G.U. Journal of Science
23(1): 53-59 (2010)

GUJS
www.gujs.org

On the Dynamics of the Recursive Sequence

Saime ZENGİN [1♠], İlhan ÖZTÜRK [2], Fatma BOZKURT [3]

[1]*Erciyes University, Faculty of Science and Arts, Department of Mathematics, 38039 Kayseri, Turkey*
[2]*Erciyes University, Kayseri Vacational School, Department of Mathematics, 38039 Kayseri, Turkey*
[1]*Erciyes University, Faculty of Education, Department of Mathematics, 38039 Kayseri, Turkey*

Received: 30.04.2009 Revised: 08.09.2009 Accepted: 23.09.2009

ABSTRACT

Our aim in this paper is to investigate the local stability of the positive solutions of the difference equation

$$y_{n+1}=\frac{\alpha-y_n}{\beta y_{n-1}}-\frac{\gamma-y_{n-1}}{\beta y_n},\quad n=0,1,2,...,$$

where the initial conditions y_{-1}, y_0 are arbitrary positive real numbers such that $y_n\neq 0$ for $n=-1,0,1,...,$ $\alpha,\beta,\gamma\in(0,\infty)$ and $\alpha>\gamma$. Furthermore we investigate the periodic nature of the mentioned difference equation.

Key Words: Difference Equations, Local Stability, Period-two Solutions.

1. INTRODUCTION

Consider the difference equation

$$y_{n+1}=\frac{\alpha-y_n}{\beta y_{n-1}}-\frac{\gamma-y_{n-1}}{\beta y_n},\quad n=0,1,2,...,\qquad(1)$$

where the initial conditions y_{-1}, y_0 are arbitrary positive real numbers such that $y_n\neq 0$ for $n=-1,0,1,...,$ $\alpha,\beta,\gamma\in(0,\infty)$ and $\alpha>\gamma$. Our aim in this paper is to investigate the local asymptotic stability of the positive equilibrium point of Eq. (1), furthermore the periodic nature of the solutions of Eq. (1) under specified conditions of the parameters α, β and γ.

Other nonlinear, rational difference equations were investigated in [1]-[6]. In [1], the global stability, the boundedness character and the periodic nature of the positive solutions of the recursive sequence

$$x_{n+1}=\alpha+\frac{x_{n-1}}{x_n},\quad n=0,1,2,...,\qquad(2)$$

was investigated by Amleh et al.[1], where $\alpha\in[0,\infty)$ and the initial conditions x_{-1} and x_0 are arbitrary positive real numbers. They showed that a necessary and sufficient condition which every positive solution of Eq. (2) be bounded is $\alpha\geq 1$. H. M. El-Owaidy et al. [2] studied the recursive sequence

$$x_{n+1}=\alpha+\frac{x_{n-k}}{x_n},\quad n=0,1,2,...,\qquad(3)$$

where $\alpha\in[1,\infty)$, $k\in\{1,2,...\}$ and the initial conditions $x_{-k},...,x_0$ are arbitrary positive real numbers.

R. DeVault et al. [3] investigated the global stability and the periodic character of solutions of the equation $y_{n+1}=(p+y_{n-k})/(qy_n+y_{n-k})$ where p and q are positive, $k\in\{1,2,...\}$ where the initial conditions $x_{-k},...,x_0$ are arbitrary positive numbers. W. S. He and W. T. Li [4] studied the global stability and the periodic character of the positive solutions of the difference equation

$$x_{n+1}=\frac{a-bx_{n-k}}{A+x_n},\quad n=0,1,2,...,\qquad(4)$$

where $a\geq 0$, $b,A>0$, $k\in\{1,2,...\}$ and initial conditions $x_{-k},...,x_0$ are arbitrary real numbers. They

Corresponding author: ozturki@erciyes.edu.tr

showed that the positive equilibrium of the equation is a global attractor with a basin that depends on certain conditions posed on the coefficients. Also, Yan et al. [5] investigated the global asymptotic stability, the global attractivity, the boundedness character, the periodic nature and chaotic behavior of solutions of the difference equation $x_{n+1} = \alpha - \frac{x_{n-1}}{x_n}$, $n = 0,1,2,...,$ where α is a real number and the initial conditions x_{-1}, x_0 are arbitrary positive real numbers.

