Hagia Sophia Journal of Geometry
Vol. 1, No. 2, 15-21, 2019
HSJG

On C-Bochner Curvature Tensor in $(LCS)_n$-Manifolds

Ümit Yildirim *, Mehmet Atçeken and Süleyman Dirik

Abstract
The object of the present paper is to study the $C-$Bochner curvature tensor in $(LCS)_n$-manifolds.

Keywords and 2010 Mathematics Subject Classification
Keywords:$(LCS)_n$-manifold— C-Bochner curvature tensor— projective tensor
MSC: 53C15, 53C25

Department of Mathematics, University of Gaziosmanpasa, 60100, Tokat, TURKEY
Department of Mathematics, University of Gaziosmanpasa, 60100, Tokat, TURKEY
Department of Statistics,University of Amasya, 05100, Amasya, TURKEY
Corresponding author: umit.yildirim@gop.edu.tr
Article History: Received 27 June 2019; Accepted 19 September 2019

1. Introduction

In 2003, Shaikh [18] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [14] and also by Mihai and Rosca [15]. Then Shaikh and Baishya [19] investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. The $(LCS)_n$-manifolds are also studied by Atçeken et. al. [1, 2, 3, 11], Hui [10], Narain and Yadav [16] many authors.

Motivated by the studies of the above authors, in this paper we classify $(LCS)_n$-manifolds, which satisfy the curvature conditions $R(\xi,X)B=0$, $B(\xi,X)P=0$, $B(\xi,X)S=0$ and C-Bochner flat, where B is the C-Bochner curvature tensor, P is the projective curvature tensor and S is the Ricci tensor.

2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p\in M$, the tensor $g_p : T_pM\times T_pM\to\mathbb{R}$ is non-degenerate inner product of signature $(-,+,...,+)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v\in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if satisfies $g_p(v,v)<0$ (resp., $\leq 0, =0, >0$) [17]. The category to which a given vector falls is called its casual chatacter.

Definition 1. *In a Lorentzian manifold (M,g), a vector field P defined by*

$$g(X,P)=A(X)$$

for any $X\in\Gamma(TM)$ is said to be a concircular vector field if

$$(\nabla_X A)Y=\alpha\{g(X,Y)+\omega(X)A(Y)\}$$

for $Y\in\Gamma(TM)$, where α is a nonzero scalar function, A is a 1-form, ω is also closed 1-form, and ∇ denotes the Levi-Civita connection on M.

Let M be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi,\xi) = -1.$$

Since ξ is a unit vector field, there exists a nonzero 1-form η such that

$$g(X,\xi) = \eta(X). \tag{1}$$

The equation of the following form holds:

$$(\nabla_X \eta)Y = \alpha\{g(X,Y)+\eta(X)\eta(Y)\}, \alpha \neq 0 \tag{2}$$

for all $X,Y \in \Gamma(TM)$, where α is nonzero scalar function satisfying

$$\nabla_X \alpha = X(\alpha) = d\alpha(X) = \rho\eta(X), \tag{3}$$

ρ being a certain scalar function given by $\rho = -\xi(\alpha)$. Let us put

$$\nabla_X \xi = \alpha\phi x, \tag{4}$$

then from (2) and (4), we can derive

$$\phi X = X + \eta(X)\xi \tag{5}$$

which tells us that ϕ is symmetric $(1,1)$-tensor. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and $(1,1)$-type tensor field ϕ is said to be a Lorentzian concircular structure manifold. A differentiable manifold M of dimension n is called (LCS)-manifold if it admits a $(1,1)$-type tensor field ϕ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\eta(\xi) = g(\xi,\xi) = -1, \tag{6}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{7}$$

$$g(X,\xi) = \eta(X)\xi, \tag{8}$$

$$\phi\xi = 0, \quad \eta\circ\phi = 0, \tag{9}$$

for all $X \in \Gamma(TM)$. Particulary, if we take $\alpha = 1$, then we can obtain the *LP*-Sasakian structure of Matsumoto [14].

