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# On The Geometry of Submanifolds of a $(k,\mu)-$ Paracontact Manifold

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**Abstract**
The object of this paper is to study submanifolds of (k,$\mu$)-paracontact manifolds. We find the necessary and sufficient conditions for a submanifold of (k, $\mu$)-paracontact manifolds to be invariant and anti-invariant. Also, we research the necessary and sufficient conditions for a submanifold of a (k,$\mu$)-paracontact to be semi-parallel and 2-semi-parallel submanifold and get interesting results.

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## 1. Introduction

In the modern geometry, the geometry of submanifolds has become a subject of growing applications in applied mathematics and physics. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous system. On the other hand, the notion of geodesics plays an important role in the theory of relativity. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Therefore, totally geodesic submanifolds are also very much important in physical sciences. The study of geometry of invariant submanifolds was initiated by Bejancu and Papaghuic[5],[6]. Later on the invariant submanifolds inherit almost all properties of the ambient manifolds. Arslan K. and et al. [1],[12] defined and studied 2-semi-paralellel surfaces in space forms. Ishıhara I.[8], Yano K. and Kon M. [17] studied anti-invariant submanifolds of a Sasakian space form. In[[4],[5],[6],[9],[11],[16]], authors studied semi-invariant and totally umbilical submanifolds in Sasakian and cosymplectic manifolds. In [2], we discussed the properties of semi-invariant submanifolds of a normal paracontact metric manifold. Motivated by the above studies, the present paper deals with the study of invariant submanifolds of a (k,$\mu$)-paracontact manifolds.

Section 1, this paper is to study. Invariant Submanifolds of a (k,$\mu$)-paracontact manifolds. Section 2, we find the necessary and sufficient conditions for a submanifold of a (k,$\mu$)-paracontact manifolds to be invariant. Also, we research the necessary and sufficient conditions for a submanifold of a (k,$\mu$)-paracontact to be semi-parallel and 2-semi-parallel submanifold and get interesting results.

## 2. Preliminaries

A contact manifold is a $C^\infty-(2n+1)$ manifold $\widetilde{M}^{2n+1}$ equipped with a global 1-form $\eta$ such that $\eta\wedge(d\eta)^n\neq 0$ everywhere on $\widetilde{M}^{2n+1}$. Given a contact form $\eta$ it is well known that there exists a unique vector field $\xi$, called the characteristic vector field of $\eta$, such that $\eta(\xi)=1$ and $d\eta(X,\xi)=0$ for every vector field $X$ on $\widetilde{M}^{2n+1}$. A Riemannian metric $g$ is said to be associated metric if there exists a tensor field $\phi$ of type $(1,1)$ such that

$$\phi^2X=-X+\eta(X)\xi,\ \ \eta(\xi)=1,\ \ \eta\circ\phi=0,\ \ \phi\xi=0 \tag{1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \;\; g(X,\xi) = \eta(X) \tag{2}$$

for all vector fields X,Y on $\widetilde{M}$. Then the structure $(\phi, \xi, \eta, g)$ on $\widetilde{M}$ is called a contact metric structure and the manifold equipped with such a structure is called a contact metric manifold [4]. We now define a $(1,1)$-tensor field $h$ by $h = \frac{1}{2}L_{\xi}\phi$, where $L$ denotes the Lie differentiation, then $h$ is symmetric and satisfies $h\phi = -\phi h$. Further, a q-dimensional distribution on a manifold $M$ is defined as a mapping $D$ on $M$ which assigns to each point $p \in M$, a q-dimensional subspace $D_p$ of $T_pM$.