We first recall some results which will be useful in the sequel.

Let $I \subset R$ and let $f : I \times I \to I$ be a continuous function. Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-1}), \quad n = 0,1,2,..., \tag{5}$$

where the initial conditions $y_{-1}, y_0 \in I$. We say that $\bar{y}$ is an equilibrium of Eq. (5) if

$$\bar{y} = f(\bar{y}, \bar{y}).$$

Let

$$s = \frac{\partial f}{\partial u}(\bar{y}, \bar{y}) \quad \text{and} \quad t = \frac{\partial f}{\partial v}(\bar{y}, \bar{y})$$

denote the partial derivatives of $f(u,v)$ evaluated at an equilibrium $\bar{y}$ of Eq. (5). Then the equation

$$x_{n+1} = s x_n + t x_{n-1}, \quad n = 0,1,2,...$$

is called the linearized equation associated with Eq. (5) about the equilibrium point $\bar{y}$ [6].

2. LOCAL ASYMPTOTIC STABILITY AND PERIOD-TWO SOLUTIONS

In this section, we discuss the local stability of the positive equilibrium point $\bar{y}$ and period-two solutions of Eq. (1). The equilibrium points of Eq. (1) are solutions of the equation

$$\beta \bar{y}^2 - (\alpha - \gamma) = 0. \tag{6}$$

So the equilibrium points of Eq. (1) are

$$\bar{y} = \pm\sqrt{\frac{\alpha - \gamma}{\beta}}, \tag{7}$$

where $\alpha > \gamma$. The linearized equation associated with Eq. (1) about the equilibrium point $\bar{y}$ is

$$x_{n+1} - \frac{\gamma - 2\bar{y}}{\alpha - \gamma} x_n + \frac{\alpha - 2\bar{y}}{\alpha - \gamma} x_{n-1} = 0, \quad n = 0,1,2,... \tag{8}$$

Hence, its characteristic equation is

$$\lambda^2 - \frac{\gamma - 2\bar{y}}{\alpha - \gamma}\lambda + \frac{\alpha - 2\bar{y}}{\alpha - \gamma} = 0. \tag{9}$$

Theorem 2.1. Suppose that $\alpha, \beta, \gamma \in (0, \infty)$ and $\alpha > \gamma$. Then the following statements are true.

(i) The positive equilibrium point $\bar{y}$ of Eq. (1) is locally asymptotically stable if

$$\frac{4(\alpha - \gamma)}{\alpha^2} < \beta < \frac{4(\alpha - \gamma)}{\gamma^2}. \tag{10}$$

(ii) The positive equilibrium point $\bar{y}$ of Eq. (1) is unstable (furthermore is a repeller point) if

$$\frac{4(\alpha - \gamma)}{\alpha^2} < \frac{4(\alpha - \gamma)}{\gamma^2} < \beta. \tag{11}$$

Proof. (i) Since

$$\beta < \frac{4(\alpha - \gamma)}{\gamma^2}, \tag{12}$$

then

$$\gamma^2 < \frac{4(\alpha - \gamma)}{\beta} \tag{13}$$

and

$$\gamma < 2\sqrt{\frac{\alpha - \gamma}{\beta}}. \tag{14}$$

Using positive equilibrium point of Eq. (1) in (7) and (14), we have

$$\gamma - 2\bar{y} < 0$$

and

$$2\alpha - \gamma - 2\bar{y} < 2\alpha - 2\gamma. \tag{15}$$

Hence from (15), we get

$$1 + \frac{\alpha - 2\bar{y}}{\alpha - \gamma} < 2. \tag{16}$$

On the other hand, using positive equilibrium point of Eq. (1) in (7) and

$$\frac{4(\alpha - \gamma)}{\alpha^2} < \beta,$$

we have

$$2\bar{y} < \alpha$$

and

$$-\frac{\gamma - 2\bar{y}}{\alpha - \gamma} < 1 + \frac{\alpha - 2\bar{y}}{\alpha - \gamma}. \tag{17}$$

