z^i \tilde{e}_{(i)}, \quad {}^{H}Z = ({}^{H}Z^\alpha) = \begin{pmatrix} Z^i \\ 0 \end{pmatrix}, \tag{2.6} $$

$$ {}^{C}Z = Z^i \tilde{e}_{(i)} - p_h \nabla_i Z^h \tilde{e}_{(\bar{i})}, \quad {}^{C}Z = \left({}^{C}Z^\alpha\right) = \begin{pmatrix} Z^i \\ -p_h \nabla_i Z^h \end{pmatrix}. \tag{2.7} $$

By (2.4), we consider local 1-forms $\tilde{\eta}^\alpha$ and vector field $\tilde{e}_\beta$ in $\pi^{-1}(U)$ given by

$$ \tilde{\eta}^\alpha = \bar{\mathrm{A}}^\alpha{}_B dx^B, \tilde{e}_\beta = \mathrm{A}_\beta{}^A \partial_A $$

where

$$ A^{-1} = (\bar{A}^\alpha{}_B) = \begin{pmatrix} \bar{A}^i{}_j & \bar{A}^i{}_{\bar{j}} \\ \bar{A}^{\bar{i}}{}_j & \bar{A}^{\bar{i}}{}_{\bar{j}} \end{pmatrix} = \begin{pmatrix} \delta_j^i & 0 \\ -p_a \Gamma_{ij}^a & \delta_i^j \end{pmatrix} \tag{2.8} $$

and

$$ A = (A_\beta{}^A) = \begin{pmatrix} A_j{}^i & A_{\bar{j}}{}^i \\ A_j{}^{\bar{i}} & A_{\bar{j}}{}^{\bar{i}} \end{pmatrix} = \begin{pmatrix} \delta_j^i & 0 \\ p_a \Gamma_{ij}^a & \delta_i^j \end{pmatrix}. \tag{2.9} $$

Also, the set $\{\tilde{\eta}^\alpha\}$ is the coframe dual to the adapted frame $\{\tilde{e}_{(\beta)}\}$, i.e. $\tilde{\eta}^\alpha(\tilde{e}_{(\beta)}) = \bar{A}^\alpha{}_B A_\beta{}^B = \delta_\beta^\alpha$.

The Lie bracket of the adapted frame $\{\tilde{e}_{(\alpha)}\}$ on T^*M is given by

$$ [\tilde{e}_\gamma, \tilde{e}_\beta] = \Omega_{\gamma\beta}{}^\alpha \tilde{e}_\alpha, $$

where

$$ \Omega_{\gamma\beta}{}^\alpha = (\tilde{e}_\gamma A_\beta{}^A - \tilde{e}_\beta A_\gamma{}^A) \bar{A}^\alpha{}_A. $$

Using (2.4), (2.8) and (2.9), we have the non-zero components of $\Omega_{\gamma\beta}{}^\alpha$ as follows

$$ \begin{cases} \Omega_{l\bar{j}}{}^{\bar{i}} = -\Omega_{\bar{j}l}{}^{\bar{i}} = -\Gamma_{li}^j, \\ \Omega_{lj}{}^{\bar{i}} = p_a R_{lji}{}^a, \end{cases} \tag{2.10} $$

where $R_{lji}{}^a$ is the local components of the curvature tensor R of ∇.

3. New metric $\breve{G}$ on T^*M

The Riemannian extension ${}^R\nabla \in \Im^0_2(T^*M)$ describes a pseudo-Riemannian metric in T^*M. The line element of the Riemannian extension ${}^R\nabla$ is determined by

$$ds^2 = 2dx^i\delta p_i,$$

where $\delta p_i = dp_i - p_h\Gamma^h_{ji}dx^i$ (see [11, 17] for details).

Using the Riemannian extension and the quadratic differential form $\sum\limits_{i,j=1}^{m} a^{ji}\delta p_j\delta p_i$, where $\delta p_i = dp_i - p_h\Gamma^h_{ji}dx^i$ and a^{ji} denote the components of a symmetric tensor field of type (2,0) on M, we have a new metric

$$\breve{G} = 2dx^j\delta p_i + \sum_{i,j=1}^{m} a^{ji}\delta p_j\delta p_i \tag{3.1}$$

on T^*M (for $a^{ji} = g^{ij}$, see [10]).

From (2.9) and (3.1), in the adapted frame $\{\tilde{e}_{(\alpha)}\}$, the metric $\breve{G}$ has the following components

$$\breve{G} = \begin{pmatrix} \breve{G}_{ji} & \breve{G}_{j\bar{i}} \\ \breve{G}_{\bar{j}i} & \breve{G}_{\bar{j}\bar{i}} \end{pmatrix} = \begin{pmatrix} 0 & \delta^i_j \\ \delta^j_i & a^{ji} \end{pmatrix}. \tag{3.2}$$

By (2.1),(2.2) and (3.2), we have

$$\begin{aligned} &\breve{G}\left({}^HV, {}^HZ\right) = 0, \\ &\breve{G}\left({}^HZ, {}^V\beta\right) = {}^V(\beta(Z)) = \beta(Z)\circ\pi, \\ &\breve{G}({}^V\omega, {}^V\beta) = {}^V\left(\tilde{a}\left(\omega,\beta\right)\right) = \tilde{a}\left(\omega,\beta\right)\circ\pi \end{aligned} \tag{3.3}$$

for any $V, Z \in \Im^1_0(M)$ and $\omega, \beta \in \Im^0_1(M)$, where $\tilde{a}$ is a symmetric tensor field of type (2,0) on M. The (0,2)-tensor field on T^*M is entirely detected by action on the vector fields of type ${}^HZ$ and ${}^V\beta$ (see [17, p.280]). So $\breve{G}$ is completely determined by the equation (3.3).

From (2.2) and (2.3), we see that the complete lift ${}^CZ$ of $Z \in \Im^1_0(M)$ is expressed by

$${}^CZ = {}^HZ - {}^V\left(p\left(\nabla Z\right)\right), \tag{3.4}$$

where $p\left(\nabla Z\right) = p_k\left(\nabla_i Z^k\right)dx^i$. Using (3.3) and (3.4), we get

$$\tilde{G}\left({}^CV, {}^CZ\right) = -{}^V\left[\left(p\left(\nabla V\right)\right)(Z) + \left(p\left(\nabla Z\right)\right)(V) + \tilde{a}\left(p\left(\nabla V\right), p\left(\nabla Z\right)\right)\right], \tag{3.5}$$

where $\tilde{a}\left(p\left(\nabla V\right), p\left(\nabla Z\right)\right) = a^{ij}\left(p_t\nabla_i V^t\right)\left(p_m\nabla_j Z^m\right)$. Then we say that this metric is completely determined with vector fields of type ${}^CV$ and ${}^CZ$ on T^*M (see [17, p.237]).

