G.U. Journal of Science
23(1): 49-52 (2010)

GUJS
www.gujs.org

The Properties of The Weak Subdifferentials

Refail KASIMBEYLİ[1], Gonca İNCEOĞLU[2♣]

[1] *Izmir University of Economics, Faculty of Computer Science , Department of Industrial System Engineering, Izmir*
[2♣] *Anadolu University, Faculty of Education, Department of Primary Education, Eskişehir*

Received: 06.03.2009 Revised: 14.03.2009 Accepted: 21.04.2009

ABSTRACT

This paper deals with the weak subdifferentials. The properties of the weak subdifferentials are examined. It is showed that the weak subdifferential of a function having a global minimum is not empty.

Key Words: *Subdifferential, Weak subgradient, Weak subdifferential.*

1. INTRODUCTION

It is well known that, a convex set has a supporting hyperline at each boudary point. This leads to one of the central notions in convex analysis, that of a subgradient of a possible nonsmooth even extended real-valued function [1,6,8]. The main reason of difficulties arising when passing from the convex analysis to the nonconvex one is that, the nonconvex cases may arise in many different forms and each case may require a special approach. The main ingredient is the method of supporting the given nonconvex set. Subgradients plays an important role in deriving of optimality conditions and duality theorems. Since a nonconvex set has no supporting hyperline at each boundary point, the notion of subgradient have been generalized by most researches on optimality conditions for nonconvex problems [4,5,9,10]. The notion of weak subdifferential which is a generalization of the classic subdifferential, is introduced by Azimov and Gasimov [2]. It uses explicitly defined supporting conic surfaces instead of supporting hyperplanes. By using this notion, a collection of zero duality gap conditons for a wide class of nonconvex optimization problems was derived [2]. In this study we give some important properties of the weak subdiffrentials.

2. WEAK SUBDIFFERENTIALS

We recall the concepts of the convex set, domain, epigraph, hypograph, subdifferential respectively [7].

Definition 1. Let $(X, \|.\|_X)$ be a real normed space and let S be a nonempty subset of X.

a) The set S is called a cone, if $x \in S, \quad \lambda \geq 0 \Rightarrow \lambda x \in S$.

b) The set S is called convex, if for every $x, y \in S$ and for all $\lambda \in [0,1]$

$$\lambda x + (1-\lambda) y \in S \ .$$

Definition 2. Let $(X, \|.\|_X)$ be a real normed space and $F : X \rightarrow R$ be a function.

a) The set $dom(F) = \{x \in X : F(x) < \infty\}$ is called the domain of F.

b) The set $epi(F) = \{(x, \alpha) \in X \times R : F(x) \leq \alpha\}$ is called the epigraph of F.

c) The set $hypo(F) = \{(x, \alpha) \in X \times R : F(x) \geq \alpha\}$ is called the hypograph of F.

Definition 3. Let $(X, \|.\|_X)$ be a real normed space, let $F : X \rightarrow R$ be a function and let $\bar{x} \in X$ be given. The set

$$\partial F(\bar{x}) = \{x^* \in X : \langle x^*, x - \bar{x} \rangle \leq F(x) - F(\bar{x}) \quad for \quad all \quad x \in X\}$$

is called the subdifferential of F at $\bar{x} \in X$.

Let $(X, \|.\|_X)$ be a real normed space and let X^* be the topological dual of X. Let $(x^*, c) \in X^* \times R_+$ where R_+ is the set of nonnegative real numbers. We define a conic surface $C(\bar{x}, x^*, c) \subset X$ with vertex $\bar{x} \in X$ as follows:

Corresponding author: inceoglugonca@gmail.com

$$C(\bar{x},x^*,c)=\{x\in X:\langle x^*,x-\bar{x}\rangle-c\|x-\bar{x}\|=0\} \quad (1)$$

Then the corresponding upper and lower conic half-spaces are respectively defined as

$$C^+(\bar{x},x^*,c)=\{x\in X:\langle x^*,x-\bar{x}\rangle-c\|x-\bar{x}\|\le 0\} \quad (2)$$

and

$$C^-(\bar{x},x^*,c)=\{x\in X:\langle x^*,x-\bar{x}\rangle-c\|x-\bar{x}\|\ge 0\} \quad (3)$$

Note that if $c=0$ the conic surface $C(\bar{x},x^*,c)$ becomes to a hyperplane. Hence a supporting cone defined below is a simple generalization of supporting hyperplane [8].