The $(k,\mu)-$nullity distribution of a contact metric manifold $\widetilde{M}(\phi,\xi,\eta,g)$ is a distribution

$$\begin{aligned} N(k,\mu) \;\; &: \;\; p \to N_p(k,\mu) = \{Z \in T_pM : \widetilde{R}(X,Y)Z \\ &= \;\; k[\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y] + \mu[\widetilde{g}(Y,Z)hX - \widetilde{g}(X,Z)hY]\} \end{aligned} \tag{3}$$

for all $X,Y \in T\widetilde{M}$. Hence if the characteristic vector field $\xi$ belongs to the $(k,\mu)-$nullity distribution, then we have

$$\widetilde{R}(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] \tag{4}$$

The contact metric manifold satisfying this relations is called $(k,\mu)-$paracontact metric manifold[5].It consists of both $k-$nullity distribution for $\mu = 0$ and Sasakian for $k = 1$. A $(k,\mu)-$paracontact metric manifold $\widetilde{M}(\phi,\xi,\eta,g)$ satisfies

$$(\widetilde{\nabla}_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \tag{5}$$

for all $X,Y \in T\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g. From (5) we have

$$\widetilde{\nabla}_X \xi = -\phi X - \phi hX \tag{6}$$

for all $X,Y \in T\widetilde{M}$. Again, if we put $\Omega(X,Y) = g(X,\phi Y)$, then $\Omega$ is a skew-symmetric (0,2) tensor field [4] . Thus we have from (6),

$$\Omega(X + hX, Y) = (\widetilde{\nabla}_X \eta)(Y) \tag{7}$$

Also from (4),it follows that

$$(\widetilde{\nabla}_Z \Omega)(X,Y) = g(X, (\widetilde{\nabla}_Z \phi)Y) = -g(((\widetilde{\nabla}_Z \phi)X, Y) \tag{8}$$

$$(\widetilde{\nabla}_Z \Omega)(X,Y) = g(Z + hZ, Y)\eta(X) - \eta(Y)g(X, Z + hZ) \tag{9}$$

for any $X,Y \in T\widetilde{M}$.

The concircular curvature tensor, conformal curvature tensor and quasi-conformal curvature tensor of a $(k,\mu)-$paracontact metric manifold $M^{2n+1}$ are, respectively, defined by

$$\widetilde{Z}(X,Y)Z = R(X,Y)Z - \frac{\tau}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{10}$$

$$\begin{aligned} C(X,Y)Z \;\; &= \;\; R(X,Y)Z - \frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- \;\; g(X,Z)QY\} + \frac{\tau}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{11}$$

$$\begin{aligned} \widetilde{C}(X,Y)Z \;\; &= \;\; \lambda R(X,Y)Z + \mu\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ &- \;\; \frac{\tau}{2n+1}\{\frac{\lambda}{2n} + 2\mu\}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{12}$$

for any $X,Y,Z \in TM$, where $R$ denotes the Riemannian curvature tensor of $M$ and $Q$ is the Ricci operator given by $g(QX,Y) = S(X,Y)$.

Also, on a $(k,\mu)-$paracontact metric manifold $M^{2n+1}$, the following relations are satisfied

$$R(\xi,Y)Z = g(Y,Z)\xi - \eta(Z)Y, \tag{13}$$

and

$$R(\xi,Y)\xi = \eta(Y)\xi - Y \tag{14}$$

for any $X,Y,Z \in TM$.

Let $M$ be a Riemannian submanifold of a $(k,\mu)-$paracontact metric manifold $\widetilde{M}$. Then the Gauss and Weingarten formulae are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{15}$$

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{16}$$

for all $X,Y \in T\widetilde{M}$ and each $V \in T^{\perp}M$, where $\nabla$ is the Levi-Civita connection on $M$, $\nabla^{\perp}$ is the normal connection on the normal bundle $T^{\perp}M$ ,$\sigma$ is the second fundamental form of $M$ and $A$ is the shape operator with respect to the normal connection $N$. Then the shape operator $A$ and the second fundamental form $\sigma$ are related by

$$g(\sigma(X,Y),V) = g(A_V X,Y) \tag{17}$$

for all $X,Y \in T\widetilde{M}$ and each $V \in T^{\perp}M$. We denote by the same symbols g both metrics on $\widetilde{M}$ and $M$.