From (3.5), we obtain the following theorem:

Theorem 3.1. *The complete lifts ${}^CV$, ${}^CZ$ of two vector fields V, Z to T^*M with metric $\breve{G}$ are orthogonal if V, Z are parallel.*

In [17, p.238 and p.277], we know that the Lie bracket for the horizontal, vertical and complete lifts of vector fields on the cotangent bundle T^*M of M satisfies the following:

$$\begin{aligned} &\left[{}^HV, {}^HZ\right] = {}^H\left[V,Z\right] + \gamma R\left(V,Z\right) = {}^H\left[V,Z\right] + {}^V\left(pR\left(V,Z\right)\right), \\ &\left[{}^HZ, {}^V\theta\right] = {}^V\left(\nabla_Z\theta\right), \\ &\left[{}^V\omega, {}^V\theta\right] = 0, \\ &\left[{}^CV, {}^HZ\right] = {}^H\left[V,Z\right] + {}^V\left(p\left(L_V\nabla\right)Z\right), \\ &\left[{}^CZ, {}^V\theta\right] = {}^V\left(L_Z\theta\right) \end{aligned} \tag{3.6}$$

for any $V, Z \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$.

Theorem 3.2. *Given an m-dimensional manifold (M,g) and its cotangent bundle $(T^*M,\breve{G})$. In the adapted frame $\{\tilde{e}_{(\alpha)}\}$, the Levi-Civita connection $\breve{\nabla}$ of the metric $\breve{G}$ satisfies the following equations:*

$$\begin{aligned} &i)\breve{\nabla}_{\tilde{e}_i}\tilde{e}_j=\left(\Gamma^l_{ij}+\tfrac{1}{2}p_sR_{ijt}{}^sa^{lt}\right)\tilde{e}_l+\left(p_sR_{lji}{}^s\right)\tilde{e}_{\bar{l}},\\ &ii)\breve{\nabla}_{\tilde{e}_i}\tilde{e}_{\bar{j}}=\left(\tfrac{1}{2}\nabla_ia^{jl}-\Gamma^l_{it}a^{lt}\right)\tilde{e}_l+\left(-\Gamma^j_{il}+p_sR_{lit}{}^sa^{jt}\right)\tilde{e}_{\bar{l}},\\ &iii)\breve{\nabla}_{\tilde{e}_{\bar{i}}}\tilde{e}_j=\left(\tfrac{1}{2}\nabla_ja^{li}\right)\tilde{e}_l+\left(\tfrac{1}{2}p_sR_{ljt}{}^sa^{it}\right)\tilde{e}_{\bar{l}},\\ &iv)\breve{\nabla}_{\tilde{e}_{\bar{i}}}\tilde{e}_{\bar{j}}=\left(-\tfrac{1}{2}\nabla_la^{ij}\right)\tilde{e}_{\bar{l}}, \end{aligned}\tag{3.7}$$

where $R_{lji}{}^s$ and Γ^l_{ij} denote the components of the curvature tensor and coefficients of ∇, respectively.

Proof. It is known that the Koszul formula for $\breve{\nabla}$ is given by

$$\begin{aligned} 2\breve{G}\left(\breve{\nabla}_ZY,V\right)&=Z\left(\breve{G}(Y,V)\right)+Y\left(\breve{G}(V,Z)\right)-V\left(\breve{G}(Z,Y)\right)\\ &\quad-\breve{G}(Z,[Y,V])+\breve{G}(Y,[V,Z])+\breve{G}(V,[Z,Y]) \end{aligned}$$

for any $V,Y,Z\in\Im^1_0(T^*M)$. In the Koszul formula, we substitute $Z=\tilde{e}_i,\tilde{e}_{\bar{i}},\quad Y=\tilde{e}_j,\tilde{e}_{\bar{j}},\quad V=\tilde{e}_k,\tilde{e}_{\bar{k}}$. Using (2.10), (3.2) and the first Bianchi identity for the curvature tensor R, we do standard calculations. □

Now we use $\breve{\nabla}_{e_\alpha}e_\beta=\breve{\Gamma}^\delta_{\alpha\beta}e_\delta$ with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$ of T^*M, where the coefficients of the Levi-Civita connection $\breve{\nabla}$ of the metric $\breve{G}$ are denoted by $\breve{\Gamma}^\delta_{\alpha\beta}$. By using Theorem 3.2, we obtain

Corollary 3.1. *Given an m-dimensional manifold (M,g) and its cotangent bundle $(T^*M,\breve{G})$. In the adapted frame $\{\tilde{e}_{(\alpha)}\}$, then the Christoffel symbols $\breve{\Gamma}^\delta_{\alpha\beta}$ of $\breve{\nabla}$ are found as follows:*

$$\begin{array}{ll} \breve{\Gamma}^k_{ij}=\Gamma^k_{ij}+\tfrac{1}{2}p_sR_{ijt}{}^sa^{tk} & \breve{\Gamma}^{\bar{k}}_{ij}=p_sR_{kji}{}^s,\\ \breve{\Gamma}^k_{i\bar{j}}=\tfrac{1}{2}\nabla_ia^{jk}-\Gamma^j_{it}a^{tk}, & \breve{\Gamma}^k_{\bar{i}j}=\tfrac{1}{2}\nabla_ja^{ik},\\ \breve{\Gamma}^{\bar{k}}_{i\bar{j}}=-\Gamma^j_{ik}+\tfrac{1}{2}p_sR_{kit}{}^sa^{jt}, & \breve{\Gamma}^{\bar{k}}_{\bar{i}j}=\tfrac{1}{2}p_sR_{kjt}{}^sa^{ti},\\ \breve{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}}=-\tfrac{1}{2}\nabla_ka^{ij}, & \breve{\Gamma}^k_{\bar{i}\bar{j}}=0. \end{array}\tag{3.8}$$

