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A New Approach for Inextensible Flows of Curves in Pseudo-Galilean Space G_3^1

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Abstract

In this paper, inextensible flows of a spacelike curve on a ruled surface of type I in 3-dimensional pseudo-Galilean space G_3^1 are researched. Firstly inextensible flows of these curves according to Darboux frame are determined then necessary and sufficient conditions for inextensible flows of the curves are expressed as a partial differential equation involving the curvature with this frame in G_3^1.

Keywords: *Darboux frame, Inextensible flows, pseudo-Galilean space.*

2010 Mathematics Subject Classification: *53A35*

1. Introduction

The flows of inextensible curve and surface are one of the tool to solve many problems in computer vision [8], [13], computer animation [2] and even structural mechanics [17]. Especially the methods used in this study are improved in [6, 7]. The differentiation between heat flows and inextensible flows of planar curves are studied by Kwon in [10]. Also, inextensible flows of curves and developable surfaces in $\mathbb{R}^3$ are revealed by Kwon in [11]. After that a lot of works have been done by some authors. Such that Latifi et al. [12] investigated inextensible flows of curves in Minkowski 3-space, Ogrenmis et al. [14] studied inextensible flows of curves in the 3-dimensional Galilean space G_3 and Oztekin et al. [15] researched this curves in the 4-dimensional Galilean space G_4.

In the differential geometry especially theory of surfaces the Darboux frame which is a natural moving frame constructed on a surface has an important role. It is the analog of the Frenet Serret frame as applied to surface geometry. After the definition of this frame in the literature, a significant number of results concerning of this frame are obtained for the different spaces, see [9, 19].

In the present study inextensible flows of a spacelike curve which is defined on a ruled surfaces of type-I according to Darboux frame in the 3-dimensional pseudo-Galilean Space G_3^1 are examined. Besides, partial differential equations in terms of inextensible flows of curves with respect to this frame in 3-dimensional pseudo-Galilean space G_3^1 are obtained. After that necessary and sufficient conditions for inextensible flows which are expressed as a partial differential equation involving the curvature are given in G_3^1.

2. Preliminaries

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries whose projective signature is (0,0,+,-). As in [3], pseudo-Galilean inner product can be written as

$$\langle v_1, v_2\rangle = \begin{cases} x_1x_2 & , if\ x_1 \neq 0 \vee x_2 \neq 0 \\ y_1y_2 - z_1z_2 & , if\ x_1 = 0 \wedge x_2 = 0 \end{cases}$$

where $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$. The pseudo-Galilean norm of the vector $v = (x, y, z)$ defined by

$$\|v\| = \begin{cases} |x| & , if\ x \neq 0 \\ \sqrt{|y^2 - z^2|} & , if\ x = 0 \end{cases}$$

In pseudo-Galilean space a curve is given by $\gamma : I \to G_3^1$

$$\gamma(t) = (x(t), y(t), z(t)) \tag{2.1}$$

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where $I \subseteq \mathbb{R}$ and $x(t), y(t), z(t) \in C^3$. A curve γ given by (2.1) is admissible if $x'(t) \neq 0$ [3]. An admissible curve in G_3^1 can be parametrized by arc length $t = s$, given as follows,

$$\gamma(s) = (s, y(s), z(s)) \ . \tag{2.2}$$

For an admissible curve $\gamma : I \subseteq \mathbb{R} \to G_3^1$, the curvature $\kappa(s)$ and the torsion $\tau(s)$ are determined by

$$\kappa(x) = \sqrt{\left|y''^2 - z''^2\right|}, \tag{2.3}$$

$$\tau(s) = \frac{1}{\kappa^2(s)} \det\left(\gamma'(s), \gamma''(s), \gamma'''(s)\right). \tag{2.4}$$

The associated trihedron is given by

$$\begin{aligned} T(s) &= \gamma'(s) = \left(1, y'(s), z'(s)\right), \\ N(s) &= \frac{1}{\kappa(s)} \gamma''(s) = \frac{1}{\kappa(s)} \left(0, y''(s), z''(s)\right), \\ B(s) &= \frac{1}{\kappa(s)} \left(0, z''(s), y''(s)\right). \end{aligned} \tag{2.5}$$

[4]. The vectors $T(s)$,$N(s)$ and $B(s)$ are called the vectors of tangent, principal normal and binormal line of γ, respectively. The curve γ given by (2.2) is timelike (resp. spacelike) if $n(s)$ is spacelike (resp. timelike) vector. For derivatives of tangent vector $T(s)$, principal normal vector $N(s)$ and binormal vector $B(s)$, respectively, the following Frenet formulas hold

$$\begin{aligned} T'(s) &= \kappa(s) N(s), \\ N'(s) &= \tau(s) B(s), \\ B'(s) &= \tau(s) N(s). \end{aligned} \tag{2.6}$$