Definition 4. A cone $C(\bar{x},x^*,c)$ is called supporting cone to the set $S\subset X$, if $S\subset C^+(\bar{x},x^*,c)$ (or $S\subset C(\bar{x},x^*,c)$) and $cl(S)\cap C(\bar{x},x^*,c)\neq\Phi$ [8].

It is clear that the lower conic half-space $C^-(\bar{x},x^*,c)$ is a convex cone with vertex at $\bar{x}\in X$.

Definiton 5. Let $F:X\to R$ be a single valued function and let $\bar{x}\in X$ be given where $F(\bar{x})$ is finite. A pair $(x^*,c)\in X^*\times R_+$ is called the weak subgardient of F at $\bar{x}\in X$ if

$$F(x)-F(\bar{x})\ge\langle x^*,x-\bar{x}\rangle-c\|x-\bar{x}\|, \text{ for all } x\in X \quad (4)$$

The set

$$\partial^W F(\bar{x})=\{(x^*,c)\in X^*\times R_+ : F(x)-F(\bar{x})\ge\langle x^*,x-\bar{x}\rangle-c\|x-\bar{x}\|, \forall x\in X\} \quad (5)$$

of all weak subgradients of F at $\bar{x}\in X$ is called the weak subdiffential of F at $\bar{x}\in X$.

If $\partial^W F(\bar{x})\neq\Phi$, then F is called weakly subdifferentiable at $\bar{x}$ [2,8].

Example 1. Let a function $F:R\to R$ is defined as $F(x)=-|x|$. Then, it follows from definition of the weak subdifferential that

$$(a,c)\in\partial^w F(0)\Leftrightarrow(a,c)\in R\times R_+ \quad and \quad -x\ge -c|x|, \quad for\, all \quad x\in R.$$

Hence, the weak subdifferential can explicity be written as

$$\partial^w F(0)=\{(a,c)\in R\times R_+ : |a\le c-1|\}.$$

Remark 1. It is obivious that, when F is subdifferentiable at $\bar{x}$, that is, if $x^*\in\partial F(\bar{x})$ the by definition $(x^*,c)\in\partial^W F(\bar{x})$ for every $c\ge 0$. It follows from Definition 5 that the pair $(x^*,c)\in X^*\times R_+$ is a weak subgradient of F at $\bar{x}\in X$ if there is a continuous (superlinear) concave function

$$g(x)=\langle x^*,x-\bar{x}\rangle+F(\bar{x})-c\|x-\bar{x}\|, \quad (6)$$

such that $g(x)\le F(x)$, for all $x\in X$ and $g(\bar{x})=F(\bar{x})$. The set $hypo(g)=\{(x,\alpha)\in X\times R: g(x)\ge\alpha\}$ is closed convex cone in $X\times R$ with vertex $(\bar{x},F(\bar{x}))$. Indeed,

$$\begin{aligned}&hypo(g)-(\bar{x},F(\bar{x}))\\&=\{(x-\bar{x},\alpha-F(\bar{x}))\in X\times R:\langle x^*,x-\bar{x}\rangle-c\|x-\bar{x}\|\ge\alpha-F(\bar{x})\}\\&=\{(u,\beta)\in X\times R:\langle x^*,u\rangle-c\|u\|\ge\beta\}\end{aligned}$$

Thus, it follows from (4) and (6) that

$$graph(g)=\{(x,\alpha)\in X\times R: g(x)=\alpha\}$$

is conic surface which is a supporting cone to

$$epi(F)=\{(x,\alpha)\in X\times R: F(x)\le\alpha\}$$

at the point $(\bar{x},F(\bar{x}))$ in the sense that

$epi(F)\subset epi(g)$, and $cl(epi(F)\cap graph(g))\neq\Phi$ [8].