Since

$$\gamma - 2\bar{y} < 0$$

we can write

$$\left|-\frac{\gamma-2\bar{y}}{\alpha-\gamma}\right|<1+\frac{\alpha-2\bar{y}}{\alpha-\gamma}\cdot \tag{18}$$

Then, one can see from (16) and (18) that, the positive equilibrium point $\bar{y}$ of Eq. (1) is locally asymptotically stable. This completes the proof of the first part in the theorem.

(ii) Since

$$\frac{4(\alpha-\gamma)}{\gamma^2}<\beta, \tag{19}$$

then

$$\frac{4(\alpha-\gamma)}{\beta}<\gamma^2 \tag{20}$$

and

$$\gamma>2\sqrt{\frac{\alpha-\gamma}{\beta}}\cdot \tag{21}$$

Using positive equilibrium point of Eq. (1) in (7) and (21), we have

$$\gamma-2\bar{y}>0$$

and

$$\alpha-2\bar{y}>\alpha-\gamma. \tag{22}$$

Thus we get

$$\left|-\frac{\alpha-2\bar{y}}{\alpha-\gamma}\right|>1\cdot \tag{23}$$

On the other hand, using positive equilibrium point of Eq. (1) in (7) and

$$\gamma<\alpha$$

we have

$$\gamma-2\bar{y}<\alpha-\gamma+\alpha-2\bar{y}$$

and

$$\frac{\gamma-2\bar{y}}{\alpha-\gamma}<1+\frac{\alpha-2\bar{y}}{\alpha-\gamma}\cdot \tag{24}$$

Since

$$\gamma-2\bar{y}>0,$$

we can write

$$\left|\frac{\gamma-2\bar{y}}{\alpha-\gamma}\right|<\left|1+\frac{\alpha-2\bar{y}}{\alpha-\gamma}\right|\cdot \tag{25}$$

Then, one can see from (23) and (25) that, the positive equilibrium point $\bar{y}$ of Eq. (1) is unstable (furthermore a repeller point). This completes the proof.

Theorem 2.2 Suppose that $\alpha,\beta,\gamma\in(0,\infty)$, $\alpha>\gamma$ and

$$\beta=\frac{4(\alpha-\gamma)}{\alpha^2}<\frac{4(\alpha-\gamma)}{\gamma^2}\cdot \tag{26}$$

Then the positive equilibrium point $\bar{y}$ of Eq. (1) is non-hyperbolic point.

Proof. Since

$$\beta=\frac{4(\alpha-\gamma)}{\alpha^2}, \tag{27}$$

then

$$\alpha=2\sqrt{\frac{\alpha-\gamma}{\beta}}\cdot \tag{28}$$

Using positive equilibrium point of Eq. (1) in (7) and (28), we have

$$\alpha-2\bar{y}=0$$

and

$$\gamma-2\bar{y}=\gamma-\alpha\cdot \tag{29}$$

Hence from (29), we can write

$$|\gamma-2\bar{y}|=|\gamma-\alpha|=|\alpha-\gamma|\cdot \tag{30}$$

On the other hand, since

$$\alpha-2\bar{y}=0,$$

we have

$$|\gamma-2\bar{y}|=|\alpha-\gamma+\alpha-2\bar{y}| \tag{31}$$

and

$$\left|\frac{\gamma-2\bar{y}}{\alpha-\gamma}\right|=\left|1+\frac{\alpha-2\bar{y}}{\alpha-\gamma}\right|\cdot \tag{32}$$

Thus, we get $|s|=|1-t|$ where $s=\frac{\gamma-2\bar{y}}{\alpha-\gamma}$ and $t=\frac{\alpha-2\bar{y}}{\alpha-\gamma}\cdot$ Then the positive equilibrium point $\bar{y}$ of Eq. (1) is non-hyperbolic point.