Let $\hat{V},\hat{Z}\in\Im^1_0(T^*M)$ and $\hat{V}=\hat{V}^\alpha\tilde{e}_\alpha,\quad\hat{Z}=\hat{Z}^\beta\tilde{e}_\beta$. In the adapted frame $\{\tilde{e}_{(\alpha)}\}$, the covariant derivative $\breve{\nabla}_{\hat{Z}}\hat{V}$ is given by

$$\breve{\nabla}_{\hat{Z}}\hat{V}^\alpha=\hat{Z}^\gamma\tilde{e}_\gamma\hat{V}^\alpha+\breve{\Gamma}^\alpha_{\gamma\beta}\hat{V}^\beta\hat{Z}^\gamma.\tag{3.9}$$

Using (2.4), (2.5), (2.6), (3.8) and (3.9), we obtain

Proposition 3.1. *Given an m-dimensional manifold (M,g) and its cotangent bundle $(T^*M,\breve{G})$. In the following, the Levi-Civita connection $\breve{\nabla}$ of the metic $\breve{G}$ provides*

$i)\breve{\nabla}_{{}^HZ}{}^HV={}^H(\nabla_ZV)+\tfrac{1}{2}{}^H(\tilde{a}\circ pR(Z,V))+{}^V\left(\tilde{V}R(Z,\tilde{p})\right),$

$ii)\breve{\nabla}_{{}^HZ}{}^V\omega=\tfrac{1}{2}{}^H((\nabla_Z\tilde{a})(\omega,))+{}^H(\tilde{a}\circ\nabla_Z\omega)+{}^V(\nabla_Z\omega)+\tfrac{1}{2}{}^V(\tilde{a}(pR(,Z),\omega)),$

$iii)\breve{\nabla}_{{}^V\omega}{}^HZ=\tfrac{1}{2}{}^H((\nabla_Z\tilde{a})(\omega,))+\tfrac{1}{2}{}^V\left(\tilde{Z}R(\tilde{\omega},\tilde{p})\right),$

$iv)\breve{\nabla}_{{}^V\omega}{}^V\theta=-\tfrac{1}{2}{}^V((\nabla\tilde{a})(\omega,\theta))$

for all $V,Z\in\Im^1_0(M)$, $\omega,\theta\in\Im^0_1(M)$, where $(\nabla_Z\tilde{a})(\omega,)=\omega_iZ^j\nabla_ja^{li}$,
${}^V(\tilde{a}(pR(,Z),\omega))=a^{jt}p_sR_{lit}{}^sZ^i\omega_j$, $\tilde{Z}=g\circ Z\in\Im^0_1(M^n)$, $\tilde{Z}R(Y,\tilde{p})\in\Im^0_1(M^n)$.

3.1. *Curvature tensor of $\breve{\nabla}$*

Now, we investigate the curvature tensor $\breve{R}$ of $(T^*M,\breve{G})$. We get

$$\breve{R}\left(\tilde{e}_{(\alpha)},\tilde{e}_{(\beta)}\right)\tilde{e}_{(\gamma)}=\breve{\nabla}_\alpha\breve{\nabla}_\beta\tilde{e}_{(\gamma)}-\breve{\nabla}_\beta\breve{\nabla}_\alpha\tilde{e}_{(\gamma)}-\Omega_{\alpha\beta}{}^\varepsilon\breve{\nabla}_\varepsilon\tilde{e}_{(\gamma)},$$

where $\breve{\nabla}_\beta = \breve{\nabla}_{\tilde{e}_{(\beta)}}$. The components of the curvature tensor $\breve{R}$ are defined by

$$\breve{R}_{\alpha\beta\gamma}{}^{\sigma} = \tilde{e}_\alpha \breve{\Gamma}^{\sigma}_{\beta\gamma} - \tilde{e}_\beta \breve{\Gamma}^{\sigma}_{\alpha\gamma} + \breve{\Gamma}^{\sigma}_{\alpha\varepsilon}\breve{\Gamma}^{\varepsilon}_{\beta\gamma} - \breve{\Gamma}^{\sigma}_{\beta\varepsilon}\breve{\Gamma}^{\varepsilon}_{\alpha\gamma} - \Omega_{\alpha\beta}{}^{\varepsilon}\breve{\Gamma}^{\sigma}_{\varepsilon\gamma}$$

with respect to the adapted frame $\left\{\tilde{e}_{(\alpha)}\right\}$.

From (2.10) and (3.8), we find the components of $\breve{R}$ as follows:

$$\begin{aligned}
\breve{R}_{kij}{}^{l} &= R_{kij}{}^{l} + \tfrac{1}{4}p_s p_m a^{nl}a^{ft}\left(R_{ktn}{}^{s}R_{ijf}{}^{m} - R_{itn}{}^{s}R_{kjf}{}^{m}\right)\\
&\quad + \tfrac{1}{2}p_s a^{lt}\left(\nabla_k R_{ijt}{}^{s} - \nabla_i R_{kjt}{}^{s}\right) - p_s a^{ml}\left(R_{tjk}{}^{a}\Gamma^{t}_{im} - R_{tji}{}^{a}\Gamma^{t}_{km}\right)\\
&\quad + \tfrac{1}{2}p_s\left(R_{ijt}{}^{s}\nabla_k a^{tl} + R_{tkj}{}^{s}\nabla_i a^{tl} + R_{tji}{}^{s}\nabla_k a^{tl} - R_{kit}{}^{s}\nabla_j a^{tl}\right),\\
\breve{R}_{\bar{k}ij}{}^{l} &= \tfrac{1}{2}R_{ijt}{}^{k}a^{tl} - \tfrac{1}{2}\nabla_i\nabla_j a^{kl} + \tfrac{1}{2}p_s\Gamma^{t}_{im}a^{ml}R_{tjf}{}^{s}a^{fk}\\
&\quad + \tfrac{1}{4}p_s\left(R_{ijm}{}^{s}a^{mt}\nabla_t a^{kl} - R_{itm}{}^{s}a^{ml}\nabla_j a^{kt} - R_{tjf}{}^{s}a^{fk}\nabla_i a^{tl}\right),\\
\breve{R}_{ki\bar{j}}{}^{l} &= R_{ikt}{}^{j}a^{tl} - \Gamma^{j}_{kt}\Gamma^{t}_{if}a^{fl} + \Gamma^{j}_{it}\Gamma^{t}_{kf}a^{fl} - \Gamma^{j}_{it}\nabla_k a^{tl}\\
&\quad + \Gamma^{t}_{im}\nabla_i a^{tl} + \tfrac{1}{2}\left(\nabla_k\nabla_i a^{jl} - \nabla_i\nabla_k a^{jl}\right)\\
&\quad + \tfrac{1}{4}p_s\left(R_{ktm}{}^{s}\left(a^{ml}\nabla_i a^{jt} + a^{mj}\nabla_i a^{tl}\right) + R_{tim}{}^{s}\left(a^{mj}\nabla_k a^{tl} + a^{ml}\nabla_k a^{jt}\right)\right)\\
&\quad + \tfrac{1}{2}p_s\left(R_{tkm}{}^{s}a^{ml}\left(\Gamma^{t}_{if}a^{fl} + \Gamma^{j}_{if}a^{ft}\right) + R_{itm}{}^{s}a^{ml}\left(\Gamma^{j}_{kf}a^{ft} + \Gamma^{t}_{kf}a^{fl}\right)\right),\\
\breve{R}_{ki\bar{j}}{}^{\bar{l}} &= R_{ikl}{}^{j} + \tfrac{1}{4}p_s p_m\left(R_{lkf}{}^{s}R_{tin}{}^{s}a^{ft}a^{nj} - R_{lif}{}^{s}R_{tkn}{}^{m}a^{tf}a^{jn}\right)\\
&\quad + \tfrac{1}{2}p_s\left(a^{mj}\nabla_k R_{lim}{}^{s} - a^{tj}\nabla_i R_{lkt}{}^{s}\right) - p_a a^{mt}\left(R_{ltk}{}^{a}\Gamma^{j}_{im} - R_{lti}{}^{a}\Gamma^{j}_{km}\right)\\
&\quad + \tfrac{1}{2}p_s\left(R_{lim}{}^{s}\nabla_k a^{mj} - R_{lkm}{}^{s}\nabla_i a^{mj} - R_{ltk}{}^{s}\nabla_i a^{jt}\right)\\
&\quad + \tfrac{1}{2}p_s\left(R_{kit}{}^{s}\nabla_l a^{tj} - R_{lti}{}^{s}\nabla_k a^{jt}\right),\\
\breve{R}_{kij}{}^{\bar{l}} &= p_s\left(\nabla_k R_{lji}{}^{s} - \nabla_i R_{ljk}{}^{s}\right) + \tfrac{1}{2}p_s p_m a^{ft}\left(R_{ltk}{}^{m}R_{ijf}{}^{s} + R_{lkf}{}^{m}R_{tji}{}^{s}\right)\\
&\quad - \tfrac{1}{2}p_s p_m a^{ft}\left(R_{itl}{}^{m}R_{kjf}{}^{s} + R_{lif}{}^{m}R_{tjk}{}^{s} + R_{kit}{}^{m}R_{ljm}{}^{s}\right),\\
\breve{R}_{\bar{k}ij}{}^{\bar{l}} &= R_{lji}{}^{k} - \tfrac{1}{2}p_s a^{tk}\nabla_i R_{ljt}{}^{s}\\
&\quad + \tfrac{1}{4}p_s p_m\left(R_{ltf}{}^{s}R_{ijn}{}^{m}a^{fk}a^{nt} - R_{lif}{}^{s}R_{tjn}{}^{m}a^{nk}a^{ft}\right)\\
&\quad - \tfrac{1}{2}p_s\left(R_{ljm}{}^{s}\nabla_i a^{mk} + R_{tji}{}^{s}\nabla_l a^{kt} - R_{lti}{}^{s}\nabla_j a^{kt}\right),\\
\breve{R}_{\bar{k}\bar{i}j}{}^{\bar{l}} &= \tfrac{1}{2}R_{ljt}{}^{k}a^{ti} - \tfrac{1}{2}R_{ljt}{}^{i}a^{tk} + \tfrac{1}{4}p_s R_{ltm}{}^{s}\left(a^{mk}\nabla_j a^{it} - a^{mi}\nabla_j a^{tk}\right)\\
&\quad + \tfrac{1}{4}p_s R_{tjm}{}^{s}\left(a^{mk}\nabla_l a^{it} - a^{mi}\nabla_l a^{kt}\right),\\
\breve{R}_{\bar{k}i\bar{j}}{}^{\bar{l}} &= \tfrac{1}{2}R_{lit}{}^{k}a^{jt} + \tfrac{1}{2}\nabla_i\nabla_l a^{kj} - \tfrac{1}{2}p_s R_{ltm}{}^{s}\Gamma^{j}_{if}a^{mk}a^{ft}\\
&\quad + \tfrac{1}{4}p_s\left(R_{lif}{}^{s}a^{fm}\nabla_t a^{kj} - R_{tim}{}^{s}a^{jm}\nabla_l a^{kt} + R_{ltm}{}^{s}a^{mk}\nabla_i a^{jt}\right),\\
\breve{R}_{\bar{k}\bar{i}\bar{j}}{}^{\bar{l}} &= \tfrac{1}{4}\left(\nabla_l a^{kt}\nabla_t a^{ij} - \nabla_l a^{it}\nabla_t a^{kj}\right),\\
\breve{R}_{k\bar{i}\bar{j}}{}^{l} &= -\tfrac{1}{4}\left(\nabla_k a^{tl}\nabla_t a^{ij} + \nabla_t a^{il}\nabla_k a^{jt}\right)\\
&\quad + \tfrac{1}{2}\left(\Gamma^{t}_{km}a^{ml}\nabla_t a^{ij} + \Gamma^{j}_{km}a^{mt}\nabla_t a^{il}\right),\\
\breve{R}_{\bar{k}\bar{i}j}{}^{l} &= \tfrac{1}{4}\left(\nabla_t a^{kl}\nabla_j a^{it} - \nabla_t a^{li}\nabla_j a^{kt}\right),\\
\breve{R}_{\bar{k}\bar{i}\bar{j}}{}^{l} &= 0.
\end{aligned}$$

We have

Theorem 3.3. *Given an m-dimensional manifold (M, g) and its cotangent bundle $(T^*M, \breve{G})$. Then $(T^*M, \breve{G})$ is flat if M is flat and $\nabla\tilde{a} = 0$.*

Proof. It immediately follows from last equations. □

4. The metric connection with respect to the metric $\breve{G}$

We know that the metric connection satisfies $\bar{\nabla}\breve{G} = 0$ and has non-trivial torsion tensor. By the definition of the horizontal lift ${}^{H}\nabla$ of any connection ∇ on T^*M, we write

$$\begin{cases} {}^{H}\nabla_{{}^{V}\beta}{}^{V}\omega = 0, & {}^{H}\nabla_{{}^{V}\beta}{}^{H}Z = 0,\\ {}^{H}\nabla_{{}^{H}Z}{}^{V}\omega = {}^{V}\left(\nabla_Z\omega\right), & {}^{H}\nabla_{{}^{H}Z}{}^{H}V = {}^{H}\left(\nabla_Z V\right) \end{cases} \tag{4.1}$$

for any $V, Z \in \Im^1_0(M)$ and $\omega, \beta \in \Im^0_1(M)$. The torsion tensor T of ${}^{H}\nabla$ determined by

$\mathrm{T}\left({}^{V}\omega, {}^{V}\beta\right) = 0, \qquad \mathrm{T}\left({}^{H}Z, {}^{V}\omega\right) = 0, \qquad \mathrm{T}\left({}^{H}V, {}^{H}Z\right) = -\gamma R(V, Z),$

where R denote the curvature tensor of ∇ and $\gamma R(V,Z)=\sum_i p_h R_{kli}{}^h V^k Z^l \frac{\partial}{\partial x^i}$ [17, p.287].