Remark 2. It follows from this remark and Definition 5 that the class of weakly subdifferentiable functions are essentially larger than the class of subdifferentiable functions. Azimov and Gasimov [3] showed that certain subclasses of lower (locally) Lipschitz functions are weakly subdifferentiable.

Now we present the definition of lower Lipshitz functions [3,8].

Definition 6. A function lower lipschitz $F:X\to(-\infty,+\infty]$ is called lower Lipschitz at $\bar{x}\in X$, if there exists a nonnegative number L (Lipschitz constant), and a neighborhood $N(\bar{x})$ of $\bar{x}$ such that

$$F(x)-F(\bar{x})\ge -L\|x-\bar{x}\| \text{ for all } x\in N(\bar{x}) \quad (7)$$

If above inequality holds true for all $x\in X$ then F is called lower Lipschitz at $\bar{x}\in X$ with Lipschitz constant L.

The following two theorems are proved in [3].

Theorem 1. For any function $F:X\to(-\infty,+\infty]$ and any point $\bar{x}$ where $F(\bar{x})$ is finite, the following properties are equivalent to each other:

a) F is weakly subdifferentiable at $\bar{x}$,

b) F is lower Lipschitz at $\bar{x}$, and

c) F is lower locally Lipschitz at $\bar{x}$, and there exist the numbers $p \geq 0$ and q

$F(x) \geq -p\|x\| + q$, for all $x \in X$.

Theorem 2. Suppose the function $F : X \to (-\infty, +\infty]$ is lower locally Lipschitz at $\bar{x} \in X$. Then F is weakly subdifferentiable at $\bar{x}$, if either one of the following statement holds:

a) F is bounded from below.

b) There is a point $u \in X$ where F subdifferentiable.

The following theorem describes an important property of the weak subdiffrential.

Theorem 3. Let the weak subdifferential $\partial^w F(\bar{x})$ of the function $F : X \to R$ is nonempty. Then the set $\partial^w F(\bar{x})$ is closed and convex.

Proof. Let $\{(x_n, c_n)\} \subset \partial^w F(\bar{x})$ and let $(x_n, c_n) \to (x^*, c)$. To prove the theorem suppose to the contrary that $(x^*, c) \notin \partial^w F(\bar{x})$. Then

$$F(x) - F(\bar{x}) \leq -c\|x - \bar{x}\| + \langle x - \bar{x}, x^* \rangle, \text{ for some } x \in X \tag{8}$$

and by the inclusion $\{(x_n, c_n)\} \subset \partial^w F(\bar{x})$,

$$F(x) - F(\bar{x}) \geq \langle x^*{}_n, x - \bar{x} \rangle - c_n\|x - \bar{x}\|, \text{ for all } x \in X. \tag{9}$$

In this inequality (9) by letting to the limit as $n \to \infty$, we obtain

$$F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\|, \text{ for all } x \in X. \tag{10}$$

But this contradics with inequality (8). Hence $\partial^w F(\bar{x})$ is closed. The convexity of $\partial^w F(\bar{x})$ is obivious.

Proposition 1. Let a function $F : X \to R$ be a weakly subdifferentiable at $\bar{x} \in X$. $\partial^w F(\alpha \bar{x}) = \alpha \partial^w F(\bar{x})$, for $\alpha > 0$.

Proof. Let be $(x^*, c) \in \partial^w (\alpha F)(\bar{x})$. Then

$$(\alpha F)(x) - (\alpha F)(\bar{x}) \geq -c\|x - \bar{x}\| + \langle x - \bar{x}, x^* \rangle$$

$$\alpha F(x) - \alpha F(\bar{x}) \geq -c\|x - \bar{x}\| + \langle x - \bar{x}, x^* \rangle$$

$$F(x) - F(\bar{x}) \geq -\frac{c}{\alpha}\|x - \bar{x}\| + \left\langle x - \bar{x}, \frac{x^*}{\alpha} \right\rangle$$