Theorem 2.3 Suppose that $\alpha,\beta,\gamma\in(0,\infty)$, $\alpha>\gamma$ and

$$\beta=\frac{4(\alpha-\gamma)}{\gamma^2}>\frac{4(\alpha-\gamma)}{\alpha^2}\cdot \tag{33}$$

Then the positive equilibrium point $\bar{y}$ of Eq. (1) is non-hyperbolic point.

Proof. Since

$$\beta=\frac{4(\alpha-\gamma)}{\gamma^2}, \tag{34}$$

then

$$\gamma = 2\sqrt{\frac{(\alpha-\gamma)}{\beta}}\,. \tag{35}$$

Using positive equilibrium point of Eq. (1) in (7) and (35), we have

$$\gamma - 2\overline{y} = 0$$

and

$$\alpha - 2\overline{y} = \alpha - \gamma\,. \tag{36}$$

Hence from (36), we can write

$$\frac{\alpha - 2\overline{y}}{\alpha-\gamma} = 1\,. \tag{37}$$

On the other hand, we have

$$t = -\frac{\alpha - 2\overline{y}}{\alpha-\gamma} = -1\,.$$

Furthermore, since $\gamma - 2\overline{y} = 0$ we get

$$|s| = \left|\frac{\gamma - 2\overline{y}}{\alpha-\gamma}\right| = 0 < 2\,. \tag{38}$$

Thus, the positive equilibrium point $\overline{y}$ of Eq. (1) is non-hyperbolic point under the condition (33).

Example 2.1. Let $\alpha = 5$, $\beta = 4$, $\gamma = 1$, $y_{-1} = 1$ and $y_0 = 0.5$. The values of α, β and γ verify (10). The graph of the first 100 iterations of Eq. (1) is given in Figure 1. The graph suggests that the solution of Eq. (1) is converging to a stable equilibrium value of about $\overline{y} = 1$. If this converge continues in the form of a limit as $n \to \infty$, future values of y_n can be predicted to be at or extremely near the equilibrium value (furthermore see Table 1).

Example 2.2. Let $\alpha = 20$, $\beta = 2.8$, $\gamma = 5$, $y_{-1} = 0.2$ and $y_0 = 0.5$. Note that the values of α, β and γ verify (11). The graph of the first 50 iterations of Eq. (1) is given in Figure 2. It does not contain any of the predictable patterns of solutions. Absent of any pattern or repetition, the general long-term behavior and specific values of y_n for large n are impossible to predict from the graph (furthermore see Figure 2 and Table 2).

Theorem 2.4. Suppose that $\{y_n\}_{n=-1}^{\infty}$ is a solution of Eq. (1) and $\alpha, \beta, \gamma \in (0, \infty)$. Then the following statements are true.

(i) If $y_{-1} = \sqrt{\frac{\alpha+\gamma}{\beta}}$ and $y_0 = -\sqrt{\frac{\alpha+\gamma}{\beta}}$ (or $y_{-1} = -\sqrt{\frac{\alpha+\gamma}{\beta}}$ and $y_0 = \sqrt{\frac{\alpha+\gamma}{\beta}}$), then Eq. (1) has period-two solutions.

(ii) Assume that $A = \alpha^2 - \frac{4(\alpha-\gamma)}{\beta} > 0$. If $y_{-1} = \frac{\alpha+\sqrt{A}}{2}$ and $y_0 = \frac{\alpha-\sqrt{A}}{2}$ (or $y_{-1} = \frac{\alpha-\sqrt{A}}{2}$ and $y_0 = \frac{\alpha+\sqrt{A}}{2}$) , then Eq. (1) has period-two solutions.

