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CLASSICAL AND STRONGLY CLASSICAL 2-ABSORBING SECOND SUBMODULES

H. ANSARI-TOROGHY AND F. FARSHADIFAR

Abstract. In this paper, we will introduce the concept of classical (resp. strongly classical) 2-absorbing second submodules of modules over a commutative ring as a generalization of 2-absorbing (resp. strongly 2-absorbing) second submodules and investigate some basic properties of these classes of modules.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and "$\subset$" will denote the strict inclusion. Further, $\mathbb{Z}$ will denote the ring of integers.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [11]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [18]. In this case $Ann_R(S)$ is a prime ideal of R.

The notion of 2-absorbing ideals as a generalization of prime ideals was introduced and studied in [7]. A proper ideal I of R is a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. The authors in [10] and [15], extended 2-absorbing ideals to 2-absorbing submodules. A proper submodule N of M is called a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [12].

In [5], the present authors introduced the dual notion of 2-absorbing submodules (that is, *2-absorbing (resp. strongly 2-absorbing) second submodules*) of M and

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investigated some properties of these classes of modules. A non-zero submodule N of M is said to be a *2-absorbing second submodule of* M if whenever $a, b \in R$, L is a completely irreducible submodule of M, and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ or $ab \in Ann_R(N)$. A non-zero submodule N of M is said to be a *strongly 2-absorbing second submodule of* M if whenever $a, b \in R$, K is a submodule of M, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$.

In [14], the authors introduced the notion of classical 2-absorbing submodules as a generalization of 2-absorbing submodules and studied some properties of this class of modules. A proper submodule N of M is called *classical 2-absorbing submodule* if whenever $a, b, c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ or $bcm \in N$ [14].

The purpose of this paper is to introduce the concepts of classical and strongly classical 2-absorbing second submodules of an R-module M as dual notion of classical 2-absorbing submodules and provide some information concerning these new classes of modules. We characterize classical (resp. strongly classical) 2-absorbing second submodules in Theorem 2.3 (resp. Theorem 3.4). Also, we consider the relationship between classical 2-absorbing and strongly classical 2-absorbing second submodules in Examples 3.9, 3.10, and Propositions 3.11. Theorem 2.14 (resp. Theorem 3.15) of this paper shows that if M is an Artinian R-module, then every non-zero submodule of M has only a finite number of maximal classical (resp. strongly classical) 2-absorbing second submodules. Further, among other results, we investigate strongly classical 2-absorbing second submodules of a finite direct product of modules in Theorem 3.19.

2. Classical 2-absorbing second submodules

We frequently use the following basic fact without further comment.

Remark 2.1. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

Definition 2.2. Let N be a non-zero submodule of an R-module M. We say that N is a *classical 2-absorbing second submodule of* M if whenever $a, b, c \in R$, L is a completely irreducible submodule of M, and $abcN \subseteq L$, then $abN \subseteq L$ or $bcN \subseteq L$ or $acN \subseteq L$. We say M is a *classical 2-absorbing second module* if M is a classical 2-absorbing second submodule of itself.

Theorem 2.3. *Let M be an R-module and N be a non-zero submodule of M. Then the following statements are equivalent:*

(a) *N is a classical 2-absorbing second submodule of M;*
(b) *For every $a, b \in R$ and completely irreducible submodule L of M with $abN \not\subseteq L$, $(L :_R abN) = (L :_R aN) \cup (L :_R bN)$;*
(c) *For every $a, b \in R$ and completely irreducible submodule L of M with $abN \not\subseteq L$, $(L :_R abN) = (L :_R aN)$ or $(L :_R abN) = (L :_R bN)$;*

(d) *For every* $a, b \in R$, *every ideal* I *of* R, *and completely irreducible submodule* L *of* M *with* $abIN \subseteq L$, *either* $abN \subseteq L$ *or* $aIN \subseteq L$ *or* $bIN \subseteq L$;

(e) *For every* $a \in R$, *every ideal* I *of* R, *and completely irreducible submodule* L *of* M *with* $aIN \not\subseteq L$, $(L :_R aIN) = (L :_R IN)$ *or* $(L :_R aIN) = (L :_R aN)$;

(f) *For every* $a \in R$, *ideals* I, J *of* R, *and completely irreducible submodule* L *of* M *with* $aIJN \subseteq L$, *either* $aIN \subseteq L$ *or* $aJN \subseteq L$ *or* $IJN \subseteq L$;

(g) *For ideals* I, J *of* R, *and completely irreducible submodule* L *of* M *with* $IJN \not\subseteq L$, $(L :_R IJN) = (L :_R IN)$ *or* $(L :_R IJN) = (L :_R JN)$;

(h) *For ideals* I_1, I_2, I_3 *of* R, *and completely irreducible submodule* L *of* M *with* $I_1I_2I_3N \subseteq L$, *either* $I_1I_2N \subseteq L$ *or* $I_1I_3N \subseteq L$ *or* $I_2I_3N \subseteq L$;

(i) *For each completely irreducible submodule* L *of* M *with* $N \not\subseteq L$, $(L :_R N)$ *is a 2-absorbing ideal of* R.