Let $M(x,v)$ be a ruled surface of type I in G_3^1 then M can be represented by

$$M(x,v) = \gamma(x) + v a(x),$$

where $\gamma(x) = (x, y(x), z(x))$ is the directrix curve and $a(x) = (1, a_2(x), a_3(x))$ is a unit generator vector field. The associated trihedron of the ruled surface of type I in G_3^1 is determined by

$$\begin{aligned} T &= (1, a_2, a_3), \\ N &= \frac{1}{\kappa} \left(0, a_2', a_3'\right), \\ B &= \frac{1}{\kappa} \left(0, a_3', a_2'\right). \end{aligned} \tag{2.7}$$

where $\kappa = \sqrt{\left|(a_2')^2 - (a_3')^2\right|}$ is the curvature and n is the central isotropic timelike normal vector field. In this paper n is taken as timelike. The following frenet formulas hold,

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.8}$$

where $\tau = \dfrac{-1}{\kappa^2} \det\left(a, a', a''\right)$ is the torsion of the ruled surface. The surface frame $\{T, S_n, S_b\}$ is defined as follows

$$T = a(x), \qquad ,S_n = \frac{M_x \wedge M_v}{\|M_x \wedge M_v\|}, \qquad ,S_b = S_n \wedge T.$$

Assuming that θ be the hyperbolic angle between the isotropic timelike vectors S_n and n. So, the following matrix form can be expressed,

$$\begin{bmatrix} T \\ S_n \\ S_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.9}$$

The Darboux equations can be written

$$\begin{bmatrix} T \\ S_n \\ S_b \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_n & \kappa_g \\ 0 & 0 & \tau_g \\ 0 & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ S_n \\ S_b \end{bmatrix}. \tag{2.10}$$

where κ_g, κ_n and τ_g are geodesic curvature, normal curvature and geodesic torsion, respectively, given by

$$\kappa_n = \kappa \cosh\theta, \qquad ,\kappa_g = -\kappa \sinh\theta, \qquad ,\tau_g = d\theta + \tau$$

[5, 1]. We refer to [16, 18] for detailed information about the pseudo-Galilean geometry.

3. Inextensible Flows of Curves with Darboux Frame in pseudo-Galilean Space G_3^1

Throughout this paper, we assume that $\gamma : [0,l]\times[0,w] \to M \subset G_3^1$ is a one parameter family of smooth spacelike curve on a ruled surface of type-I in 3-dimensional pseudo-Galilean space G_3^1, where l is the arc length of the initial curve and u is the curve parametrization variable, $0 \leq u \leq l$. The arc length of γ is given by

$$s(u) = \int_0^u \left|\frac{\partial \gamma}{\partial u}\right| du, \tag{3.1}$$

where

$$\left|\frac{\partial \gamma}{\partial u}\right| = \left|\left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u}\right\rangle\right|^{\frac{1}{2}}. \tag{3.2}$$

The operator $\dfrac{\partial}{\partial s}$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u},$$

where $v = \left|\dfrac{\partial \gamma}{\partial u}\right|$ and the arc length parameter is $ds = vdu$. Arbitrary flow of γ can be represented as

$$\frac{\partial \gamma}{\partial t} = f_1 T + f_2 S_n + f_3 S_b \tag{3.3}$$

where $\{T, S_n, S_b\}$ is Darboux frame of the spacelike curve γ on a ruled surfaces of type-I in G_3^1 and f_1, f_2, f_3 are scalar speeds of the curve γ. Let the arc length variation be

$$s(u,t) = \int_0^u vdu\,.$$

In the 3-dimensional pseudo-Galilean space G_3^1 the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial v}{\partial t} du = 0, \tag{3.4}$$

for all $u \in [0,l]$.