Proof. Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

be a period-two solutions of Eq. (1). Then,

$$\phi = \frac{\alpha-\psi}{\beta\phi} - \frac{\gamma-\phi}{\beta\psi} = \frac{\alpha\psi - \psi^2 - \gamma\phi + \phi^2}{\beta\phi\psi}\,, \tag{39}$$

$$\psi = \frac{\alpha-\phi}{\beta\psi} - \frac{\gamma-\psi}{\beta\phi} = \frac{\alpha\phi - \phi^2 - \gamma\psi + \psi^2}{\beta\phi\psi}\,. \tag{40}$$

Subtracting (40) from (39), we have

$$(\phi-\psi)[\beta\phi\psi + \alpha + \gamma - 2(\phi+\psi)] = 0\,. \tag{41}$$

Since ϕ and ψ period-two solutions of Eq.(1), then $\psi \neq \phi$. Hence from (41), we have

$$\beta\phi\psi + \alpha + \gamma - 2(\phi+\psi) = 0\,. \tag{42}$$

Thus period-two solutions of Eq.(1) are also the solutions of (42). From (39) and (40), we get

$$\frac{\phi}{\psi} = \frac{\alpha\psi - \psi^2 - \gamma\phi + \phi^2}{\alpha\phi - \phi^2 - \gamma\psi + \psi^2}$$

and

$$(\phi-\psi)(\phi+\psi)(\alpha-\phi-\psi) = 0\,. \tag{43}$$

Firstly, assume that $\phi + \psi = 0$ in (43). From this, we get $\psi = -\phi$. Using $\psi = -\phi$ in (42), we obtain

$$\phi_{1,2} = \pm\sqrt{\frac{\alpha+\gamma}{\beta}}\,. \tag{44}$$

So the period-two solutions must be of the form

$$\ldots, -\sqrt{\frac{\alpha+\gamma}{\beta}}, \sqrt{\frac{\alpha+\gamma}{\beta}}, \ldots \tag{45}$$

Table 1. Values of the iteration solutions of Eq. (1) for $\alpha=5$, $\beta=4$, $\gamma=1$, $y_{-1}=1$ and $y_0=0.5$.

Terms	Values	Terms	Values	Terms	Values	Terms	Values	Terms	Values
y_1	1.1250	y_{11}	0.9156	y_{21}	0.9868	y_{31}	1.0059	y_{41}	1.0004
y_2	1.8264	y_{12}	1.0708	y_{22}	0.9821	y_{32}	0.9972	y_{42}	1.0012
y_3	0.7224	y_{13}	1.0532	y_{23}	1.0145	y_{33}	0.9963	y_{43}	0.9994
y_4	0.8715	y_{14}	0.9383	y_{24}	1.0101	y_{34}	1.0030	y_{44}	0.9993
y_5	1.3492	y_{15}	0.9783	y_{25}	0.9868	y_{35}	1.0020	y_{45}	1.0006
y_6	1.0234	y_{16}	1.0558	y_{26}	0.9958	y_{36}	0.9972	y_{46}	1.0004
y_7	0.8221	y_{17}	1.0028	y_{27}	1.0111	y_{37}	0.9992	y_{47}	0.9994
y_8	1.0277	y_{18}	0.9604	y_{28}	1.0004	y_{38}	1.0023	y_{48}	0.9998
y_9	1.1646	y_{19}	1.0078	y_{29}	0.9917	y_{39}	1.0000	y_{49}	1.0005
y_{10}	0.9390	y_{20}	1.0294	y_{30}	1.0018	y_{40}	0.9983	y_{50}	1.0000