**Definition 1.** *A submanifold M is said to be*
*(i) totally geodesic in $\widetilde{M}$ if*

$$\sigma = 0 \ \text{ or equivalently } \ A_V = 0 \tag{18}$$

*for each $V \in T^{\perp}M$.*
*(ii) Minimal in $\widetilde{M}$ if the curvature vector H satisfies*

$$H = \frac{T\tau(\sigma)}{\dim M} = 0 \tag{19}$$

*and*
*(iii) totally umbilical if*

$$\sigma(X,Y) = g(X,Y)H. \tag{20}$$

*Let M be a submanifold of a $(k,\mu)$-paracontact manifold $\widetilde{M}$ for any tangent vector field X, we can write $\phi X = TX + NX$, where $TX$ and $NX$ are the tangent and normal components of $\phi X$, respectively. Similarly $\phi V = tV + nV$ for any normal vector field V. where $tV$ is tangent component and $nV$ normal component of $\phi V$.*

*$\overline{M}$ is said to be an invariant submanifold if $N = 0$ [7]. Throughout this paper, we assume that $\widetilde{M}$ is an invariant submanifold of a $(k,\mu)$-paracontact manifold M. In this case, we have $\phi(T\widetilde{M}) \subseteq T\widetilde{M}$ and $\phi T^{\perp}\widetilde{M} \subseteq T^{\perp}\widetilde{M}$ [10].*

**Lemma 2.** *Let M be a submanifold of a $(k,\mu)$-paracontact manifold $\widetilde{M}(\phi,\xi,\eta,g)$, then*

$$(\nabla_X T)Y = t\sigma(X,Y) + A_{NY}X + g(X+hX,Y)\xi - \eta(Y)(X+hX) \tag{21}$$

*and*

$$(\nabla_X N)Y = -\sigma(X,TY) + n\sigma(X,Y) \tag{22}$$

*for any vector fields $X,Y \in TM$.*

Let $(M,g)$ be a Riemannian manifold and $\widetilde{M}$ be a submanifold of $M$. We denote the Levi-Civita connection of $g$ and the second fundamental form of $\widetilde{M}$ by $\nabla$ and $\sigma$, respectively. The submanifold $\widetilde{M}$ said to be semiparallel if

$$R(X,Y)\cdot\sigma=0, \tag{23}$$

for any $X,Y\in T\widetilde{M}$, where $R$ denotes the Riemannian curvature tensor of $M$ and $R(X,Y)\cdot\sigma=0$ is defined by

$$(R(X,Y)\cdot\sigma)(Z,U)=R^{\perp}(X,Y)\sigma(Z,U)-\sigma(R(X,Y)Z,U)-\sigma(Z,R(X,Y)U), \tag{24}$$

for any $X,Y,Z,U\in T\widetilde{M}$.

In [1], Arslan et al. defined and studied 2-semiparallel submanifolds. Such submanifolds are defined as, a Riemannian submanifold $\widetilde{M}$ is said to be 2-semiparallel if the following relation holds

$$R(X,Y)\cdot\nabla\sigma=0, \tag{25}$$

for any $X,Y\in T\widetilde{M}$, where

$$\begin{aligned}(R(X,Y)\cdot\nabla\sigma)(Z,U,W) &= R^{\perp}(X,Y)(\nabla_Z\sigma)(U,W)-(\nabla_{R(X,Y)Z}\sigma)(U,W)\\ &- (\nabla_Z\sigma)(R(X,Y)U,W)-(\nabla_Z\sigma)(U,R(X,Y)W),\end{aligned} \tag{26}$$

for any $X,Y,Z,U,W\in T\widetilde{M}$.

Now, let us assume that $(k,\mu)$ paracontact metric manifold $M^{2n+1}$ is conformal flat. Then from (11), we have

$$\begin{aligned}R(X,Y)\xi &= \frac{1}{2n-1}\{S(Y,\xi)X-S(X,\xi)Y+\eta(Y)QX-\eta(X)QY\}\\ &- \frac{\tau}{2n(2n-1)}\{\eta(Y)X-\eta(X)Y\}.\end{aligned} \tag{27}$$

## 3. Submanifolds of a $(k,\mu)-$ paracontact manifold

In this section, we define invariant and anti-invariant submanifolds of $(k,\mu)-$paracontact manifold. A submanifold $M$ of a $(k,\mu)$-paracontact manifold $\widetilde{M}$ is said to be invariant (resp. anti-invariant) submanifold of $\widetilde{M}$ if for each $x\in M$, $\phi(T_xM)\subset T_xM$ ( resp. $\phi(T_xM)\subset T_x^{\perp}M$), here $T_xM$ and $T_x^{\perp}M$ are the tangent and normal bundles.