Also, in an $(LCS)_n$-manifold M, the following conditions are satisfied

$$\eta(R(X,Y)Z) = (\alpha^2-\rho)\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big], \tag{10}$$

$$R(\xi,X)Y = (\alpha^2-\rho)\big[g(X,Y)\xi - \eta(Y)X\big], \tag{11}$$

$$R(X,Y)\xi = (\alpha^2-\rho)\big[\eta(Y)X - \eta(X)Y\big], \tag{12}$$

$$(\nabla_X\phi)Y = \alpha\big[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\big], \tag{13}$$

$$S(X,\xi) = (n-1)(\alpha^2-\rho)\eta(X), \tag{14}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)(\alpha^2-\rho)\eta(X)\eta(Y) \tag{15}$$

for all X,Y,Z on M, where R is the Riemannian curvature tensor and S is the Ricci tensor. Q is also the Ricci operator given by $S(X,Y) = g(QX,Y)$ [18].

S. Bochner [5] introduced a K$\ddot{a}$hler analogue of the Weyl conformal curvature tensor by purely formal considerations, which is now well known as the Bochner curvature tensor. A geometric meaning of the Bochner curvature tensor was given by D. E. Blair [4]. By using the Boothby-Wangs fibration [6], M. Matsumoto and G. Chuman [13] constructed the C-Bochner curvature tensor from the Bochner curvature tensor.

The C-Bochner curvature tensor is given by

$$\begin{aligned} B(X,Y)Z &= R(X,Y)Z+\frac{1}{n+3}\big[S(X,Z)Y-S(Y,Z)X+g(X,Z)QY \\ &- g(Y,Z)QX+S(\phi X,Z)\phi Y-S(\phi Y,Z)\phi X \\ &+ g(\phi X,Z)Q\phi Y-g(\phi Y,Z)Q\phi X+2S(\phi X,Y)\phi Z \\ &+ 2g(\phi X,Y)Q\phi Z-S(X,Z)\eta(Y)\xi \\ &+ S(Y,Z)\eta(X)\xi-\eta(X)\eta(Z)QY+\eta(Y)\eta(Z)QX\big] \\ &- \frac{p+n-1}{n+3}\big[g(\phi X,Z)Y-g(\phi Y,Z)\phi X+2g(\phi X,Y)\phi Z\big] \\ &- \frac{p-4}{n+3}\big[g(X,Z)Y-g(Y,Z)X\big] \\ &+ \frac{p}{n+3}\big[g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi \\ &+ \eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\big], \end{aligned} \tag{16}$$

where S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator defined by $g(QX,Y)=S(X,Y)$ and $p=\frac{n+r-1}{n+1}$, r is the scalar curvature of the manifold.

The projective curvature tensor P of n-dimensional Riemann manifold is defined by

$$P(X,Y)Z=R(X,Y)Z-\frac{1}{(n-1)}\big[S(Y,Z)X-S(X,Z)Y\big], \tag{17}$$

where S is the Ricci tensor of the manifold [21].

In $(LCS)_n$-manifold M, the following conditions are satisfied

$$\begin{aligned} B(\xi,Y)Z &= \Big[\frac{4(\alpha^2-\rho)+2p-4}{n+3}\Big]\big[g(Y,Z)\xi-\eta(Z)Y\big] \\ &+ \frac{2}{n+3}\big[\eta(Z)QY-S(Y,Z)\xi\big], \end{aligned} \tag{18}$$

$$\begin{aligned} B(X,Y)\xi &= \Big[\frac{4(\alpha^2-\rho)+2p-4}{n+3}\Big]\big[\eta(Y)X-\eta(X)Y\big] \\ &+ \frac{2}{n+3}\big[\eta(X)QY-\eta(Y)QX\big], \end{aligned} \tag{19}$$

$$\begin{aligned} B(\xi,Y)\xi &= \Big[\frac{4(\alpha^2-\rho)+2p-4}{n+3}\Big]\big[\eta(Y)\xi+Y\big] \\ &- \frac{2}{n+3}\big[QY+(n-1)(\alpha^2-\rho)\eta(Y)\xi\big]. \end{aligned} \tag{20}$$

$$P(\xi,Y)Z=(\alpha^2-\rho)g(Y,Z)\xi-\frac{1}{n-1}S(Y,Z)\xi \tag{21}$$

and

$$P(X,Y)\xi=P(\xi,Y)\xi=0. \tag{22}$$

Theorem 2. *If an $(LCS)_n$-manifold M is C-Bochner flat, then M reduces to an η-Einstein Manifold.*