Definition 3.1. *Let $\gamma(u,t)$ be a curve evolution and $\dfrac{\partial \gamma}{\partial t}$ be its flow in 3-dimensional pseudo-Galilean space G_3^1. A curve evolution $\gamma(u,t)$ is inextensible if*

$$\frac{\partial}{\partial t}\left|\frac{\partial \gamma}{\partial u}\right| = 0.$$

Lemma 3.2. *Let $\dfrac{\partial \gamma}{\partial t} = f_1 T + f_2 S_n + f_3 S_b$ be a smooth flow of the curve γ in G_3^1. Then the flow is inextensible if and only if*

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u}. \tag{3.5}$$

Proof. Assuming that $\dfrac{\partial \gamma}{\partial t}$ be a smooth flow of the curve γ in G_3^1. Using definition of γ, we have

$$v^2 = \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u}\right\rangle \tag{3.6}$$

Since $\dfrac{\partial}{\partial u}$ and $\dfrac{\partial}{\partial t}$ are commute we get

$$v\frac{\partial v}{\partial t} = \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial}{\partial u}(f_1 T + f_2 S_n + f_3 S_b)\right\rangle.$$

Using Darboux frame, we obtain

$$v\frac{\partial v}{\partial t} = \left\langle T, \frac{\partial f_1}{\partial u} T + \left(\frac{\partial f_2}{\partial u} + f_1\kappa_n + f_3\tau_g\right) S_n + \left(\frac{\partial f_3}{\partial u} + f_1\kappa_g + f_2\tau_g\right) S_b\right\rangle.$$

After necessary calculations from above equation, we have (3.5), which proves the lemma. □

Theorem 3.3. *Let $\dfrac{\partial \gamma}{\partial t} = f_1 T + f_2 S_n + f_3 S_b$ be a smooth flow of the curve γ in G_3^1. Then the flow is inextensible if and only if*

$$\frac{\partial f_1}{\partial s} = 0. \tag{3.7}$$

Proof. From (3.4), we have

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u \frac{\partial v}{\partial t}du=\int_0^u \frac{\partial f_1}{\partial u}=0. \tag{3.8}$$

Substituting (3.5) into (3.8) completes the proof. □

After this arc length parametrized curves are used that is, $v=1$ and the local coordinate u corresponds to the curve arc length s.

Lemma 3.4. *Let* $\frac{\partial \gamma}{\partial t}=f_1T+f_2S_n+f_3S_b$ *be a inextensible flow of the curve* γ *in* G_3^1*. Then,*

$$\begin{aligned}
\frac{\partial T}{\partial t} &= \left(\frac{\partial f_2}{\partial s}+f_1\kappa_n+f_3\tau_g\right)S_n+\left(\frac{\partial f_3}{\partial s}+f_1\kappa_g+f_2\tau_g\right)S_b,\\
\frac{\partial S_n}{\partial t} &= \left(\frac{\partial f_2}{\partial s}+f_1\kappa_n+f_3\tau_g\right)T,\\
\frac{\partial S_b}{\partial t} &= \left(-\frac{\partial f_3}{\partial s}-f_1\kappa_g-f_2\tau_g\right)T.
\end{aligned} \tag{3.9}$$

Proof. Nothing that,

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \gamma}{\partial s}=\frac{\partial}{\partial s}\left(f_1T+f_2S_n+f_3S_b\right).$$

Thus, it is seen that

$$\frac{\partial T}{\partial t}=\frac{\partial f_1}{\partial s}T+\left(\frac{\partial f_2}{\partial s}+f_1\kappa_n+f_3\tau_g\right)S_n+\left(\frac{\partial f_3}{\partial s}+f_1\kappa_g+f_2\tau_g\right)S_b. \tag{3.10}$$

On the other hand substituting (3.7) into the equation (3.10), we get

$$\frac{\partial T}{\partial t}=\left(\frac{\partial f_2}{\partial s}+f_1\kappa_n+f_3\tau_g\right)S_n+\left(\frac{\partial f_3}{\partial s}+f_1\kappa_g+f_2\tau_g\right)S_b.$$

The differentiation of the Darboux frame with respect to t is as follows:

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\langle T,S_n\rangle=-\left(\frac{\partial f_2}{\partial s}+f_1\kappa_n+f_3\tau_g\right)+\left\langle T,\frac{\partial S_n}{\partial t}\right\rangle,\\
0 &= \frac{\partial}{\partial t}\langle T,S_b\rangle=\left(\frac{\partial f_3}{\partial s}+f_1\kappa_g+f_2\tau_g\right)+\left\langle T,\frac{\partial S_b}{\partial t}\right\rangle.
\end{aligned}$$