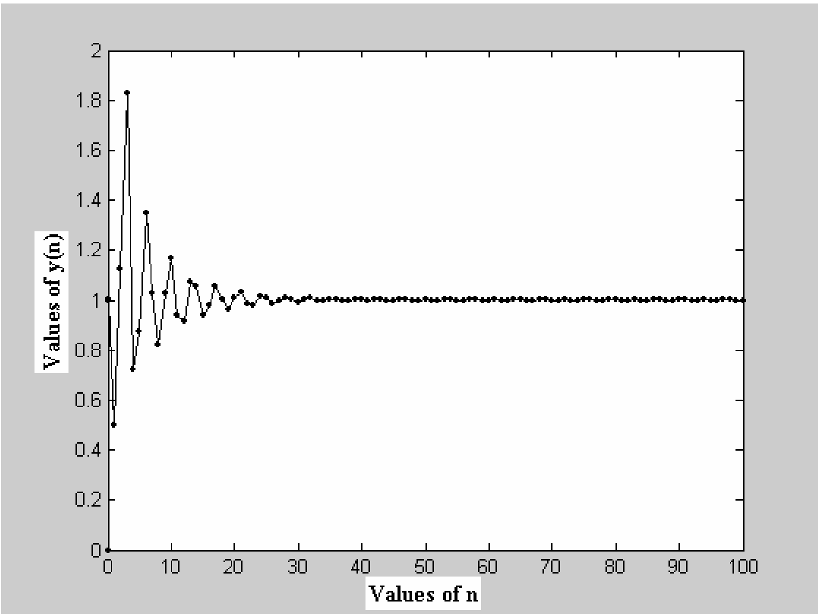

Figure 1. Graph of the iteration solutions of Eq. (1) for $\alpha=5$, $\beta=4$, $\gamma=1$, $y_{-1}=1$ and $y_0=0.5$.

Table 2. Values of the iteration solutions of Eq. (1) for $\alpha=20$, $\beta=2.8$, $\gamma=5$, $y_{-1}=0.2$ and $y_0=0.5$.

Terms	Values	Terms	Values	Terms	Values	Terms	Values	Terms	Values
y_1	31.3929	y_{11}	-3.9822	y_{21}	-1.3682	y_{31}	-0.5262	y_{41}	-13.1619
y_2	-8.1889	y_{12}	4.7432	y_{22}	0.5611	y_{32}	1.8422	y_{42}	-2.5499
y_3	-0.8304	y_{13}	-2.0446	y_{23}	-9.1278	y_{33}	- 13.3959	y_{43}	1.9319
y_4	4.7641	y_{14}	1.7047	y_{24}	18.7150	y_{34}	6.5586	y_{44}	-3.9264
y_5	-6.9900	y_{15}	-4.6716	y_{25}	-0.3199	y_{35}	-1.3601	y_{45}	4.7023
y_6	2.0354	y_{16}	5.4207	y_{26}	-14.9249	y_{36}	0.7539	y_{46}	-2.0694
y_7	-3.0217	y_{17}	-1.7518	y_{27}	-38.8659	y_{37}	- 8.0670	y_{47}	1.7276
y_8	4.3900	y_{18}	1.3474	y_{28}	-1.2255	y_{38}	13.4849	y_{48}	-4.6149
y_9	-2.4976	y_{19}	-5.5924	y_{29}	12.5883	y_{39}	-0.6345	y_{49}	5.3420
y_{10}	1.9175	y_{20}	7.0171	y_{30}	-2.3365	y_{40}	-4.2293	y_{50}	-1.7772