**Lemma 3.** *For a submanifold $M$ of a $(k,\mu)$-paracontact manifold $\widetilde{M}$ , we have*

$$-\phi X-\phi hX=\nabla_X\xi+\sigma(X,\xi),\ \ \xi\in TM \tag{28}$$

,

$$-\phi X-\phi hX=-A_{\xi}X+\nabla^{\perp}_X\xi,\ \ \xi\in T^{\perp}M \tag{29}$$

,

$$\eta(A_NX)=0,\ \ \xi\in T^{\perp}M \tag{30}$$

*and*

$$\eta(A_NX)=-g(\phi X+\phi hX,N),\ \ \xi\in TM \tag{31}$$

*for each $X\in TM$ and $N\in T^{\perp}M$.*

*Proof.* From (6)and (15), we get (36). Also from (6) and (16), we obtain (37). Again, in view of (2), (38)is obvious. Now for $\xi\in TM$ and in view of (2), (6) and (16) we get

$$\eta(A_NX)=g(\xi,A_NX)=-g(\xi,\nabla_XN)=-g(\phi X+\phi hX,N),\ \ \ \xi\in TM \tag{32}$$

■

This completes the proof of our lemma.

**Theorem 4.** *Let $M$ be a submanifold of a $(k,\mu)$-paracontact manifold $\widetilde{M}$ such that the structure vector field $\xi$ is tangent to $M$. Then $M$ is invariant if and only if $\sigma(X,\xi)=0$, and $M$ is anti-invariant if and only if $\nabla_X\xi=0$.*

Since it is trivial from Lemma 3.1., we omit to prove our theorem.

**Theorem 5.** *If $M$ is a totally umbilical submanifold of a $(k,\mu)$-paracontact manifold $\widetilde{M}$ such that the structure vector field $\xi$ is tangent to $M$. Then*
*(i) $M$ is necessarily minimal and consequently totally geodesic and*
*(ii) $M$ is an invariant submanifold of $\widetilde{M}$ and $\nabla_X\xi\neq 0$.*

*Proof.* Let M be a totally umbilical. Using (1), (2) and (36) in (20), we get

$$0=\sigma(\xi,\xi)=g(\xi,\xi)H=H. \tag{33}$$

Hence in view of (19) and (20), we obtain (i). The second part follows from Theorem 3.1. and the above (i). ■

**Theorem 6.** *A submanifold $M$ of a $(k,\mu)$-paracontact manifold $\widetilde{M}$ with structure vector field $\xi$ normal to $M$ is anti-invariant in $\widetilde{M}$ if and only if $A_\xi X=0$. Consequently, if $M$ is totally geodesic, then it is anti-invariant.*

*Proof.* Since $\xi$ is normal to M, by virtue of (16) and (37) yields

$$-g(\phi X+\phi hX,Y)=g(A_\xi X,Y)=g(\sigma(X,Y),\xi),X,Y\in TM \tag{34}$$

which provides the proof of our theorem. ■

## 4. Invariant Submanifolds of a $(k,\mu)-$ paracontact manifold

In this section, we study of invariant submanifolds of a $(k,\mu)$-paracontact manifold satisfying the $\widetilde{Z}(X,Y)\cdot\sigma=0$ and $\widetilde{Z}(X,Y)\cdot\nabla\sigma=0$. Finally we see that these conditions are satisfied if and only if invariant submanifold is totally geodesic.