Using (3.3) and (4.1), we have

$$\begin{aligned}
\left({}^H\nabla_{{}^H Z}\breve{G}\right)\left({}^V\beta,{}^V\varepsilon\right) &= {}^H\nabla_{{}^H Z}\breve{G}\left({}^V\beta,{}^V\varepsilon\right)-\breve{G}\left({}^H\nabla_{{}^H Z}{}^V\beta,{}^V\varepsilon\right)-\breve{G}\left({}^V\beta,{}^H\nabla_{{}^H Z}{}^V\varepsilon\right)\\
&= {}^H\nabla_{{}^H Z}{}^V\left(\tilde{a}\left(\beta,\varepsilon\right)\right)-\breve{G}\left({}^V\left(\nabla_Z\beta\right),{}^V\varepsilon\right)-\breve{G}\left({}^V\beta,{}^V\left(\nabla_Z\varepsilon\right)\right)\\
&= {}^V\left(\nabla_Z\left(\tilde{a}\left(\beta,\varepsilon\right)\right)\right)-{}^V\left(\tilde{a}\left(\nabla_Z\beta,\varepsilon\right)\right)-{}^V\left(\tilde{a}\left(\beta,\nabla_Z\varepsilon\right)\right)\\
&= {}^V\left(Z\tilde{a}\left(\beta,\varepsilon\right)\right)-{}^V\left(\tilde{a}\left(\nabla_Z\beta,\varepsilon\right)\right)-{}^V\left(\tilde{a}\left(\beta,\nabla_Z\varepsilon\right)\right)\\
&= {}^V\left(\left(\nabla_Z\tilde{a}\right)\left(\beta,\varepsilon\right)\right)
\end{aligned}$$

and the others are zero. Then we have the following theorem:

Theorem 4.1. *The horizontal lift ${}^H\nabla$ of ∇ is a metric connection of the metric $\breve{G}$ if and only if the symmetric (2,0)-tensor field $\tilde{a}$ on (M,g) is parallel with respect to ∇.*

5. Geodesics on $(T^*M,\breve{G})$

Now, let us investigate the geodesics of the $(T^*M,\breve{G})$. Firstly, let $C: x^h=x^h(t)$ be a curve in M and $\omega_h(t)$ be a covector field along C. We suppose that $\tilde{C}$ is a curve on T^*M and locally given by

$$x^h=x^h(t)\,,\,x^{\bar{h}}\overset{def}{=}p_h=\omega_h(t)\,. \tag{5.1}$$

The horizontal lift of the curve C in M satisfies the equation

$$\tfrac{\delta\omega_h}{dt}=\tfrac{d\omega_h}{dt}-\Gamma^i_{jh}\tfrac{dx^j}{dt}\omega_i=0.$$

Hence, if the initial condition $\omega_h=\omega_h^0$ for $\omega_h=\omega_h^0$ is offered, then there exists a unique horizontal lift given by (5.1).

The differential equation of the geodesic in $(T^*M,\breve{G})$ is expressed by the form

$$\frac{\delta^2x^A}{dt^2}=\frac{d^2x^A}{dt^2}+\breve{\Gamma}^A_{CB}\frac{dx^C}{dt}\frac{dx^B}{dt}=0 \tag{5.2}$$

with respect to the induced coordinates $\left(x^i,x^{\bar{i}}\right)=\left(x^i,p_i\right)$ in T^*M, where t is the arc length of a curve $x^B=x^B(t)\,,B=(h,\bar{h})$ in T^*M and $\breve{\Gamma}^A_{CB}$ are components of $\breve{\nabla}$ defined by (3.8).

By using the adapted frame $\left\{\tilde{e}_{(\alpha)}\right\}$, we can easily write the equation (5.2). Using (2.8), we get

$$\eta^\alpha=\bar{A}^\alpha{}_A dx^A,$$

i.e.

$$\eta^h=\bar{A}^h{}_A dx^A=\delta^h_i dx^i=dx^h$$

for $\alpha=h$ and

$$\eta^{\bar{h}}=\bar{A}^{\bar{h}}{}_A dx^A=-p_a\Gamma^a_{hj}dx^j+\delta^h_j dx^j=\delta p_h$$

for $\alpha=\bar{h}$. Also we put

$$\begin{aligned}
\tfrac{\eta^h}{dt}&=\bar{A}^h{}_A\tfrac{dx^A}{dt}=\tfrac{dx^h}{dt},\\
\tfrac{\eta^{\bar{h}}}{dt}&=\bar{A}^{\bar{h}}{}_A\tfrac{dx^A}{dt}=\tfrac{\delta p_h}{dt}
\end{aligned}$$

along a curve $x^B=x^B(t)$ in T^*M. So, we get the equation (5.2) which is equal to the following

$$\tfrac{d}{dt}\left(\tfrac{\eta^\alpha}{dt}\right)+\breve{\Gamma}^\alpha_{\gamma\beta}\tfrac{\eta^\gamma}{dt}\tfrac{\eta^\beta}{dt}=0$$

with respect to adapted frame $\left\{\tilde{e}_{(\alpha)}\right\}$. From (3.8), we obtain

$$\begin{aligned}
&a)\tfrac{\delta^2x^h}{dt^2}+\tfrac{1}{2}p_mR_{ijt}{}^m a^{th}\tfrac{dx^i}{dt}\tfrac{dx^j}{dt}+\tfrac{1}{2}\nabla_j a^{ih}\tfrac{\delta p_i}{dt}\tfrac{dx^j}{dt}+\tfrac{1}{2}\left(\nabla_i a^{jh}-\Gamma^j_{it}a^{th}\right)\tfrac{dx^i}{dt}\tfrac{\delta p_j}{dt}=0,\\
&b)\tfrac{\delta^2p_h}{dt^2}+p_mR_{hji}{}^m\tfrac{dx^i}{dt}\tfrac{dx^j}{dt}+\tfrac{1}{2}p_mR_{hjt}{}^m a^{it}\tfrac{\delta p_i}{dt}\tfrac{dx^j}{dt}+p_mR_{hit}{}^s a^{tj}\tfrac{dx^i}{dt}\tfrac{\delta p_j}{dt}\\
&\quad-\tfrac{1}{2}\nabla_h a^{ij}\tfrac{\delta p_i}{dt}\tfrac{\delta p_j}{dt}=0.
\end{aligned}$$