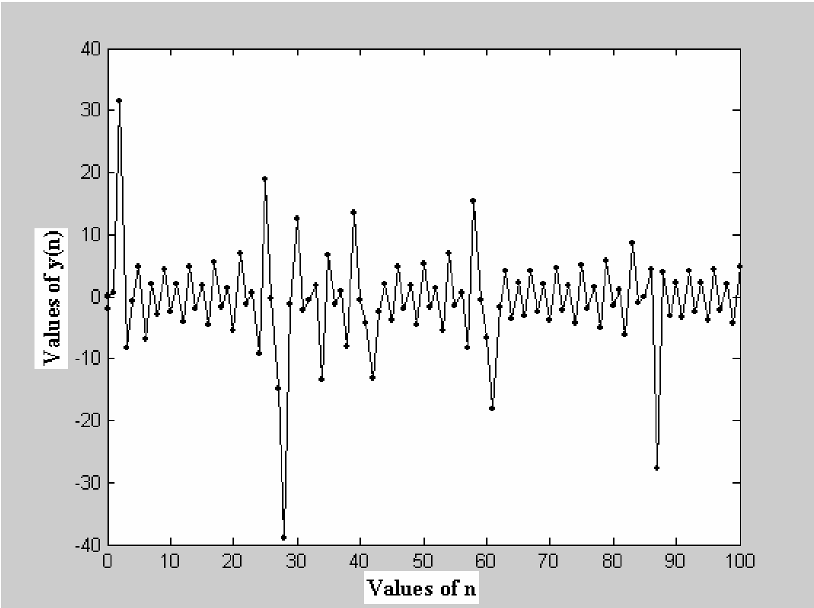

Figure 2. Graph of the iteration solutions of Eq. (1) for $\alpha = 20$, $\beta = 2.8$, $\gamma = 5$, $y_{-1} = 0.2$ and $y_0 = 0.5$.

Table 3. Values of the iteration solutions of Eq. (1) for $\alpha = 6$, $\beta = 0.2$, $\gamma = 5$, $y_{-1} = 5$, $y_0 = 1$.

Terms	Values	Terms	Values	Terms	Values	Terms	Values
y_1	5	y_6	1	y_{11}	5	y_{16}	1
y_2	1	y_7	5	y_{12}	1	y_{17}	5
y_3	5	y_8	1	y_{13}	5	y_{18}	1
y_4	1	y_9	5	y_{14}	1	y_{19}	5
y_5	5	y_{10}	1	y_{15}	5	y_{20}	1

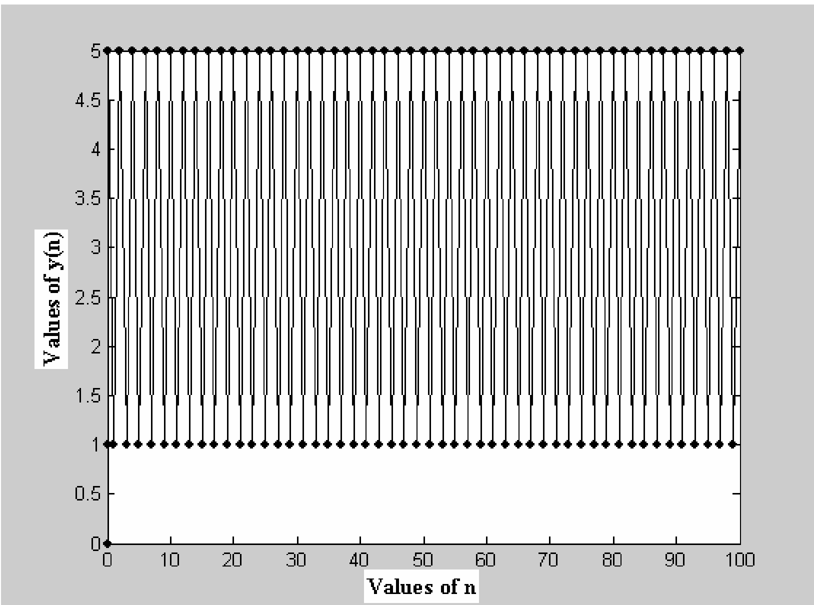

Figure 3. Graph of the period-two solutions of Eq. (1) for $\alpha = 6$, $\beta = 0.2$, $\gamma = 5$, $y_{-1} = 5$, $y_0 = 1$