Proof. $(a) \Rightarrow (b)$ Let $t \in (L :_R abN)$. Then $tabN \subseteq L$. Since $abN \not\subseteq L$, $atN \subseteq L$ or $btN \subseteq L$ as needed.

$(b) \Rightarrow (c)$ This follows from the fact that if an ideal is the union of two ideals, then it is equal to one of them.

$(c) \Rightarrow (d)$ Let for some $a, b \in R$, an ideal I of R, and completely irreducible submodule L of M, $abIN \subseteq L$. Then $I \subseteq (L :_R abN)$. If $abN \subseteq L$, then we are done. Assume that $abN \not\subseteq L$. Then by part (c), $I \subseteq (L :_R bN)$ or $I \subseteq (L :_R aN)$ as desired.

$(d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h)$ The proofs are similar to that of the previous implications.

$(h) \Rightarrow (a)$ Trivial.

$(h) \Leftrightarrow (i)$ This is straightforward. □

We recall that an R-module M is said to be a *cocyclic module* if $Soc_R(M)$ is a large and simple submodule of M [19]. (Here $Soc_R(M)$ denotes the sum of all minimal submodules of M.) A submodule L of M is a completely irreducible submodule of M if and only if M/L is a cocyclic R-module [12].

Corollary 2.4. Let N be a classical 2-absorbing second submodule of a cocyclic R-module M. Then $Ann_R(N)$ is a 2-absorbing ideal of R.

Proof. This follows from Theorem 2.3 $(a) \Rightarrow (i)$, because (0) is a completely irreducible submodule of M. □

Example 2.5. For any prime integer p, let $M = \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module and $G_i = \langle 1/p^i + \mathbb{Z} \rangle$ for $i \in \mathbb{N}$. Then G_i is not a classical 2-absorbing second submodule of M for each integers $i \geq 3$.

Lemma 2.6. Every 2-absorbing second submodule of M is a classical 2-absorbing second submodule of M.

Proof. Let N be a 2-absorbing second submodule of M, $a, b, c \in R$, L a completely irreducible submodule of M, and $abcN \subseteq L$. Then $abN \subseteq (L :_M c)$. Thus $aN \subseteq$

$(L :_M c)$ or $bN \subseteq (L :_M c)$ or $abN = 0$ because by [6, Lemma 2.1], $(L :_M c)$ is a completely irreducible submodule of M. Hence $acN \subseteq L$ or $bcN \subseteq L$ or $abN \subseteq L$ as needed. □

Example 2.7. Consider $M = \mathbb{Z}_{|||} \oplus \mathbb{Q}$ as a $\mathbb{Z}$-module, where p, q are prime integers. Then M is a classical 2-absorbing second module which is not a strongly 2-absorbing second module.

Proposition 2.8. Let N be a classical 2-absorbing second submodule of an R-module M. Then we have the following.

(a) If $a \in R$, then $a^n N = a^{n+1} N$, for all $n \geq 2$.

(b) If L is a completely irreducible submodule of M such that $N \not\subseteq L$, then $\sqrt{(L :_R N)}$ is a 2-absorbing ideal of R.

Proof. (a) It is enough to show that $a^2 N = a^3 N$. It is clear that $a^3 N \subseteq a^2 N$. Let L be a completely irreducible submodule of M such that $a^3 N \subseteq L$. Since N is a classical 2-absorbing second submodule, $a^2 N \subseteq L$. This implies that $a^2 N \subseteq a^3 N$.

(b) Assume that $a, b, c \in R$ and $abc \in \sqrt{(L :_R N)}$. Then there is a positive integer t such that $a^t b^t c^t N \subseteq L$. By hypotheses, N is a classical 2-absorbing second submodule of M, thus $a^t b^t N \subseteq L$ or $b^t c^t N \subseteq L$ or $a^t c^t N \subseteq L$. Therefore, $ab \in \sqrt{(L :_R N)}$ or $bc \in \sqrt{(L :_R N)}$ or $ac \in \sqrt{(L :_R N)}$. □

Theorem 2.9. *Let N be a submodule of an R-module M. Then we have the following.*

(a) *If N is a classical 2-absorbing second submodule of M, then IN is a classical 2-absorbing second submodule of M for all ideals I of R with $I \not\subseteq Ann_R(N)$.*

(b) *If N is a classical 2-absorbing submodule of M, then $(N :_R I)$ is a classical 2-absorbing submodule of M for all ideals I of R with $I \not\subseteq (N :_R M)$.*

(c) *Let $f : M \to \acute{M}$ be a monomorphism of R-modules. If $\acute{N}$ is a classical 2-absorbing second submodule of $f(M)$, then $f^{-1}(\acute{N})$ is a classical 2-absorbing second submodule of M.*

Proof. (a) Let I be an ideal of R with $I \not\subseteq Ann_R(N)$, $a, b, c \in R$, L be a completely irreducible submodule of M, and $abcIN \subseteq L$. Then $acN \subseteq L$ or $cbIN \subseteq L$ or $abIN \subseteq L$ by Theorem 2.3 $(a) \Rightarrow (d)$. If $cbIN \subseteq L$ or $abIN \subseteq L$, then we are done. If $acN \subseteq L$, then $acIN \subseteq acN$ implies that $acIN \subseteq L$, as needed. Since $I \not\subseteq Ann_R(N)$, we have IN is a non-zero submodule of M.