Taking account of the local components of the curvature tensor R, i.e. ${R_{ijk}}^t = \partial_i \Gamma^t_{jk} - \partial_j \Gamma^t_{ik} + \Gamma^t_{im}\Gamma^m_{jk} - \Gamma^t_{jm}\Gamma^m_{ik}$ and antisymmetry with respect to i and j, we find ${R_{(ij)t}}^m = 0$. Since ${R_{(ij)t}}^m = 0$, we have ${R_{ijt}}^m \frac{dx^i}{dt}\frac{dx^j}{dt} = 0$. So, we obtain

$$\begin{array}{l} a) \frac{\delta^2 x^h}{dt^2} + \frac{1}{2}\nabla_j a^{ih} \frac{\delta p_i}{dt}\frac{dx^j}{dt} + \frac{1}{2}\left(\nabla_i a^{jh} - \Gamma^j_{it} a^{th}\right)\frac{dx^i}{dt}\frac{\delta p_j}{dt} = 0, \\ b) \frac{\delta^2 p_h}{dt^2} + p_m {R_{hji}}^m \frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{1}{2} p_m {R_{hjt}}^m a^{it}\frac{\delta p_i}{dt}\frac{dx^j}{dt} + p_m {R_{hit}}^s a^{tj}\frac{dx^i}{dt}\frac{\delta p_j}{dt} \\ \quad - \frac{1}{2}\nabla_h a^{ij}\frac{\delta p_i}{dt}\frac{\delta p_j}{dt} = 0. \end{array} \tag{5.3}$$

Theorem 5.1. *Let $\tilde{C}$ be a curve in T^*M expressed locally by $x^h = x^h(t)$, $p_h = \omega_h(t)$ with respect to the induced coordinates $(x^i, x^{\bar{i}}) = (x^i, p_i)$ in T^*M. The curve $\tilde{C}$ is a geodesic of $\breve{G}$, if it satisfies the equation (5.3).*

Let now $\tilde{C} : x^h = x^h(t), x^{\bar{h}} = p_h(t) = \omega_h(t)$ be a horizontal lift $\left(\frac{\delta p_h}{dt} = \frac{\delta \omega_h}{dt} = \frac{d\omega_h}{dt} - \Gamma^i_{jh}\frac{dx^j}{dt}\omega_i = 0\right)$ of the geodesic $C : x^h = x^h(t)$ $(\frac{\delta^2 x^h}{dt^2} = 0)$ in M of ∇. Due to the non-vanishing second term of equation (5.3,b), the geodesic equation (5.2) does not provide.

Theorem 5.2. *The horizontal lift of a geodesic on (M, g) needs not be a geodesic on T^*M with respect to the connection $\breve{\nabla}$.*

6. Para-Nordenian structures on $(T^*M, \breve{G})$

An almost product structure $P \in \Im^1_1(M)$ is defined by $P^2 = I$. Therefore, the pair (M, g) is called an almost product manifold. An almost paracomplex manifold is an almost product manifold $(M, P), P^2 = I$, such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues +1 and -1 of P, respectively, have the same rank. We know that the dimension of an almost paracomplex manifold has to be even. Using the paracomplex structure F, we get the set $\{I, P\}$ on M, which is an isomorphic representation of the algebra of order 2, which is defined the algebra of paracomplex (or double) numbers and is given by $R(j), j^2 = 1$ [6].

If a tensor field $\vartheta \in \Im^0_q(M^{2m})$ satisfies

$$\vartheta(PZ_1, Z_2, ..., Z_q) = \vartheta(Z_1, PZ_2, ..., Z_q) = ... = \vartheta(Z_1, Z_2, ..., PZ_q)$$

for all $Z_1, Z_2, ..., Z_q \in \Im^1_0(M^{2m})$, then ϑ is called pure with respect to the paracomplex structure P.

By means of the paracomplex structure P and the pure tensor field ϑ, the operator Φ_P defined in [16] is

$$\begin{array}{c} (\Phi_P \vartheta)(Y, Z_1, ..., Z_q) = (PY)(\vartheta(Z_1, ..., Z_q)) - Y(\vartheta(PZ_1, Z_2, ..., Z_q)) \\ + \vartheta((L_{Z_1}P)Y, Z_2, ..., Z_q) + ... + \vartheta\left(Z_1, Z_2, ..., \left(L_{Z_q}P\right)Y\right), \end{array}$$

where L_Y is the Lie derivative with respect to Y and $\Phi_P \vartheta \in \Im^0_{q+1}(M^{2m})$.

A tensor field ϑ is called an almost paraholomorphic with respect to the paracomplex algebra $R(j)$, if $\Phi_P \vartheta = 0$ (see [9, 15]).

The pair (P, g) is a para-Nordenian structure where P is an almost paracomplex structure and g is a pure tensor field with respect to P, i.e. $g(PV, Z) = g(V, PZ)$. Then a 2m-dimensional pseudo-Riemannian manifold M with an almost para-Nordenian structure is called to be an almost para-Nordenian manifold. Furthermore, the almost para-Nordenian manifold is para-Kähler $(\nabla_g P = 0)$ if and only if g is paraholomorphic $(\Phi_P g = 0)$ (see [14, 15]).

Given the cotangent bundle T^*M with the metric $\breve{G}$. A tensor field $P \in \Im^1_1(T^*M)$ is expressed by

$$\begin{cases} P^H Z = -^H Z, \\ P^V \theta = -^V \theta \end{cases} \tag{6.1}$$

for any $Z \in \Im^1_0(M)$ and $\theta \in \Im^0_1(M)$. By virtue of (6.1), we have

$$\begin{array}{c} P^2\left({}^H Z\right) = P\left(P^H Z\right) = P\left(-^H Z\right) = {}^H Z \\ P^2\left({}^V \theta\right) = P\left(P^V \theta\right) = P\left(-^V \theta\right) = {}^V \theta \end{array}$$

for any $Z \in \Im^1_0(M)$ and $\theta \in \Im^0_1(M)$, i.e. $P^2 = I$.