Secondly, assume that $(\alpha - \phi - \psi) = 0$. By using $\psi = \alpha - \phi$ in (42) we obtain

$$\phi_{3,4} = \frac{\alpha \pm \sqrt{A}}{2}. \qquad (46)$$

(i) Let $B = \sqrt{\frac{\alpha + \gamma}{\beta}}$, $y_{-1} = B$ and $y_0 = -B$. Then from Eq. (1), we have

$$y_1 = \frac{\alpha - y_0}{\beta y_{-1}} - \frac{\gamma - y_{-1}}{\beta y_0} = \frac{\alpha + B}{\beta B} + \frac{\gamma - B}{\beta B} = B,$$

$$y_2 = \frac{\alpha - y_1}{\beta y_0} - \frac{\gamma - y_0}{\beta y_1} = \frac{\alpha - B}{-\beta B} - \frac{\gamma + B}{\beta B} = -B,$$

$$y_3 = \frac{\alpha - y_2}{\beta y_1} - \frac{\gamma - y_1}{\beta y_2} = \frac{\alpha + B}{\beta B} + \frac{\gamma - B}{\beta B} = B,$$

$$y_4 = \frac{\alpha - y_3}{\beta y_2} - \frac{\gamma - y_2}{\beta y_3} = \frac{\alpha - B}{-\beta B} - \frac{\gamma + B}{\beta B} = -B, \ \ldots$$

The case $y_{-1} = -B$ and $y_0 = B$ is similar and will be omitted.

(ii) Using $y_{-1} = \frac{\alpha + \sqrt{A}}{2}$ and $y_0 = \frac{\alpha - \sqrt{A}}{2}$ in Eq. (1), we have

$$y_1 = \frac{\alpha - y_0}{\beta y_{-1}} - \frac{\gamma - y_{-1}}{\beta y_0} = \frac{\alpha + \sqrt{A}}{2},$$

$$y_2 = \frac{\alpha - y_1}{\beta y_0} - \frac{\gamma - y_0}{\beta y_1} = \frac{\alpha - \sqrt{A}}{2}$$

$$y_3 = \frac{\alpha - y_2}{\beta y_1} - \frac{\gamma - y_1}{\beta y_2} = \frac{\alpha + \sqrt{A}}{2},$$

$$y_4 = \frac{\alpha - y_3}{\beta y_2} - \frac{\gamma - y_2}{\beta y_3} = \frac{\alpha - \sqrt{A}}{2}, \ \ldots$$

The case $y_{-1} = \frac{\alpha - \sqrt{A}}{2}$ and $y_0 = \frac{\alpha + \sqrt{A}}{2}$ is similar and will be omitted. This completes the proof.

Example 2.3. Let $\alpha = 6$, $\beta = 0.2$, $\gamma = 5$. Using the initial conditions $y_{-1} = 5$ and $y_0 = 1$, we have the the graph of the first 100 iterations of Eq. (1) and this graph is shown in Figure 3. The graphs show that, the solution of Eq. (1) oscillates between two values of 5 and 1. It is easy to predict future values of y_n (furthermore see Figure 3 and Table 3).

REFERENCES

[1] Amleh, A.M., Grove, E.A., Ladas, G., "On the Recursive Sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$", ***Journal of Mathematical Analysis and Applications***, 233: 790-798 (1999).

[2] El-Owaidy, H.M., Ahmed, A.M., Mousa, M.S., "On asymptotic behavior of the difference equation $x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}$", ***Applied Mathematics and Computations***, 147: 163-167 (2004).

[3] DeVault, R., Kosmala, W., Ladas, G., Schultz, S.W., "Global Behavior of $y_{n+1} = \frac{p + y_{n-k}}{q y_n + y_{n-k}}$", ***Nonlinear Analysis***, 47: 4743-4751 (2001).

[4] He, W.S., Li, W.T., "Attractivity in a Nonlinear Delay Difference Equation", ***Applied Mathematics***, E-Notes, 4: 48-53 (2004).

[5] Yan, X.X., Li, W.T., "Dynamic behavior of a recursive sequence", ***Applied Mathematics and Computation***, 157: 713-727 (2004).

[6] Gibbons, C.H., Kulenović, M.R.S., Ladas, G., Voulov, H.D., "On the Trichotomy Character of $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + x_n)$", ***Journal of Difference Equations and Applications***, 8 (1): 75-92 (2002).