Proof. Suppose that an $(LCS)_n$-manifold M is C-Bochner flat. Then we have,

$$B(X,Y)Z = 0. \tag{23}$$

In (16), putting $Z = \xi$, we have

$$\begin{aligned} 0 &= R(X,Y)\xi + \frac{1}{n+3}\big[S(X,\xi)Y - S(Y,\xi)X \\ &+ g(X,\xi)QY - g(Y,\xi)QX - S(X,\xi)\eta(Y)\xi \\ &+ S(Y,\xi)\eta(X)\xi + \eta(X)QY - \eta(Y)QX\big] \\ &- \frac{p-4}{n+3}\big[g(X,\xi)Y - g(Y,\xi)X\big] \\ &+ \frac{p}{n+3}\big[g(X,\xi)\eta(Y)\xi - g(Y,\xi)\eta(X)\xi \\ &+ \eta(Y)X - \eta(X)Y\big]. \end{aligned} \tag{24}$$

In (24), by using the equations (6),(8),(9),(12) and (14), we obtain

$$\begin{aligned} 0 &= \big[\alpha^2 - \rho + \frac{2p-4}{n+3} - \frac{(n-1)(\alpha^2-\rho)}{n+3}\big]\big[\eta(Y)X - \eta(X)Y\big] \\ &+ \frac{2}{n+3}\big[\eta(X)QY - \eta(Y)QX\big]. \end{aligned} \tag{25}$$

Putting $X = \xi$ in (25) and by using (14), we obtain

$$\begin{aligned} \frac{2}{n+3}QY &= \big[\alpha^2 - \rho + \frac{2p-4}{n+3} - \frac{3(n-1)(\alpha^2-\rho))}{n+3}\big]\eta(Y)\xi \\ &+ \big[\alpha^2 - \rho + \frac{2p-4}{n+3} - \frac{(n-1)(\alpha^2-\rho)}{n+3}\big]Y, \end{aligned} \tag{26}$$

which is equivalent to

$$QY = \big[2p - 4 + 4(\alpha^2-\rho)\big]Y + \big[2p - 4 - (\alpha^2-\rho)(2n-6)\big]\eta(Y)\xi. \tag{27}$$

Inner product both sides of the equation by $W \in \chi(M)$ and taking into account $p = \frac{n+r-1}{n+1}$, we conclude

$$\begin{aligned} S(Y,W) &= \big[2(\alpha^2-\rho) - \big(1 + \frac{r}{n+1}\big)\big]g(Y,W) \\ &+ \big[(3-n)(\alpha^2-\rho) - \big(1 + \frac{r}{n+1}\big)\big]\eta(Y)\eta(W). \end{aligned}$$

■

Theorem 3. *Let M be an $(LCS)_n$-manifold. Then, $R(\xi,Y)B$ is always identically zero, for any $Y \in \chi(M)$.*

Proof. For any $X,Y,U,W,Z \in \chi(M)$ on M, we have

$$\begin{aligned} (R(X,Y)B)(U,W,Z) &= R(X,Y)B(U,W)Z - B(R(X,Y)U,W)Z \\ &- B(U,R(X,Y)W)Z - B(U,W)R(X,Y)Z. \end{aligned} \tag{28}$$

In (28), for $X = \xi$, we have

$$\begin{aligned} (R(\xi,Y)B)(U,W,Z) &= R(\xi,Y)B(U,W)Z - B(R(\xi,Y)U,W)Z \\ &- B(U,R(\xi,Y)W)Z - B(U,W)R(\xi,Y)Z. \end{aligned} \tag{29}$$