$$\Rightarrow \left(\frac{x^*}{\alpha}, \frac{c}{\alpha}\right) \in \partial^w F(\bar{x})$$

$$\Rightarrow (x^*, c) \in \partial^w F(\bar{x}).$$

Hence, we obtain

$$\partial^w F(\alpha \bar{x}) \subset \alpha \partial^w F(\bar{x}) \tag{11}$$

Now we prove that the converse content. Let $(x^*, c) \in \alpha \partial^w F(\bar{x})$. Then we have $\frac{1}{\alpha}(x^*, c) \in \partial^w F(\bar{x})$. Thus

$$F(x) - F(\bar{x}) \geq -\frac{c}{\alpha}\|x - \bar{x}\| + \left\langle x - \bar{x}, \frac{x^*}{\alpha} \right\rangle$$

$$\alpha F(x) - \alpha F(\bar{x}) \geq -c\|x - \bar{x}\| + \langle x - \bar{x}, x^* \rangle$$

$$(\alpha F)(x) - (\alpha F)(\bar{x}) \geq -c\|x - \bar{x}\| + \langle x - \bar{x}, x^* \rangle$$

$$\Rightarrow (x^*, c) \in \partial^w (\alpha F)(\bar{x})$$

$$\Rightarrow \alpha \partial^w F(\bar{x}) \subset \partial^w (\alpha F)(\bar{x}) \tag{12}$$

From (11) and (12) we obtain that $\partial^w F(\alpha \bar{x}) = \alpha \partial^w F(\bar{x})$, for $\alpha > 0$.

Proposition 2. Let $F, G : X \to R$ and $F + G : X \to R$ single valued functions being weakly subdifferentiable at $\bar{x} \in X$. Then $\partial^w F(\bar{x}) + \partial^w G(\bar{x}) \subset \partial^w (F + G)(\bar{x})$.

Proof. Let $\partial^w F(\bar{x}) \neq \Phi$ and let $\partial^w G(\bar{x}) \neq \Phi$. Take arbitrary $(x^*{}_1, c_1) \in \partial^w F(\bar{x})$, $(x^*{}_2, c_2) \in \partial^w G(\bar{x})$. Since $(x^*{}_1, c_1) \in \partial^w F(\bar{x})$, $(x^*{}_2, c_2) \in \partial^w G(\bar{x})$, we have by definition of the weak subgradient

$$F(x) - F(\bar{x}) \geq -c_1\|x - \bar{x}\| + \langle x - \bar{x}, x^*{}_1 \rangle, \quad \forall x \in X \tag{13}$$

and

$$G(x) - G(\bar{x}) \geq -c_2\|x - \bar{x}\| + \langle x - \bar{x}, x^*{}_2 \rangle, \quad \forall x \in X \tag{14}$$

By collecting side by side inequalities (13) and (14), we obtain

$$(F(x) + G(x)) - (F(\bar{x}) + G(\bar{x})) \geq -(c_1 + c_2)\|x - \bar{x}\| + \langle x - \bar{x}, x^*{}_1 \rangle$$
$, \forall x \in X$.

Thus $\left(x_1^* + x_2^*, c_1 + c_2\right) \in \partial^w (F+G)(\bar{x})$, and then we obtain $\partial^w F(\bar{x}) + \partial^w G(\bar{x}) \subset \partial^w (F+G)(\bar{x})$.

Proposition 3. Let a function $F : X \to (-\infty, +\infty]$ be weakly subdifferentiable at $\bar{x} \in X$ and having a global minimum at $\bar{x} \in X$. Then $(0,0) \in \partial^w F(\bar{x})$.