**Theorem 7.** *Let $\widetilde{M}$ be an invariant submanifold of a $(k,\mu)$-paracontact manifold $M$. Then the following relations holds:*
*(1) $\nabla_X\xi=-TX+ThX,\;\;\sigma(X,\xi)=0$*
*(2) $n\sigma(X,Y)=\sigma(X,TY)$*
*(3) $(\nabla_X T)Y=t\sigma(X,Y)+A_{NY}X+g(X+hX,Y)\xi-\eta(Y)(X+hX)$,*
*for any $X,Y\in T\widetilde{M}$.*

*Proof.* By using (6) and taking into account of $\widetilde{M}$ being invariant submanifold, (1) statement is obvious. On the other hand, making use of (5) and (15), we have

$$\begin{aligned}(\widetilde{\nabla}_X\phi)Y &= \widetilde{\nabla}_X\phi Y-\phi\widetilde{\nabla}_X Y\\ g(X+hX,Y)\xi-\eta(Y)(X+hX) &= \sigma(X,TY)+\nabla_X TY-\phi h(X,Y)-T\nabla_X Y\end{aligned}$$

for any $X,Y\in T\widetilde{M}$, which proves(2) and (3) statements. ■

Thus we have the following conclusion.

**Corollary 8.** *Every invariant submanifold of a $(k,\mu)$-paracontact manifold has a $(k,\mu)$-paracontact metric structure.*

**Theorem 9.** *Let $\widetilde{M}$ be an invariant submanifold of a $(k,\mu)$-paracontact manifold $M$. Then the second fundamental form of $\widetilde{M}$ is parallel if and only if $\widetilde{M}$ is a totally geodesic submanifold.*

*Proof.* If the second fundamental form $\sigma$ of $\widetilde{M}$ is parallel, then we have

$$\nabla^\perp_X\sigma(Y,Z)-\sigma(\nabla_X Y,Z)-\sigma(Y,\nabla_X Z)=0,$$

for any $X,Y,Z\in\chi(\widetilde{M})$. Setting $Z=\xi$ in (18) and taking into account that Theorem 4.1., we get $\sigma(Y,\nabla_X\xi)=-\sigma(\phi X+\phi hX,Y)=0$, which implies that $\widetilde{M}$ is a totally geodesic submanifold. The converse statement is obvious. ■

**Theorem 10.** *Let $\widetilde{M}$ be an invariant submanifold of a $(k,\mu)$-paracontact manifold $M$. Then $\widetilde{M}$ is semiparallel if and only if $\widetilde{M}$ is a totally geodesic submanifold.*

*Proof.* If $\widetilde{M}$ is semiparallel, then $R\cdot\sigma=0$. This implies that

$$\begin{aligned}(R(X,Y)\cdot\sigma)(Z,U) &= R^{\perp}(X,Y)\sigma(Z,U)-\sigma(R(X,Y)Z,U)\\ &- \sigma(Z,R(X,Y)U),\end{aligned}\tag{35}$$

for any $X,Y,Z,U\in\chi(\widetilde{M})$. Putting $X=U=\xi$ in (35), we obtain

$$R^{\perp}(\xi,Y)\sigma(Z,\xi)-\sigma(R(\xi,Y)Z,\xi)-\sigma(Z,R(\xi,Y)\xi)=0.$$

Here taking into account of (4) and (5), we get

$$\eta(Z)\sigma(Y,\xi)-g(Y,Z)\sigma(\xi,\xi)+\sigma(Y,Z)-\eta(Y)\sigma(Z,\xi)=0.$$

Here, we conclude $\sigma(Y,Z)=0$, that is, the submanifold is a totally geodesic. Conversely, if $\sigma=0$, then $M$ is semiparallel. ■

**Theorem 11.** *Let $\widetilde{M}$ be an invariant submanifold of a $(k,\mu)$-paracontact manifold $M$. Then $\widetilde{M}$ is 2-semiparallel if and only if $\widetilde{M}$ is a totally geodesic submanifold.*

