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Structural Stability for a Class of Nonlinear Wave Equations

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ABSTRACT

In this paper we discuss the structural stability of an initial value problem defined for the equation

$$u_t - u_{txx} + \alpha u u_x = \beta u_x u_{xx} + u u_{xxx} \qquad \text{(i.1)}$$

where α, β are constants, $x \in \mathbb{R}$, $t \in \mathbb{R}^+$. For the choices of α and β , (i.1) describe the nonlinear shallow water waves. Upper and lower bounds are derived for energy decay rate in every finite interval $[0,T]$ which reveals that only the lower bound of the energy decays exponentially.

Key Words: *Degasperis-Procesi equation, Camassa-Holm equation, traveling wave*

1. INTRODUCTION

The Degasperis-Procesi (D-P) equation

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \quad x \in \mathbb{R},\ t > 0 \qquad (1.1)$$

was proposed in [1] as one out of three integral equations within a certain family of third-order nonlinear dispersive partial differential equations; the other two being the well-known Korteweg-de Vries (KdV)

$$u_t - 6uu_x + u_{xxx} = 0 \quad x \in \mathbb{R},\ t > 0 \qquad (1.2)$$

and Camassa-Holm (C-H) equation [2]

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \quad x \in \mathbb{R},\ t > 0 \qquad (1.3)$$

which models the shallow water waves.

All weak traveling wave solutions of the D-P equations are classified by Lenells [3]. Similar classification for C-H has also been done in [4]. Degasperis, Holm and Hone [5] investigated D-P equation using the method of asymptotic integrability. This equation has a form similar to C-H shallow water wave equation. The exact integrability of the equation (1.1) investigated in [5]. The solitary wave solutions for modified forms of the equations D-P and C-H are developed by Wazwaz [6].

In this work we are interested in the structural stability of the equations D-P and C-H besides the upper and lower bounds of the energy for these equations. For the structural stability, it is fundamental that one wishes to know if a small change in a coefficient of the equation or boundary data, or small change of the equations themselves will lead to a drastic change in the solution or not. In this article we have proved that

$$u_t - u_{txx} + \alpha uu_x = \beta u_x u_{xx} + uu_{xxx} \quad x \in \mathbb{R},\ t > 0 \qquad (1.4)$$

is structurally stable with respect to the coefficients α and β . D-P and C-H equations are attained for the choices $\alpha = 4$, $\beta = 3$ and $\alpha = 3$, $\beta = 2$ respectively. We obtain that upper and lower bounds of the energy for the solutions of equations D-P and C-H are derived in every finite interval $[0,T]$ which shows that only the lower bound of the energy decays exponentially.

2. STRUCTURAL STABILITY

Now we consider the problem

$$u_t - u_{txx} + \alpha uu_x = \beta u_x u_{xx} + uu_{xxx} \quad u \in C_0^{4,1}(\mathbb{R} \times \mathbb{R}^+),$$

$$0 < t < T \quad \text{for fixed } T \qquad (2.1)$$

$$u(x,0) = u_0(x) \quad x \in \mathbb{R} \qquad (2.2)$$

where $\alpha, \beta > 1$ are constants, $C_0^{4,1}(\mathbb{R} \times \mathbb{R}^+)$ is the space of functions having compact support which have fourth order and first order derivative with respect to x and t respectively. To do this, we let (u, α_1, β_1) be the solution of the following problem

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$$u_t - u_{txx} + \alpha_1 u u_x = \beta_1 u_x u_{xx} + u u_{xxx} \quad u \in C_0^{4,1}(\mathbb{R}\times\mathbb{R}^+),\ 0<t<T \ \text{ for fixed } T \tag{2.3}$$

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R} \tag{2.4}$$

and (v, α_2, β_2) be the solution of

$$v_t - v_{txx} + \alpha_2 v v_x = \beta_2 v_x v_{xx} + v v_{xxx} \quad v \in C_0^{4,1}(\mathbb{R}\times\mathbb{R}^+),\ 0<t<T \ \text{ for fixed } T \tag{2.5}$$

$$v(x,0) = u_0(x) \qquad x \in \mathbb{R} \tag{2.6}$$

where $\alpha_1,\ \beta_1, \alpha_2,\ \beta_2 > 1$ are constants. Now, we define the difference of these solutions by $w = u - v$, $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$ where we assume that $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$. Then from (2.3)-(2.6), we find that (w, α, β) satisfies the following initial value problem

$$w_t - w_{txx} + \alpha u u_x + \alpha_2 (w u_x + v w_x) - \beta u_x u_{xx} - \beta_2 (w_x u_{xx} + v_x w_{xx}) - (w u_{xxx} + v w_{xxx}) = 0 \tag{2.7}$$

$$w(x,0) = 0 \tag{2.8}$$

We may state our result on structural stability for the problem defined by (2.1)-(2.2) as :

Theorem 1: Let w be the solution of the problem (2.7) and (2.8). Then w satisfies the estimate

$$\|w\|^2 + 2\|w_x\|^2 + 2\|w_{xx}\|^2 + \|w_{xxx}\|^2 \le (\alpha K_1 + \beta K_2)\left(\frac{e^{\gamma T} - 1}{\gamma}\right) \tag{2.9}$$

for fixed T where K_1, K_2 and γ are positive constants and $\|\cdot\|$ denotes the L_2 norm of functions.