(b) Use the technique of part (a) and apply [14, Theorem 2].

(c) If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = ff^{-1}(\acute{N}) = f(0) = 0$. Thus $\acute{N} = 0$, a contradiction. Therefore, $f^{-1}(\acute{N}) \neq 0$. Now let $a, b, c \in R$, L be a completely irreducible submodule of M, and $abcf^{-1}(\acute{N}) \subseteq L$. Then

$$abc\acute{N} = abc(f(M) \cap \acute{N}) = abcff^{-1}(\acute{N}) \subseteq f(L).$$

By [5, Lemma 3.14], $f(L)$ is a completely irreducible submodule of $f(M)$. Thus as $\acute{N}$ is a classical 2-absorbing second submodule, $ab\acute{N} \subseteq f(L)$ or $bc\acute{N} \subseteq f(L)$ or

$ac\acute{N} \subseteq f(L)$. Therefore, $abf^{-1}(\acute{N}) \subseteq f^{-1}f(L) = L$ or $bcf^{-1}(\acute{N}) \subseteq f^{-1}f(L) = L$ or $acf^{-1}(\acute{N}) \subseteq f^{-1}f(L) = L$, as desired. □

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [8].

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [2].

Corollary 2.10. Let M be an R-module. Then we have the following.

(a) If M is a multiplication classical 2-absorbing second R-module, then every non-zero submodule of M is a classical 2-absorbing second submodule of M.

(b) If M is a comultiplication module and the zero submodule of M is a classical 2-absorbing submodule, then every proper submodule of M is a classical 2-absorbing submodule of M.

Proof. This follows from parts (a) and (b) of Lemma 2.9. □

Proposition 2.11. Let M be an R-module and $\{K_i\}_{i\in I}$ be a chain of classical 2-absorbing second submodules of M. Then $\sum_{i\in I} K_i$ is a classical 2-absorbing second submodule of M.

Proof. Let $a, b, c \in R$, L be a completely irreducible submodule of M, and $abc \sum_{i\in I} K_i \subseteq L$. Assume that $ab\sum_{i\in I} K_i \not\subseteq L$ and $ac\sum_{i\in I} K_i \not\subseteq L$. Then there are $m, n \in I$ where $abK_n \not\subseteq L$ and $acK_m \not\subseteq L$. Hence, for every $K_n \subseteq K_s$ and every $K_m \subseteq K_d$ we have that $abK_s \not\subseteq L$ and $acK_d \not\subseteq L$. Therefore, for each submodule K_h such that $K_n \subseteq K_h$ and $K_m \subseteq K_h$, we have $bcK_h \subseteq L$. Hence $bc\sum_{i\in I} K_i \subseteq L$, as needed. □

Definition 2.12. We say that a classical 2-absorbing second submodule N of an R-module M is a *maximal classical 2-absorbing second submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a classical 2-absorbing second submodule T of M such that $N \subset T \subset K$.

Lemma 2.13. Let M be an R-module. Then every classical 2-absorbing second submodule of M is contained in a maximal classical 2-absorbing second submodule of M.

Proof. This is proved easily by using Zorn's Lemma and Proposition 2.11. □

Theorem 2.14. *Let M be an Artinian R-module. Then every non-zero submodule of M has only a finite number of maximal classical 2-absorbing second submodules.*

Proof. Suppose that there exists a non-zero submodule N of M such that it has an infinite number of maximal classical 2-absorbing second submodules. Let S be a submodule of M chosen minimal such that S has an infinite number of maximal classical 2-absorbing second submodules because M is an Artinian R-module. Then S is not a classical 2-absorbing second submodule. Thus there exist $a, b, c \in R$ and a completely irreducible submodule L of M such that $abcS \subseteq L$ but $abS \not\subseteq L$, $acS \not\subseteq L$, and $bcS \not\subseteq L$. Let V be a maximal classical 2-absorbing second submodule of M contained in S. Then $abV \subseteq L$ or $acV \subseteq L$ or $bcV \subseteq L$. Thus $V \subseteq (L :_M ab)$ or $V \subseteq (L :_M ac)$ or $V \subseteq (L :_M bc)$. Therefore, $V \subseteq (L :_S ab)$ or $V \subseteq (L :_S ac)$ or $V \subseteq (L :_S bc)$. By the choice of S, the modules $(L :_S ab)$, $(L :_S ac)$, and $(L :_S bc)$ have only finitely many maximal classical 2-absorbing second submodules. Therefore, there is only a finite number of possibilities for the module S, which is a contradiction. $\square$

3. Strongly classical 2-absorbing second submodules

Definition 3.1. Let N be a non-zero submodule of an R-module M. We say that N is a *strongly classical 2-absorbing second submodule of* M if whenever $a, b, c \in R$, L_1, L_2, L_3 are completely irreducible submodules of M, and $abcN \subseteq L_1 \cap L_2 \cap L_3$, then $abN \subseteq L_1 \cap L_2 \cap L_3$ or $bcN \subseteq L_1 \cap L_2 \cap L_3$ or $acN \subseteq L_1 \cap L_2 \cap L_3$. We say M is a *strongly classical 2-absorbing second module* if M is a strongly classical 2-absorbing second submodule of itself.