By using (11) in (29), we obtain

$$\begin{aligned}(R(\xi,Y)B)(U,W,Z) &= \big(\alpha^2-\rho\big)\big[g(Y,B(U,W)Z)\xi-\eta\big(B(U,W)Z\big)Y \\ &- B\big(g(Y,U)\xi-\eta(U)Y,W\big)Z \\ &- B\big(U,g(Y,W)\xi-\eta(W)Y\big)Z \\ &- B(U,W)\big(g(Y,Z)\xi-\eta(Z)Y\big)\big]. \end{aligned} \tag{30}$$

Now, by using (18),(19) and choosing $U=Z=\xi$, we obtain

$$\begin{aligned}(R(\xi,Y)B)(\xi,W,\xi) &= g\big(Y,A\eta(W)+AW-\frac{2}{n+3}QW-D\eta(W)\xi\big)\xi \\ &- \eta\big(A\eta(W)\xi+AW-\frac{2}{n+3}QW-D\eta(W)\big)Y \\ &- 2\eta(Y)\big(A\eta(W)\xi+AW-\frac{2}{n+3}QW-D\eta(W)\big) \\ &- A\eta(W)Y+2A\eta(Y)W-\frac{4}{n+3}\eta(Y)QW+\frac{2}{n+3}\eta(W)QY \\ &+ \eta(W)\big[A\eta(Y)\xi+AY-\frac{2}{n+3}-D\eta(Y)\xi\big] \\ &- Ag(W,Y)\xi+\frac{2}{n+3}S(W,Y)\xi, \end{aligned} \tag{31}$$

where, $A=\frac{4(\alpha^2-\rho)+2p-4}{n+3}$ and $D=\frac{2(n-1)(\alpha^2-\rho)}{n+3}$.

We easily obtain from (31) that

$$(R(\xi,Y)B)(\xi,W,\xi)=0. \tag{32}$$

■

3. $(LCS)_n$-Manifolds Satisfying Conditions $(B,\xi)P=0$ and $(B,\xi)S=0$

Theorem 4. *Let M be an $(LCS)_n$-manifold. Then the manifold satisfies $B(\xi,Y)P=0$ if and only if there is the following relations*

$$\|Q\|^2=n\big[(n-1)(\alpha^2-\rho)\big]^2\big[2(\alpha^2-\rho)+p-2\big]+r\big[(\alpha^2-\rho)(n+1)+p-2\big].$$

Proof. In order to prove our theorem, we assume that $B((\xi,Y)P)(U,W)Z=0$, for all $\xi,Y,U,W,Z\in\chi(M)$. Then we have

$$\begin{aligned}0 &= B(\xi,Y)P(U,W)Z-P(B(\xi,Y)U,W)Z \\ &- P(U,B(\xi,Y)W)Z-P(U,W)B(\xi,Y)Z \end{aligned} \tag{33}$$

In (33), by using the equation (18) we obtain

$$\begin{aligned}0 &= \big[\frac{4(\alpha^2-\rho)+2p-4}{n+3}\big]\big[g(Y,P(U,W)Z)\xi-\eta(P(U,W)Z)Y \\ &- g(Y,U)P(\xi,W)Z+\eta(U)P(Y,W)Z \\ &- g(Y,W)P(U,\xi)Z+\eta(W)P(U,Y)Z \\ &- g(Y,Z)P(U,W)\xi+\eta(Z)P(U,W)Y\big] \\ &+ \frac{2}{n+3}\big[\eta(P(U,W)Z)QY-S(Y,P(U,W)Z)\xi \\ &- \eta(U)P(QY,W)Z+S(Y,U)P(\xi,W)Z \\ &- \eta(W)P(U,QY)Z+S(Y,W)P(U,\xi)Z \\ &- \eta(Z)P(U,W)QY+S(Y,Z)P(U,W)\xi\big]. \end{aligned} \tag{34}$$