Theorem 6.1. *The triple $\left(T^*M, \breve{G}, P\right)$ is an almost para-Nordenian manifold.*

Proof. Using purity condition

$$W(V,Z)=\breve{G}(PV,Z)-\breve{G}(V,PZ)$$

for any $V,Z\in\Im_0^1(T^*M)$, from (3.3) and (6.1) we have

$$W\left({}^V\omega,{}^V\theta\right)=\breve{G}\left(P^V\omega,{}^V\theta\right)-\breve{G}\left({}^V\omega,P^V\theta\right)=0,$$
$$W\left({}^HZ,{}^V\theta\right)=\breve{G}\left(P^HZ,{}^V\theta\right)-\breve{G}\left({}^HZ,P^V\theta\right)=-\breve{G}\left({}^HZ,{}^V\theta\right)+\breve{G}\left({}^HZ,{}^V\theta\right)=0,$$
$$W\left({}^V\theta,{}^HZ\right)=-W\left({}^HZ,{}^V\theta\right)=0,$$
$$W\left({}^HV,{}^HZ\right)=\breve{G}\left(P^HV,{}^HZ\right)-\breve{G}\left({}^HV,P^HZ\right)=-\breve{G}\left({}^HV,{}^HZ\right)+\breve{G}\left({}^HV,{}^HZ\right)=0$$

i.e. $\breve{G}$ is pure with respect to P as defined by (6.1). Hence, Theorem 6.1 is proved. □

In view of Proposition 3.1 and (6.1), the covariant derivative of P with respect to the metric $\breve{G}$ is

$$\left(\breve{\nabla}_{{}^HV}P\right)\left({}^V\theta\right)=\breve{\nabla}_{{}^HV}\left(P^V\theta\right)-P\left(\breve{\nabla}_{{}^HV}{}^V\theta\right)=-\breve{\nabla}_{{}^HV}\left({}^V\theta\right)-P\left(\breve{\nabla}_{{}^HV}{}^V\theta\right)$$
$$=-\tfrac{1}{2}{}^H\left(\left(\nabla_V\tilde{a}\right)(\omega,)\right)-{}^H\left(\tilde{a}\circ\nabla_V\omega\right)-{}^V\left(\nabla_V\omega\right)-\tfrac{1}{2}{}^V\left(\tilde{a}\left(pR\left(,V\right),\omega\right)\right)$$
$$-\left(-\tfrac{1}{2}{}^H\left(\left(\nabla_V\tilde{a}\right)(\omega,)\right)-{}^H\left(\tilde{a}\circ\nabla_V\omega\right)-{}^V\left(\nabla_Y\omega\right)-\tfrac{1}{2}{}^V\left(\tilde{a}\left(pR\left(,V\right),\omega\right)\right)\right)=0,$$
$$\left(\breve{\nabla}_{{}^HV}P\right)\left({}^HZ\right)=\breve{\nabla}_{{}^HV}\left(P^HZ\right)-P\left(\breve{\nabla}_{{}^HV}{}^HZ\right)=-\breve{\nabla}_{{}^HV}{}^HZ-P\left(\breve{\nabla}_{{}^HV}{}^HZ\right)$$
$$=-\left({}^H\left(\nabla_VZ\right)+\tfrac{1}{2}{}^H\left(\tilde{a}\circ pR\left(V,Z\right)\right)+{}^V\left(\tilde{Z}R\left(V,\tilde{p}\right)\right)\right)$$
$$-\left(-{}^H\left(\nabla_VZ\right)-\tfrac{1}{2}{}^H\left(\tilde{a}\circ pR\left(V,Z\right)\right)-{}^V\left(\tilde{Z}R\left(V,\tilde{p}\right)\right)\right)=0,$$
$$\left(\breve{\nabla}_{{}^V\omega}P\right)\left({}^HZ\right)=\breve{\nabla}_{{}^V\omega}\left(P^HZ\right)-P\left(\breve{\nabla}_{{}^V\omega}{}^HZ\right)=\breve{\nabla}_{{}^V\omega}\left(-{}^HZ\right)-P\left(\breve{\nabla}_{{}^V\omega}{}^HZ\right)$$
$$=-\tfrac{1}{2}{}^H\left(\left(\nabla_Z\tilde{a}\right)(\omega,)\right)-\tfrac{1}{2}{}^V\left(\tilde{Z}R\left(\tilde{\omega},\tilde{p}\right)\right)+\tfrac{1}{2}{}^H\left(\left(\nabla_Z\tilde{a}\right)(\omega,)\right)+\tfrac{1}{2}{}^V\left(\tilde{Z}R\left(\tilde{\omega},\tilde{p}\right)\right)=0,$$
$$\left(\breve{\nabla}_{{}^V\omega}P\right)\left({}^V\theta\right)=\breve{\nabla}_{{}^V\omega}\left(P^V\theta\right)-P\left(\breve{\nabla}_{{}^V\omega}{}^V\theta\right)=\tfrac{1}{2}{}^V\left(\left(\nabla\tilde{a}\right)(\omega,\theta)\right)$$
$$-\tfrac{1}{2}{}^V\left(\left(\nabla\tilde{a}\right)(\omega,\theta)\right)=0.$$

Theorem 6.2. *The triple* $\left(T^*M,\breve{G},P\right)$ *is a para-Kählerian manifold.*

The Nijenhuis tensor is given by the formula

$$N_P(V,Z)=[PV,PZ]-P[PV,Z]-P[V,PZ]+P^2[V,Z] \tag{6.2}$$

and the vanishing of the Nijenhuis tensor is characterized integrability of the almost paracomplex structure. If P is integrable, we say that the almost para-Nordenian manifold is a para-Nordenian manifold (see [14]). Using (3.6), (6.1) and (6.2), we find

$$N_P\left({}^HV,{}^HZ\right)=N_P\left({}^HV,{}^V\theta\right)=N_P\left({}^V\omega,{}^HZ\right)=N_P\left({}^V\omega,{}^V\theta\right)=0$$

i.e. $\left(T^*M,\breve{G},P\right)$ is integrable. Then we have the following theorem:

Theorem 6.3. *The triple* $\left(T^*M,\breve{G},P\right)$ *is a para-Nordenian manifold.*

If the Lie derivative of a vector field $\tilde{Z}\in\Im_0^1(T^*M)$ satisfies the condition $L_{\tilde{Z}}P=0$, then the vector field is said to be an almost paraholomorphic (see [9]).