Clearly every strongly classical 2-absorbing second submodule is a classical 2-absorbing second submodule.

Question 3.2. Let M be an R-module. Is every classical 2-absorbing second submodule of M a strongly classical 2-absorbing second submodule of M?

Example 3.3. The $\mathbb{Z}$-module $\mathbb{Z}$ has no strongly classical 2-absorbing second submodule.

Theorem 3.4. *Let M be an R-module and N be a non-zero submodule of M. Then the following statements are equivalent:*

(a) *N is strongly classical 2-absorbing second;*
(b) *If $a, b, c \in R$, K is a submodule of M, and $abcN \subseteq K$, then $abN \subseteq K$ or $bcN \subseteq K$ or $acN \subseteq K$;*
(c) *For every $a, b, c \in R$, $abcN = abN$ or $abcN = acN$ or $abcN = bcN$;*
(d) *For every $a, b \in R$ and submodule K of M with $abN \not\subseteq K$, $(K :_R abN) = (K :_R aN) \cup (K :_R bN)$;*
(e) *For every $a, b \in R$ and submodule K of M with $abN \not\subseteq K$, $(K :_R abN) = (K :_R aN)$ or $(K :_R abN) = (K :_R bN)$;*
(f) *For every $a, b \in R$, every ideal I of R, and submodule K of M with $abIN \subseteq K$, either $abN \subseteq K$ or $aIN \subseteq K$ or $bIN \subseteq K$;*
(g) *For every $a \in R$, every ideal I of R, and submodule K of M with $aIN \not\subseteq K$, $(K :_R aIN) = (K :_R IN)$ or $(K :_R aIN) = (K :_R aN)$;*
(h) *For every $a \in R$, ideals I, J of R, and submodule K of M with $aIJN \subseteq K$, either $aIN \subseteq K$ or $aJN \subseteq K$ or $IJN \subseteq K$;*

(i) *For ideals I, J of R, and submodule K of M with $IJN \not\subseteq K$, $(K :_R IJN) = (K :_R IN)$ or $(K :_R IJN) = (K :_R JN)$;*
(j) *For ideals I_1, I_2, I_3 of R, and submodule K of M with $I_1I_2I_3N \subseteq K$, either $I_1I_2N \subseteq K$ or $I_1I_3N \subseteq K$ or $I_2I_3N \subseteq K$;*
(k) *For each submodule K of M with $N \not\subseteq K$, $(K :_R N)$ is a 2-absorbing ideal of R.*

Proof. $(a) \Rightarrow (b)$ Let $a, b, c \in R$, K is a submodule of M, and $abcN \subseteq K$. Assume on the contrary that $abN \not\subseteq K$, $bcN \not\subseteq K$, and $acN \not\subseteq K$. Then there exist completely irreducible submodules L_1, L_2, L_3 of M such that K is a submodule of them but $abN \not\subseteq L_1$, $bcN \not\subseteq L_2$, and $acN \not\subseteq L_3$. Now we have $abcN \subseteq L_1 \cap L_2 \cap L_3$. Thus by part (a), $abN \subseteq L_1 \cap L_2 \cap L_3$ or $bcN \subseteq L_1 \cap L_2 \cap L_3$ or $acN \subseteq L_1 \cap L_2 \cap L_3$. Therefore, $abN \subseteq L_1$ or $bcN \subseteq L_2$ or $acN \subseteq L_3$ which are contradictions.

$(b) \Rightarrow (c)$ Let $a, b, c \in R$. Then $abcN \subseteq abcN$ implies that $abN \subseteq abcN$ or $bcN \subseteq abcN$ or $acN \subseteq abcN$ by part (b). Thus $abN = abcN$ or $bcN = abcN$ or $acN = abcN$ because the reverse inclusions are clear.

$(c) \Rightarrow (d)$ Let $t \in (K :_R abN)$. Then $tabN \subseteq K$. Since $abN \not\subseteq K$, $atN \subseteq K$ or $btN \subseteq K$ as needed.

$(d) \Rightarrow (e)$ This follows from the fact that if an ideal is the union of two ideals, then it is equal to one of them.

$(e) \Rightarrow (f)$ Let for some $a, b \in R$, an ideal I of R, and submodule K of M, $abIN \subseteq K$. Then $I \subseteq (K :_R abN)$. If $abN \subseteq K$, then we are done. Assume that $abN \not\subseteq K$. Then by part (d), $I \subseteq (K :_R bN)$ or $I \subseteq (K :_R aN)$ as desired.