Proof. If F have global minimum at $\bar{x} \in X$, we have

$$F(x) \geq F(\bar{x}), \quad \forall x \in X$$

$$F(x) \geq F(\bar{x}) - 0\|x - \bar{x}\| + \langle x - \bar{x}, 0 \rangle, \quad \forall x \in X$$

$$\Rightarrow (0,0) \in \partial^w F(\bar{x}).$$

Proposition 4. Let $F, G : X \to (-\infty, +\infty]$ functions be given. Let $F + G : X \to (-\infty, +\infty]$ functions being weakly subdifferentiable at $\bar{x} \in X$ and $(-G)$ function being both subdifferentiable and derivable at $\bar{x} \in X$. Then

$$(x^*, c) \in \partial^w (F+G)(\bar{x}) \Rightarrow (x^* - G'(\bar{x}), c) \in \partial^w F(\bar{x}).$$

Proof. Let $(x^*, c) \in \partial^w (F+G)(\bar{x})$. Then, by the definition of the weak subgradient we have

$$(F+G)(x) - (F+G)(\bar{x}) \geq -c\|x - \bar{x}\| + \langle x - \bar{x}, x^* \rangle, \ \forall x \in X$$

$$F(x) + G(x) - F(\bar{x}) - G(\bar{x}) \geq -c\|x - \bar{x}\| + \langle x - \bar{x}, x^* \rangle, \ \forall x \in X. \tag{15}$$

Since G function is subdifferentiable at $\bar{x} \in X$, we have

$$-G(x) \geq -G(\bar{x}) + \langle x - \bar{x}, -G'(\bar{x}) \rangle, \ \forall x \in X$$

and then

$$G(x) \leq G(\bar{x}) + \langle x - \bar{x}, G'(\bar{x}) \rangle, \ \forall x \in X \tag{16}$$

From inequalities (15) and (16), we have

$$-c\|x - \bar{x}\| + \langle x - \bar{x}, x^* \rangle \leq F(x) + G(x) - F(\bar{x}) - G(\bar{x})$$
$$-c\|x - \bar{x}\| + \langle x - \bar{x}, x^* \rangle \leq F(x) + G(\bar{x}) + \langle x - \bar{x}, G'(\bar{x}) \rangle - F(\bar{x}) - G(\bar{x})$$
$$-c\|x - \bar{x}\| + \langle x - \bar{x}, x^* \rangle \leq F(x) - F(\bar{x}) + \langle x - \bar{x}, G'(\bar{x}) \rangle$$
$$-c\|x - \bar{x}\| + \langle x - \bar{x}, x^* - G'(\bar{x}) \rangle \leq F(x) - F(\bar{x}),$$

and then

$$(x^* - G'(\bar{x}), c) \in \partial^w F(\bar{x}).$$

3. CONCLUSION

In this paper, the important properties of the weak subdifferentials are presented. Convexity and closedness of the weak subdiffrential set are proved.

REFERENCES

[1] Aubin, J-P., Ekeland, I., "Applied Nonlinear Analysis", ***Wiley-İnterscience***, New York, USA, 159-200 (1984).

[2] Azimov, A. Y., Gasimov R. N., "On weak conjugacy, weak subdifferentials and duality with zero gap in nonconvex optimization", ***International Journal of Applied Mathematics***, 1:171-192 (1999).

[3] Azimov, A. Y., Gasimov R. N., "Stability and duality of nonconvex problems via augmented lagrangian", ***Cybenetics and Systems Analysis***, 38:412-421 (2002).

[4] Clarke F.H., "A new approach to lagrange multipliers", ***Mathematics Operations Research***, 1:165-174 (1976).

[5] Clarke F.H., "Optimization and Nonsmooth Analysis", ***Wiley-İnterscience***, New York, USA, 9-30 (1983).

[6] Ekeland I., Temam R., "Convex Analysis and Variational Problems", ***Elsevier Norht-Holland***, Amsterdam, 3-34 (1983).

[7] Rocafellar R. T., "Convex Analysis", ***Princeton University*****,** New Jersey, USA, 215-227 (1970).

[8] Kasımbeyli R. N., "Radial Epiderivatives and Set Valued Optimization", ***Optimization***, 58(5):521-534 (2009).

[9] Rocafellar R. T., "Proximal subgradients, marginal values and augmented lagrangian in nonconvex optimization**"**, ***Math. Oper. Res***., 6:427-437, (1981).

[10] Rocafellar R. T., "Lagrange multipliers and optimality", ***SIAM Review***, 35:183-238 (1993).