Here, substituting $U=\xi$ in (34), we have

$$\begin{aligned} 0 &= [\frac{4(\alpha^2-\rho)+2p-4}{n+3}]\big[g(Y,P(\xi,W)Z)\xi-\eta(P(\xi,W)Z)Y \\ &- \eta(Y)P(\xi,W)Z-P(Y,W)Z+P(\xi,Y)Z+\eta(Z)P(\xi,W)Y\big] \\ &+ \frac{2}{n+3}\big[\eta(P(\xi,W)Z)QY-S(Y,P(\xi,W)Z)\xi \\ &+ P(QY,W)Z-\eta(W)P(\xi,QY)Z-\eta(Z)P(\xi,W)QY\big] \\ &+ \frac{2(n-1)(\alpha^2-\rho)}{n+3}\eta(Y)P(\xi,W)Z. \end{aligned} \tag{35}$$

Let $Z=\xi$ be in (35), then also by using (6), (21) and (22), we obtain

$$[\frac{4(\alpha^2-\rho)+2p-4}{n+3}]P(\xi,W)QY+\frac{2}{n+3}P(\xi,W)Y=0. \tag{36}$$

Again by using (21) in (36), we get

$$\begin{aligned} 0 &= [\frac{4(\alpha^2-\rho)+2p-4}{n+3}]\big[(\alpha^2-\rho)g(W,Y)\xi-\frac{1}{n-1}S(W,Y)\xi\big] \\ &- \frac{2}{n+3}\big[(\alpha^2-\rho)g(W,QY)\xi-\frac{1}{n-1}S(W,QY)\xi\big] \end{aligned}$$

which implies that

$$\begin{aligned} S(W,QY) &= \big[(\alpha^2-\rho)(n+1)+p-2\big]S(Y,W) \\ &- (n-1)(\alpha^2-\rho)\big[2(\alpha^2-\rho)+p-2\big]g(W,Y). \end{aligned} \tag{37}$$

Now, for $\big[e_1,e_2,...,e_{n-1},\xi\big]$ orthonormal basis of M from (37), we conclude

$$\|Q\|^2=n\big[(n-1)(\alpha^2-\rho)\big]^2\big[2(\alpha^2-\rho)+p-2\big]+r\big[(\alpha^2-\rho)(n+1)+p-2\big],$$

which proves our assertion. The converse is obvious. ∎

Theorem 5. *Let M be an $(LCS)_n$-manifold. Then $B(\xi,Y)S=0$ if and only if there is the following relations*

$$\|Q\|^2=n\big[(n-1)(\alpha^2-\rho)\big]^2\big[2(\alpha^2-\rho)+p-2\big]+r\big[(\alpha^2-\rho)(n+1)+p-2\big].$$

Proof. We suppose that $(B(\xi,Y)S)(U,W)=0$. Then for all $\xi,Y,U,W\in\chi(M)$ we have

$$S(B(\xi,Y)U,W)+S(U,B(\xi,Y)W)=0. \tag{38}$$

In (38), by using (18) we get

$$\begin{aligned} 0 &= [\frac{4(\alpha^2-\rho)+2p-4}{n+3}]\big[g(Y,U)S(\xi,W)-\eta(U)S(Y,W) \\ &+ g(Y,W)S(U,\xi)-\eta(W)S(U,Y)\big] \\ &+ \frac{2}{n+3}\big[\eta(U)S(QY,W)-S(Y,U)S(\xi,W) \\ &+ \eta(W)S(U,QY)-S(Y,W)S(U,\xi)\big]. \end{aligned} \tag{39}$$

Now, in (39) substituting $U=\xi$ we obtain

$$\begin{aligned} S(QY,W) &= \big[(\alpha^2-\rho)(n+1)+p-2\big]S(Y,W) \\ &+ \big[(\alpha^2-\rho)(n-1)\big]\big[2(\alpha^2-\rho)+p-2\big]g(Y,W). \end{aligned} \tag{40}$$

Again for $\big[e_1,e_2,...,e_{n-1},\xi\big]$ orthonormal basis of M from (40), we conclude

$$\|Q\|^2=n\big[(n-1)(\alpha^2-\rho)\big]^2\big[2(\alpha^2-\rho)+p-2\big]+r\big[(\alpha^2-\rho)(n+1)+p-2\big],$$

∎

4. Conclusion

In the present paper, we have studied the C-Bochner curvature tensor of $(LCS)_n$-manifolds satisfying the conditions C-Bochner flat, $R.B = 0$, $B.P = 0$ and $B.S = 0$. According these cases, we classified $(LCS)_n$-manifolds. The same classification can be made for other curvature tensors.