$(g) \Rightarrow (h) \Rightarrow (i) \Rightarrow (h) \Rightarrow (j)$ Have proofs similar to that of the previous implications.

$(j) \Rightarrow (a)$ Trivial.

$(j) \Leftrightarrow (k)$ This is straightforward. □

Let N be a submodule of an R-module M. Then Theorem 3.4 $(a) \Leftrightarrow (c)$ shows that N is a strongly classical 2-absorbing second submodule of M if and only if N is a strongly classical 2-absorbing second module.

Corollary 3.5. Let N be a strongly classical 2-absorbing second submodule of an R-module M and I be an ideal of R. Then $I^nN = I^{n+1}N$, for all $n \geq 2$.

Proof. It is enough to show that $I^2N = I^3N$. By Theorem 3.4, $I^2N = I^3N$. □

Example 3.6. Clearly every strongly 2-absorbing second submodule is a strongly classical 2-absorbing second submodule. But the converse is not true in general. For example, consider $M = \mathbb{Z}_{\ntriangleleft} \oplus \mathbb{Q}$ as a $\mathbb{Z}$-module. Then M is a strongly classical 2-absorbing second module. But M is not a strongly 2-absorbing second module.

A non-zero submodule N of an R-module M is said to be a *weakly second submodule* of M if $rsN \subseteq K$, where $r, s \in R$ and K is a submodule of M, implies either $rN \subseteq K$ or $sN \subseteq K$ [1].

Proposition 3.7. Let M be an R-module. Then we have the following.

(a) If M is a comultiplication R-module and N is a strongly classical 2-absorbing second submodule of M, then N is a strongly 2-absorbing second submodule of M.
(b) If N_1, N_2 are weakly second submodules of M, then $N_1 + N_2$ is a strongly classical 2-absorbing second submodule of M.
(c) If N is a strongly classical 2-absorbing second submodule of M, then IN is a strongly classical 2-absorbing second submodule of M for all ideals I of R with $I \not\subseteq Ann_R(N)$.
(d) If M is a multiplication strongly classical 2-absorbing second R-module, then every non-zero submodule of M is a classical 2-absorbing second submodule of M.
(e) If M is a strongly classical 2-absorbing second R-module, then every non-zero homomorphic image of M is a classical 2-absorbing second R-module.

Proof. (a) By Theorem 3.4 $(a) \Rightarrow (k)$, $Ann_R(N)$ is a 2-absorbing ideal of R. Now the result follows from [5, Theorem 3.10].

(b) Let N_1, N_2 be weakly second submodules of M and $a, b, c \in R$. Since N_1 is a weakly second submodule, we may assume that $abcN_1 = aN_1$. Likewise, assume that $abcN_2 = bN_2$. Hence $abc(N_1 + N_2) = ab(N_1 + N_2)$ which implies $N_1 + N_2$ is a classical 2-absorbing second submodule by Theorem 3.4 $(c) \Rightarrow (a)$.

(c) Use the technique of the proof of Theorem 2.9 (a).

(d) This follows from part (c).

(e) This is straightforward. □

For a submodule N of an R-module M the *second radical* (or *second socle*) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [9] and [3]).

Theorem 3.8. *Let M be a finitely generated comultiplication R-module. If N is a strongly classical 2-absorbing second submodule of M, then $sec(N)$ is a strongly 2-absorbing second submodule of M.*

Proof. Let N be a strongly classical 2-absorbing second submodule of M. By Proposition 3.7 (a), $Ann_R(N)$ is a 2-absorbing ideal of R. Thus by [7, Theorem 2.1], $\sqrt{Ann_R(N)}$ is a 2-absorbing ideal of R. By [4, Theorem 2.12], $Ann_R(sec(N)) = \sqrt{Ann_R(N)}$. Therefore, $Ann_R(sec(N))$ is a 2-absorbing ideal of R. Now the result follows from [5, Theorem 3.10]. □

The following examples show that the two concepts of classical 2-absorbing submodules and strongly classical 2-absorbing second submodules are different in general.

Example 3.9. The submodule $2\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ is a classical 2-absorbing submodule which is not a strongly classical 2-absorbing second module.

Example 3.10. The submodule $\langle 1/p + \mathbb{Z}\rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is a a strongly classical 2-absorbing second module which is not a classical 2-absorbing submodule of $\mathbb{Z}_{p^\infty}$.

A commutative ring R is said to be a *u-ring* provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a *um-ring* is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them [16].

In the following proposition, we investigate the relationships between strongly classical 2-absorbing second submodules and classical 2-absorbing submodules.

Proposition 3.11. Let M be a non-zero R-module. Then we have the following.