Taking account of (3.6) and (6.1), we have

$$\left(L_{{}^CZ}P\right){}^V\theta=L_{{}^CZ}P^V\theta-P\left(L_{{}^CZ}{}^V\theta\right)$$
$$=-L_{{}^CZ}{}^V\theta-P\left({}^V\left(L_Z\theta\right)\right)=-{}^V\left(L_Z\theta\right)+\left({}^V\left(L_Z\theta\right)\right)=0,$$
$$\left(L_{{}^CZ}P\right){}^HV=L_{{}^CZ}P^HV-P\left(L_{{}^CZ}{}^HV\right)=-L_{{}^CZ}{}^HV-P\left(L_{{}^CZ}{}^HV\right)$$
$$=-\left({}^H[Z,V]+{}^V\left(p\left(L_Z\nabla\right)Y\right)\right)-P\left({}^H[Z,V]+{}^V\left(p\left(L_Z\nabla\right)Y\right)\right)=0,$$
$$\left(L_{{}^V\omega}P\right){}^V\theta=L_{{}^V\omega}P^V\theta-P\left(L_{{}^V\omega}{}^V\theta\right)=0,$$
$$\left(L_{{}^V\omega}P\right){}^HZ=L_{{}^V\omega}P^HZ-P\left(L_{{}^V\omega}{}^HZ\right)=-L_{{}^V\omega}{}^HZ-P\left(L_{{}^V\omega}{}^HZ\right)$$
$$={}^V\left(\nabla_Z\omega\right)-{}^V\left(\nabla_Z\omega\right)=0,$$
$$\left(L_{{}^HZ}P\right){}^V\theta=L_{{}^HZ}P^V\theta-P\left(L_{{}^HZ}{}^V\theta\right)=-\left[{}^HZ,{}^V\omega\right]-P\left(\left[{}^HZ,{}^V\omega\right]\right)$$
$$=-{}^V\left(\nabla_Z\omega\right)-P\left({}^V\left(\nabla_Z\omega\right)\right)=0,$$
$$\left(L_{{}^HZ}P\right){}^HV=L_{{}^HZ}P^HV-P\left(L_{{}^HZ}{}^HV\right)=-\left[{}^HZ,{}^HV\right]-P\left(\left[{}^HZ,{}^HV\right]\right)$$
$$=-\left({}^H[Z,V]+{}^V\left(pR\left(Z,V\right)\right)\right)-P\left({}^H[Z,V]+{}^V\left(pR\left(Z,V\right)\right)\right)=0.$$

Hence, we can conclude the following theorem:

Theorem 6.4. *The complete and horizontal lifts* ${}^{C}Z, {}^{H}Z \in \Im_0^1(T^*M)$ *of* $Z \in \Im_0^1(M)$ *and the vertical lift* ${}^{V}\omega \in \Im_0^1(T^*M)$ *of* $\omega \in \Im_1^0(M)$ *are the almost paraholomorphic vector fields with respect to the almost para-Nordenian structure* $\left(P, \breve{G}\right)$.

References

[1] Aslanci, S., Cakan, R., On a cotangent bundle with deformed Riemannian extension, *Mediterr. J. Math.* 11 (2014), 1251-1260.
[2] Aslanci, S., Kazimova, S., Salimov, A.A., Some Remarks Concerning Riemannian Extensions, *Ukrainian. Math. J.* 62, (2010), 661-675.
[3] Bejan, C.L., Eken, Ş., A characterization of the Riemann extension in terms of harmonicity, *Czech. Math. J.* 67, (2017), 197-206.
[4] Bejan, C.L., Meriç, Ş. E., Kılıç, E., Einstein Metrics Induced by Natural Riemann Extensions, *Adv. Appl. Clifford Algebras.* 27, (2017), 2333-2343.
[5] Calviño-Louzao, E., García-Río, E., Gilkey, P., Vázquez-Lorenzo A., The Geometry of Modified Riemannian Extensions, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 465, (2009), 2023-2040.
[6] Cruceanu, V., Fortuny, P., Gadea, M., A survey on paracomplex Geometry, *Rocky Mountain J. Math.* 26, (1995), 83-115.
[7] Dryuma, V., The Riemann Extensions in Theory of Differential Equations and their Applications, *Mat. Fiz. Anal. Geom.* 10, (2003), 307-325.
[8] Gezer, A., Bilen, L., Cakmak, A., Properties of Modified Riemannian Extensions, *Zh. Mat. Fiz. Anal. Geom.* 11, (2015), 159-173.
[9] Kruckovic, GI., Hypercomplex structures on manifolds I. *Trudy. Sem. Vektor Tenzor Anal.* 16, (1972), 174-201(in Russian).
[10] Ocak, F., Kazimova, S., On a new metric in the cotangent bundle, *Transactions of NAS of Azerbaijan Series of Physical-Technical and Mathematical Sciences.* 38, (2018), 128-138.
[11] Patterson, E.M., Walker, A.G., Riemann Extensions, *Quart. J. Math. Oxford Ser.* 3, (1952), 19–28.
[12] Salimov, A., Tensor Operators and Their Applications, Nova Science Publishers, New York, USA, 2012.
[13] Salimov, A., Cakan, R., On deformed Riemannian extensions associated with twin Norden metrics, *Chinese Annals of Mathematics Series B.* 36, (2015), 345-354.
[14] Salimov, A.A., Iscan, M., Akbulut, K., Notes on para-Norden–Walker 4-manifolds, *International Journal of Geometric Methods in Modern Physics.* 7, (2010), 1331-1347.
[15] Salimov, A. A., Iscan, M. and Etayo, F., Paraholomorphic B-manifold and its properties, *Topology Appl.* 154,(2007), 925-933.
[16] Yano, K., Ako, M., On certain operators associated with tensor fields, *Kodai Math. Sem. Rep.* 20, (1968), 414-436.
[17] Yano, K. and Ishihara, S., Tangent and Cotangent Bundles, Pure and Applied Mathematics, 16, Marcel Dekker, Inc., New York, 1973.

Affiliations

FILIZ OCAK
ADDRESS: Karadeniz Technical University, Faculty of Sciences, Dept. of Mathematics, Trabzon, Turkey.
E-MAIL: filiz_math@hotmail.com, filiz.ocak@ktu.edu.tr
http://orcid.org/0000-0003-4157-6404