Considering the above equations, pseudo-Galilean inner product and the following statement

$$\left\langle\frac{\partial S_n}{\partial t},S_n\right\rangle=\left\langle\frac{\partial S_b}{\partial t},S_b\right\rangle=\left\langle\frac{\partial S_n}{\partial t},S_b\right\rangle=\left\langle\frac{\partial S_b}{\partial t},S_n\right\rangle=0,$$

we obtain

$$\begin{aligned}
\frac{\partial S_n}{\partial t} &= \left(\frac{\partial f_2}{\partial s}+f_1\kappa_n+f_3\tau_g\right)T,\\
\frac{\partial S_b}{\partial t} &= \left(-\frac{\partial f_3}{\partial s}-f_1\kappa_g-f_2\tau_g\right)T.
\end{aligned}$$

□

Corollary. Let $\frac{\partial \gamma}{\partial t}=f_1T+f_2S_n+f_3S_b$ be a inextensible flow of the curve γ in G_3^1. Then, if γ is a geodesic curve (not straight line) on the surface then $\kappa_g=0$. Therefore using this statement and the equation (3.9) we get the following differential equation system

$$\begin{aligned}
\frac{\partial T}{\partial t} &= \left(\frac{\partial f_2}{\partial s}+f_1\kappa+f_3\tau\right)S_n+\left(\frac{\partial f_3}{\partial s}+f_2\tau\right)S_b,\\
\frac{\partial S_n}{\partial t} &= \left(\frac{\partial f_2}{\partial s}+f_1\kappa+f_3\tau\right)T,\\
\frac{\partial S_b}{\partial t} &= \left(-\frac{\partial f_3}{\partial s}-f_2\tau\right)T.
\end{aligned} \tag{3.11}$$

Theorem 3.5. *Let* $\frac{\partial \gamma}{\partial t}=f_1T+f_2S_n+f_3S_b$ *be a inextensible flow of the curve* γ *in* G_3^1*. Then, the following system of partial differential equations holds:*

$$\begin{aligned}
\frac{\partial \kappa_n}{\partial t} &= \frac{\partial^2 f_2}{\partial s^2}+f_1\frac{\partial \kappa_n}{\partial s}+2\frac{\partial f_3}{\partial s}\tau_g+f_3\frac{\partial \tau_g}{\partial s}+f_1\kappa_g\tau_g+f_2\left(\tau_g\right)^2,\\
\frac{\partial \kappa_g}{\partial t} &= \frac{\partial^2 f_3}{\partial s^2}+f_1\frac{\partial \kappa_g}{\partial s}+2\frac{\partial f_2}{\partial s}\tau_g+f_2\frac{\partial \tau_g}{\partial s}+f_1\kappa_n\tau_g+f_3\left(\tau_g\right)^2,\\
\frac{\partial \tau_g}{\partial t} &= \frac{\partial f_2}{\partial s}\kappa_g+f_1\kappa_n\kappa_g+f_3\kappa_g\tau_g.
\end{aligned} \tag{3.12}$$

Proof. Considering the equation (3.9) we obtain,

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial T}{\partial t} &= \left(\frac{\partial^2 f_2}{\partial s^2}+\frac{\partial f_1}{\partial s}\kappa_n+f_1\frac{\partial \kappa_n}{\partial s}+\frac{\partial f_3}{\partial s}\tau_g+f_3\frac{\partial \tau_g}{\partial s}\right)S_n \\ &+ \left(\frac{\partial f_2}{\partial s}+f_1\kappa_n+f_3\tau_g\right)\left(\tau_g S_b\right) \\ &+ \left(\frac{\partial^2 f_3}{\partial s^2}+\frac{\partial f_1}{\partial s}\kappa_g+f_1\frac{\partial \kappa_g}{\partial s}+\frac{\partial f_2}{\partial s}\tau_g+f_2\frac{\partial \tau_g}{\partial s}\right)S_b \\ &+ \left(\frac{\partial f_3}{\partial s}+f_1\kappa_g+f_2\tau_g\right)\left(\tau_g S_n\right). \end{aligned} \tag{3.13}$$

On the other hand we have,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial T}{\partial s} &= \frac{\partial}{\partial t}\left(\kappa_n S_n+\kappa_g S_b\right) \\ &= \left(\frac{\partial f_2}{\partial s}\kappa_n+f_1\left(\kappa_n\right)^2+f_3\tau_g\kappa_n-\frac{\partial f_3}{\partial s}\kappa_g-f_1\left(\kappa_g\right)^2-f_2\kappa_g\tau_g\right)T \\ &+ \left(\frac{\partial \kappa_n}{\partial t}\right)S_n+\left(\frac{\partial \kappa_g}{\partial t}\right)S_b. \end{aligned} \tag{3.14}$$

Hence from (3.7), (3.14) and (3.15), we get the desired result. Using the same method, the last equation of (3.13) can be obtained. □

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