Proof. Taking the inner product of (2.7) by w yields

$$(w_t, w) - (w_{txx}, w) + (\alpha u u_x, w) + (\alpha_2 (w u_x + v w_x), w) - (\beta u_x u_{xx}, w) - (\beta_2 (w_x u_{xx} + v_x w_{xx}), w) - (w u_{xxx} + v w_{xxx}, w) = 0 \tag{2.10}$$

which gives

$$\frac{1}{2}\frac{d}{dt}\left\{\|w\|^2 + \|w_x\|^2\right\} = -\alpha \int u u_x w dx - \alpha_2 \int (w u_x + v w_x) w dx + \beta \int u_x u_{xx} w dx + \beta_2 \int (w_x u_{xx} + v_x w_{xx}) w dx + \int (w u_{xxx} + v w_{xxx}) w dx \tag{2.11}$$

Since $u, v \in C_0^{4,1}(\mathbb{R}\times\mathbb{R}^+)$ then there exists a generic constant such that the functions u and v together with their derivatives are all bounded by a generic constant D. For the first integral on the right hand side of (2.11) we obtain,

$$-\alpha \int u u_x w dx \le \alpha C\left\{\|w\|^2 + \|u_x\|^2\right\} \tag{2.12}$$

utilizing Cauchy and Hölder inequalities. For the second integral in the right hand side of (2.11) we get

$$-\alpha_2 \int (w u_x + v w_x) w dx \le \alpha_2 C\left\{\|w\|^2 + \|w_x\|^2\right\}. \tag{2.13}$$

In a similar way , we can compute the estimates for the other terms as

$$\beta \int u_x u_{xx} w dx \le \beta C\left\{\|w\|^2 + \|u_{xx}\|^2\right\}, \tag{2.14}$$

$$\beta_2 \int (w_x u_{xx} + v_x w_{xx}) w dx \le \beta_2 C\left\{\|w\|^2 + \|w_x\|^2 + \|w_{xx}\|^2\right\}, \tag{2.15}$$

$$\int (w u_{xxx} + v w_{xxx}) w dx \le C\left\{\|w\|^2 + \|w_{xxx}\|^2\right\}. \tag{2.16}$$

Substituting the estimates (2.12)-(2.16) in (2.11) we find

$$\frac{d}{dt}\left\{\|w\|^2+\|w_x\|^2\right\}\le C\left\{\alpha\|u_x\|^2+\beta\|u_{xx}\|^2\right\}+C\{\alpha+\beta+\alpha_2+\beta_2+1\}\|w\|^2+C\left\{(\alpha_2+\beta_2)\|w_x\|^2+\beta_2\|w_{xx}\|^2+\|w_{xxx}\|^2\right\} \tag{2.17}$$

Similarly taking the inner product of (2.7) by w_{xx}, we find

$$\frac{d}{dt}\left\{\|w_x\|^2+\|w_{xx}\|^2\right\}\le D\left\{\alpha\|u_x\|^2+\beta\|u_{xx}\|^2\right\}+C\left\{(\alpha_2+1)\|w\|^2+\alpha_2\|w_x\|^2\right\}+\left\{C(\alpha_2+\beta_2+1)+\beta D\right\}\|w_{xx}\|^2+C\|w_{xxx}\|^2 \tag{2.18}$$

Now let us differentiate equation of (2.7) with respect to x :

$$w_{tx}-w_{txxx}+\alpha(u_x^2+uu_{xx})+\alpha_2(w_xu_x+wu_{xx}+v_xw_x+vw_{xx})-\beta(u_xu_{xxx}+u_{xx}^2)-\beta_2(w_{xx}u_{xx}+w_xu_{xxx}+v_{xx}w_{xx}+v_xw_{xxx})$$
$$-(w_xu_{xxx}+wu_{xxxx}+v_xw_{xxx}+vw_{xxxx})=0 \,. \tag{2.19}$$