(a) If M is a finitely generated strongly classical 2-absorbing second R-module, then the zero submodule of M is a classical 2-absorbing submodule.
(b) If M is a multiplication strongly classical 2-absorbing second R-module, then the zero submodule of M is a classical 2-absorbing submodule.
(c) Let R be a *um*-ring. If M is a Artinian R-module and the zero submodule of M is a classical 2-absorbing submodule, then M is a strongly classical 2-absorbing second R-module.
(d) Let R be a *um*-ring. If M is a comultiplication R-module and the zero submodule of M is a classical 2-absorbing submodule, then M is a strongly classical 2-absorbing second R-module.

Proof. (a) Let $a, b, c \in R$, $m \in M$, and $abcm = 0$. By Theorem 3.4, we can assume that $abcM = acM$. Since M is finitely generated, by using [13, Theorem 76], $Ann_R(abM) + Rc = R$. It follows that $(0 :_M abc) = (0 :_M ab)$. This implies that $abm = 0$, as needed.

(b) Let $a, b, c \in R$, $m \in M$, and $abcm = 0$. Then by Theorem 3.4, we can assume that $abcM = acM$. Thus

$$0 = abc((0 :_M abc) :_R M)M = (((0 :_M abc) :_R M)M)ab.$$

Since M is a multiplication module, $((0 :_M abc) :_R M)M = (0 :_M abc)$. Therefore, $(0 :_M abc)ab = 0$. It follows that $(0 :_M abc) \subseteq (0 :_M ab)$. Thus $(0 :_M abc) = (0 :_M ab)$ because the reverse inclusion is clear. Hence $abm = 0$, as required.

(c) Let $a, b, c \in R$. Then by [14, Theorem 4], we can assume that $(0 :_M abc) = (0 :_M ab)$. Hence $(0 :_{M/(0:_M ab)} c) = 0$. Since M is Artinian, it follows that $cM + (0 :_M ab) = M$. Therefore, $abcM = abM$. Thus by Theorem 3.4 $(c) \Rightarrow (a)$, M is a classical 2-absorbing second R-module.

(d) Let $a, b, c \in R$. Then by [14, Theorem 4], we can assume that $(0 :_M abc) = (0 :_M ab)$. Since M is a comultiplication R-module, this implies that

$$M = ((0 :_M abc) :_M Ann_R(abcM) = ((0 :_M ab) :_M Ann_R(abcM)) = (abcM :_M ab).$$

It follows that $abM \subseteq abcM$. Thus $abM = abcM$ because the reverse implication is clear and this completed the proof. $\square$

Proposition 3.12. Let M be an R-module and $\{K_i\}_{i \in I}$ be a chain of strongly classical 2-absorbing second submodules of M. Then $\sum_{i \in I} K_i$ is a strongly classical 2-absorbing second submodule of M.

Proof. Use the technique of Proposition 2.11. □

Definition 3.13. We say that a strongly classical 2-absorbing second submodule N of an R-module M is a *maximal strongly classical 2-absorbing second submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a strongly classical 2-absorbing second submodule T of M such that $N \subset T \subset K$.

Lemma 3.14. Let M be an R-module. Then every strongly classical 2-absorbing second submodule of M is contained in a maximal strongly classical 2-absorbing second submodule of M.

Proof. This is proved easily by using Zorn's Lemma and Proposition 3.12. □

Theorem 3.15. *Let M be an Artinian R-module. Then every non-zero submodule of M has only a finite number of maximal strongly classical 2-absorbing second submodules.*

Proof. Use the technique of Theorem 2.14 any apply Lemma 3.14. □

Theorem 3.16. *Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then we have the following.*

(a) *If N is a strongly classical 2-absorbing second submodule of M, then $f(N)$ is a strongly classical 2-absorbing second submodule of $\acute{M}$.*
(b) *If $\acute{N}$ is a strongly classical 2-absorbing second submodule of $f(M)$, then $f^{-1}(\acute{N})$ is a strongly classical 2-absorbing second submodule of M.*

Proof. (a) Since $N \neq 0$ and f is a monomorphism, we have $f(N) \neq 0$. Let $a, b, c \in R$. Then by Theorem 3.4 $(a) \Rightarrow (c)$, we can assume that $abcN = abN$. Thus

$$abcf(N) = f(abcN) = f(abN) = abf(N).$$

Hence $f(N)$ is a strongly classical 2-absorbing second submodule of $\acute{M}$ by Theorem 3.4 $(c) \Rightarrow (a)$.

(b) If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = ff^{-1}(\acute{N}) = f(0) = 0$. Thus $\acute{N} = 0$, a contradiction. Therefore, $f^{-1}(\acute{N}) \neq 0$. Now let $a, b, c \in R$, K be a submodule of M, and $abcf^{-1}(\acute{N}) \subseteq K$. Then

$$abc\acute{N} = abc(f(M) \cap \acute{N}) = abcff^{-1}(\acute{N}) \subseteq f(K).$$

Thus as $\acute{N}$ is a strongly classical 2-absorbing second submodule, $ab\acute{N} \subseteq f(K)$ or $bc\acute{N} \subseteq f(K)$ or $ac\acute{N} \subseteq f(K)$. Therefore, $abf^{-1}(\acute{N}) \subseteq f^{-1}f(K) = K$ or $bcf^{-1}(\acute{N}) \subseteq f^{-1}f(K) = K$ or $acf^{-1}(\acute{N}) \subseteq f^{-1}f(K) = K$, as desired. □