References

[1] Atceken, M. and Hui, S. K., Slant and pseudo-slant submanifolds of $(LCS)_n$manifolds, Czechoslovak Math. J., 63 (2013), 177-190.

[2] Atceken, M., On geometry of submanifolds of $(LCS)_n$-manifolds, Int. J. Math. and Math. Sci., 2012, doi:10.1155/2012/304647.

[3] Atçeken, M. and Yıldırım, Ü., Weakly symmetric and weakly Ricci symmetric conditions on $(LCS)_n$-manifolds, African Journal of Mathematics and Computer Science Research, Vol. 6(6), (2013), 129-134.

[4] Blair, D. E., On the geometric meaning of the Bochner tensor, Geom. Dedicata, 4, 33-38, 1975.

[5] Bochner, S., Curvature and Betti numbers, Ann. of Math. 50 (1949)77-93.

[6] Boothby, W. M. and Wang, H. C., On contact manifolds. Annals of Math., 68 (1958), 721-734

[7] De, U.C. Samui, S., E-Bochner curvature tensor on (κ,μ)-contact metric manifolds, Int. Electron. J. Geom. 7(1)(2014)143-153.

[8] De, U.C.and Ghosh, S., E-Bochner curvature tensor on $N(k)$-contact metric mnifolds, Hacettepe Journal of Mathematics and Statistics, 43(3), (2014), 365-374.

[9] Endo, H., On K-contact Reimannian manifolds with vanishing E-contact Bochner curvature tensor, Colloq. Math. 62(1991), 293-297.

[10] Hui, S. K., On ϕ-pseudo symmetries of (LCS)n-manifolds, Kyungpook Math. J., 53 (2013), 285-294.

[11] Hui, S. K. and Atceken, M., Contact warped product semi-slant submanifolds of (LCS)n-manifolds, Acta Univ. Sapientiae Mathematica, 3(2) (2011), 212-224.

[12] Kaneyuki,S. and Williams, F. L., Almost paracontact and parahodge structures on manifolds, Nagoya Math. J. 99 (1985), 173-187.

[13] Matsumoto, M. and Chuman, G., On the C-Bochner curvature tensor,TRUMath.5(1969)21-30

[14] Matsumoto, K., On Lorentzian paracontact manifolds, Bulletin of Yamagata University, vol. 12, no. 2, pp. 151-156, 1989.

[15] Mihai, I. and Rosca, R., On Lorentzian para-Sasakian manifolds, Classical Anal., World Sci. Publ., Singapore, (1992), 155-169.

[16] Narain, D. and Yadav, S., On weak concircular symmetries of $(LCS)_{2n+1}$- manifolds, Global J. Sci. Frontier Research, 12 (2012), 85-94.

[17] O' Neill, B., Semi-Riemannian Geometry, Academic Press, New York, 1983.

[18] Shaikh, A. A., On Lorentzian almost paracontact manifolds with a structure of the concircular type, Kyungpook Mathematical Journal, vol. 43, no. 2, pp. 305-314, 2003.

[19] Shaikh, A. A. and Baishya, K. K., On concircular structure spacetimes II, American J. Appl. Sci., 3(4) (2006), 1790-1794.

[20] Shaikh, A. A., Some results on $(LCS)_n$-manifolds, J. Korean Math. Soc. 46 (2009), No. 3, pp. 449-461.

[21] Yano, K. and Kon, M., Structures of manifolds, World Scientific Publishing, Singapore 1984.