*Proof.* Let us suppose $\widetilde{M}$ be 2-semiparallel. This implies that

$$\begin{aligned}(R(X,Y)\cdot\nabla\sigma)(Z,U,W) &= R^{\perp}(X,Y)(\nabla_Z\sigma)(U,W)-(\nabla_{R(X,Y)Z}\sigma)(U,W)\\ &- (\nabla_Z\sigma)(R(X,Y)U,W)-(\nabla_Z\sigma)(U,R(X,Y)W),\end{aligned}\tag{36}$$

for all $X,Y,Z,U,W\in\chi(\widetilde{M})$. Here taking $X=U=\xi$ and we calculate each expression as follows

$$\begin{aligned}R^{\perp}(\xi,Y)(\nabla_Z\sigma)(\xi,W) &= R^{\perp}(\xi,Y)\{\nabla^{\perp}_Z\sigma(\xi,W)-\sigma(\nabla_Z\xi,W)-\sigma(\xi,\nabla_ZW)\}\\ &= R^{\perp}(\xi,Y)\sigma(fZ,W),\end{aligned}\tag{37}$$

$$\begin{aligned}(\nabla_{R(\xi,Y)Z}\sigma)(\xi,W) &= \nabla^{\perp}_{R(\xi,Y)Z}\sigma(\xi,W)-\sigma(\nabla_{R(\xi,Y)Z}\xi,W)-\sigma(\nabla_{R(\xi,Y)Z}W,\xi)\\ &= -\sigma(\nabla_{-\eta(Z)Y+g(Y,Z)\xi}\xi,W)\\ &= \eta(Z)\sigma(\nabla_Y\xi,W)-g(Y,Z)\sigma(\nabla_\xi\xi,W)\\ &= -\eta(Z)\sigma(\phi Y+\phi hY,W),\end{aligned}\tag{38}$$

$$\begin{aligned}(\nabla_Z\sigma)(R(\xi,Y)\xi,W) &= \nabla^{\perp}_Z\sigma(R(\xi,Y)\xi,W)-\sigma(\nabla_ZR(\xi,Y)\xi,W)\\ &- \sigma(R(\xi,Y)\xi,\nabla_ZW)\\ &= -\nabla^{\perp}_Z\sigma(Y,W)+\nabla^{\perp}_Z\sigma(\eta(Y)\xi,W)+\sigma(\nabla_ZY,W)\\ &- \sigma(\eta(Y)\xi,W)+\sigma(Y,\nabla_ZW)-\sigma(\nabla_ZW,\eta(Y)\xi)\\ &= -(\nabla_Z\sigma)(Y,W),\end{aligned}\tag{39}$$

and

$$\begin{aligned}(\nabla_Z\sigma)(\xi,R(\xi,Y)W) &= \nabla^{\perp}_Z\sigma(\xi,R(\xi,Y)W)-\sigma(\nabla_Z\xi,R(\xi,Y)W)\\ &\quad -\sigma(\nabla_ZR(\xi,Y)W,\xi)\\ &= -\sigma(\nabla_Z\xi,-\eta(W)Y+g(Y,W)\xi)\\ &= -\eta(W)\sigma(TZ,Y).\end{aligned}\tag{40}$$

Thus, by combining (37),(38),(39) and (40), we derive

$$\begin{aligned}(R(\xi,Y)\cdot\nabla\sigma)(Z,\xi,W) &= R^{\perp}(\xi,Y)\sigma(TZ,W)+\eta(Z)\sigma(fY,W)\\ &+ (\nabla_Z\sigma)(Y,W)+\eta(W)\sigma(TZ,Y).\end{aligned}$$

Since $\widetilde{M}$ is 2-semiparallel and for $W=\xi$, we obtain $\sigma(TZ,Y)=0$. This proves our assertion. The converse is obvious. ■

## 5. Conclusions

In this present study submanifolds of a (k,mü)-paracontact manifolds is examined. Firstly. After that, we found the necessary and sufficient conditions for a submanifold of a (k,mü)-paracontact manifolds to be invariant and anti-invariant. The method and the results are given in this study. Also we found for a submanifold of a (k,mü)-paracontact manifolds to be semi-parallel and 2- semi-parallel submanifold. We believe that these results will lead to further studies.

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