Let R_i be a commutative ring with identity and M_i be an R_i-module for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Theorem 3.17. *Let $R = R_1 \times R_2$ be a decomposable ring and let $M = M_1 \times M_2$ be an R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times N_2$ is a non-zero submodule of M. Then the following conditions are equivalent:*

(a) *N is a strongly classical 2-absorbing second submodule of M;*

(b) *Either $N_1 = 0$ and N_2 is a strongly classical 2-absorbing second submodule of M_2 or $N_2 = 0$ and N_1 is a strongly classical 2-absorbing second submodule of M_1 or N_1, N_2 are weakly second submodules of M_1, M_2, respectively.*

Proof. $(a) \Rightarrow (b)$. Suppose that N is a strongly classical 2-absorbing second submodule of M such that $N_2 = 0$. From our hypothesis, N is non-zero, so $N_1 \neq 0$. Set $\acute{M} = M_1 \times 0$. One can see that $\acute{N} = N_1 \times 0$ is a strongly classical 2-absorbing second submodule of $\acute{M}$. Also observe that $\acute{M} \cong M_1$ and $\acute{N} \cong N_1$. Thus N_1 is a strongly classical 2-absorbing second submodule of M_1. Suppose that $N_1 \neq 0$ and $N_2 \neq 0$. We show that N_1 is a weakly second submodule of M_1. Since $N_2 \neq 0$, there exists a completely irreducible submodule L_2 of M_2 such that $N_2 \not\subseteq L_2$. Let $abN_1 \subseteq K$ for some $a, b \in R_1$ and submodule K of M_1. Thus $(a, 1)(b, 1)(1, 0)(N_1 \times N_2) = abN_1 \times 0 \subseteq K \times L_2$. So either $(a, 1)(b, 1)(N_1 \times N_2) = abN_1 \times N_2 \subseteq K \times L_2$ or $(a, 1)(1, 0)(N_1 \times N_2) = aN_1 \times 0 \subseteq K \times L_2$ or $(b, 1)(1, 0)(N_1 \times N_2) = bN_1 \times 0 \subseteq K \times L_2$. If $abN_1 \times N_2 \subseteq K \times L_2$, then $N_2 \subseteq L_2$, a contradiction. Hence either $aN_1 \subseteq K$ or $bN_1 \subseteq K$ which shows that N_1 is a weakly second submodule of M_1. Similarly, we can show that N_2 is a weakly second submodule of M_2.

$(b) \Rightarrow (a)$. Suppose that $N = N_1 \times 0$, where N_1 is a strongly classical 2-absorbing (resp. weakly) second submodule of M_1. Then it is clear that N is a strongly classical 2-absorbing (resp. weakly) second submodule of M. Now, assume that $N = N_1 \times N_2$, where N_1 and N_2 are weakly second submodules of M_1 and M_2, respectively. Hence $(N_1 \times 0) + (0 \times N_2) = N_1 \times N_2 = N$ is a strongly classical 2-absorbing second submodule of M, by Proposition 3.7 (b). □

Lemma 3.18. Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a decomposable ring and $M = M_1 \times M_2 \cdots \times M_n$ be an R-module where for every $1 \leq i \leq n$, M_i is an R_i-module, respectively. A non-zero submodule N of M is a weakly second submodule of M if and only if $N = \times_{i=1}^{n} N_i$ such that for some $k \in \{1, 2, ..., n\}$, N_k is a weakly second submodule of M_k, and $N_i = 0$ for every $i \in \{1, 2, ..., n\} \setminus \{k\}$.

Proof. $(\Rightarrow)$ Let N be a weakly second submodule of M. We know $N = \times_{i=1}^{n} N_i$ where for every $1 \leq i \leq n$, N_i is a submodule of M_i, respectively. Assume that N_r is a non-zero submodule of M_r and N_s is a non-zero submodule of M_s for some $1 \leq r < s \leq n$. Since N is a weakly second submodule of M,

$$(0, \cdots, 0, 1_{R_r}, 0, \cdots, 0)(0, \cdots, 0, 1_{R_s}, 0, \cdots, 0)N = (0, \cdots, 0, 1_{R_r}, 0, \cdots, 0)N$$

or

$$(0,\cdots,0,1_{R_r},0,\cdots,0)(0,\cdots,0,1_{R_s},0,\cdots,0)N=(0,\cdots,0,1_{R_s},0,\cdots,0)N.$$

Thus $N_r=0$ or $N_s=0$. This contradiction shows that exactly one of the N_i's is non-zero, say N_k. Now, we show that N_k is a weakly second submodule of M_k. Let $a,b\in R_k$. Since N is a weakly second submodule of M,

$$(0,\cdots,0,a,0,\cdots,0)(0,\cdots,0,b,0,\cdots,0)N=(0,\cdots,0,a,0,\cdots,0)N$$

or

$$(0,\cdots,0,a,0,\cdots,0)(0,\cdots,0,b,0,\cdots,0)N=(0,\cdots,0,b,0,\cdots,0)N$$

Thus $abN_k=aN_k$ or $abN_k=bN_k$ as needed.

($\Leftarrow$) This is clear. $\square$

Theorem 3.19. *Let $R=R_1\times R_2\times\cdots\times R_n$ $(2\le n<\infty)$ be a decomposable ring and $M=M_1\times M_2\cdots\times M_n$ be an R-module, where for every $1\le i\le n$, M_i is an R_i-module, respectively. Then for a non-zero submodule N of M the following conditions are equivalent:*

(a) *N is a strongly classical 2-absorbing second submodule of M;*
(b) *Either $N=\times_{i=1}^{n}N_i$ such that for some $k\in\{1,2,...,n\}$, N_k is a strongly classical 2-absorbing second submodule of M_k, and $N_i=0$ for every $i\in\{1,2,...,n\}\setminus\{k\}$ or $N=\times_{i=1}^{n}N_i$ such that for some $k,m\in\{1,2,...,n\}$, N_k is a weakly second submodule of M_k, N_m is a weakly second submodule of M_m, and $N_i=0$ for every $i\in\{1,2,...,n\}\setminus\{k,m\}$.*

Proof. We use induction on n. For $n=2$ the result holds by Theorem 3.17. Now suppose that the result is valid when $K=M_1\times\cdots\times M_t$ for each $t<n$. We show that the result holds when $M=K\times M_n$. By Theorem 3.17, N is a strongly classical 2-absorbing second submodule of M if and only if either $N=L\times 0$ for some strongly classical 2-absorbing second submodule L of K or $N=0\times L_n$ for some strongly classical 2-absorbing second submodule L_n of M_n or $N=L\times L_n$ for some weakly second submodule L of K and some weakly second submodule L_n of M_n. Note that by Lemma 3.18, a non-zero submodule L of K is a weakly second submodule of K if and only if $L=\times_{i=1}^{n-1}N_i$ such that for some $k\in\{1,2,...,n-1\}$, N_k is a weakly second submodule of M_k and $N_i=0$ for every $i\in\{1,2,...,n-1\}\setminus\{k\}$. Hence the claim is proved. $\square$

Example 3.20. Let R be a Noetherian ring and let $E=\oplus_{m\in Max(R)}E(R/m)$. Then for each 2-absorbing ideal P of R, $(0:_E P)$ is a strongly classical 2-absorbing second submodule of E.

Proof. By using [17, p. 147], $Hom_R(R/P,E)\neq 0$. Now since $(0:_E P)\cong Hom_R(R/P,E)$, $(0:_E P)$ is a strongly 2-absorbing second submodule of E by [5, Theorem 3.27]. Now the result follows from Example 3.6. $\square$

Theorem 3.21. *Let R be a um-ring and M be an R-module. If E is an injective R-module and N is a classical 2-absorbing submodule of M such that $Hom_R(M/N, E) \neq 0$, then $Hom_R(M/N, E)$ is a strongly classical 2-absorbing second R-module.*

Proof. Let $a, b, c \in R$. Since N is a classical 2-absorbing submodule of M, we can assume that $(N :_M abc) = (N :_M ab)$ by [14, Theorem 4]. Since E is an injective R-module, by replacing M with M/N in [1, Theorem 3.13 (a)], we have $Hom_R(M/(N :_M r), E) = rHom_R(M/N, E)$ for each $r \in R$. Therefore,

$$abcHom_R(M/N, E) = Hom_R(M/(N :_M abc), E) =$$
$$Hom_R(M/(N :_M ab), E) = abHom_R(M/N, E),$$

as needed □

Theorem 3.22. *Let M be a strongly classical 2-absorbing second R-module and F be a right exact linear covariant functor over the category of R-modules. Then $F(M)$ is a strongly classical 2-absorbing second R-module if $F(M) \neq 0$.*

Proof. This follows from [1, Lemma 3.14] and Theorem 3.4 $(a) \Rightarrow (c)$. □

Corollary 3.23. Let M be an R-module, S be a multiplicative subset of R and N be a strongly classical 2-absorbing second submodule of M. Then $S^{-1}N$ is a strongly classical 2-absorbing second submodule of $S^{-1}M$ if $S^{-1}N \neq 0$.

Proof. This follows from Theorem 3.22. □

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Current address: H. Ansari-Toroghy: Department of pure Mathematics Faculty of mathematical Sciences University of Guilan P. O. Box 41335-19141, Rasht, Iran.

E-mail address: ansari@guilan.ac.ir

ORCID Address: https://orcid.org/0000-0001-7310-0924

Current address: F. Farshadifar (Corresponding Author): Assistant Professor, Department of Mathematics, Farhangian University, Tehran, Iran., School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran

E-mail address: f.farshadifar@cfu.ac.ir

ORCID Address: https://orcid.org/0000-0001-7600-994X