Taking the inner product of (2.19) by w_{xxx}, we find

$$\frac{d}{dt}\left\{\|w_{xx}\|^2+\|w_{xxx}\|^2\right\}\le C\left\{\alpha\|u_{xx}\|^2+\beta\|u_{xxx}\|^2\right\}+C\left\{(\alpha_2+1)\|w\|^2+(\alpha_2+\beta_2+1)\|w_x\|^2+(\alpha+\alpha_2+\beta+\beta_2)\|w_{xx}\|^2\right\}$$
$$+C(\alpha+\alpha_2+\beta+\beta_2+1)\|w_{xxx}\|^2 \,. \tag{2.20}$$

Adding up the inequalities (2.17), (2.18), (2.20) we have

$$\frac{d}{dt}\left\{\|w\|^2+2\|w_x\|^2+2\|w_{xx}\|^2+\|w_{xxx}\|^2\right\}\le\alpha K_1+\beta K_2+\gamma\left\{\|w\|^2+2\|w_x\|^2+2\|w_{xx}\|^2+\|w_{xxx}\|^2\right\} \tag{2.21}$$

where $K_1=(C+D)\|u_x\|^2+C\|u_{xx}\|^2$, $K_2=(C+D)\|u_{xx}\|^2+C\|u_{xxx}\|^2$ and

$$\gamma=C\max\{\ (\alpha+\beta+3\alpha_2+3+\beta_2),(3\alpha_2+1+2\beta_2),(\alpha+\beta+\alpha_2+2\beta_2),(\alpha+\beta+\alpha_2+\beta_2+2)\ \}$$

Thus, from (2.20) we have

$$\frac{d}{dt}\Psi(t)-\gamma\,\Psi(t)\le\alpha K_1+\beta K_2 \tag{2.22}$$

where $\Psi(t)=\|w\|^2+2\|w_x\|^2+2\|w_{xx}\|^2+\|w_{xxx}\|^2$. Solving the differential inequality (2.22), we arrive at

$$\|w\|^2+2\|w_x\|^2+2\|w_{xx}\|^2+\|w_{xxx}\|^2\le(\alpha K_1+\beta K_2)\left(\frac{e^{\gamma T}-1}{\gamma}\right)$$

for fixed T . And so we have completed the proof of the theorem.

Remark 2. w and its x derivatives of order up to 3 , tends to zero as $\alpha\to 0,\ \beta\to 0$ for finite T

So, the solutions of (2.22) depend continuously on α and β in $C_0^{4,1}(\mathbb{R}\times\mathbb{R}^+)$ which means that (2.1)-(2.2) is structurally stable with respect to the coefficients α and β .

3. UPPER AND LOWER BOUNDS ON THE ENERGY

Let u be a solution to the initial-value problem (2.1), (2.2) with $\alpha>1$ and $\beta>1$. By similar computations given in Section 2, we find

$$\frac{d}{dt}\left\{\|u\|^2+2\|u_x\|^2+2\|u_{xx}\|^2+\|u_{xxx}\|^2\right\}=(2-\beta-\alpha)\int u^2u_{xxx}dx-(2\beta+1)\int u_xu_{xxx}^2dx+(1-2\beta-5\alpha)\int u_xu_{xx}^2dx \tag{3.1}$$

If we use Cauchy and Hölder inequalities in (3.1) we obtain

$$\frac{d}{dt}\left\{\|u\|^2+2\|u_x\|^2+2\|u_{xx}\|^2+\|u_{xxx}\|^2\right\}\geq(1-\frac{\beta}{2}-\frac{\alpha}{2})\max u^2\|u\|^2+2(-\frac{5\alpha}{4}-\beta)\|u_x\|^2+2\left(\frac{1}{4}-\frac{\beta}{2}-\frac{5\alpha}{4}\right)\max u_{xx}^2\|u_{xx}\|^2$$

$$+\left(1-\frac{\beta}{2}-\frac{\alpha}{2}-\left(\beta+\frac{1}{2}\right)\max u_{xxx}^2\right)\|u_{xxx}\|^2 \tag{3.2}$$

Taking

$$\eta=\max\left\{\left(1-\frac{\beta}{2}-\frac{\alpha}{2}\right)\max u^2,\ \left(-\frac{5\alpha}{4}-\beta\right),\ \left(\frac{1}{4}-\frac{\beta}{2}-\frac{5\alpha}{4}\right)\max u_{xx}^2\ ,\left(1-\frac{\beta}{2}-\frac{\alpha}{2}-\left(\beta+\frac{1}{2}\right)\max u_{xxx}^2\right)\right\}$$

and

$$Y(t)=\|u\|^2+2\|u_x\|^2+2\|u_{xx}\|^2+\|u_{xxx}\|^2,$$

we have

$$\frac{d}{dt}Y(t)-\eta\ \mathrm{Y}(t)\geq 0 \tag{3.3}$$

Solving the inequality (3.3) we arrive at

$$e^{\eta T}\left\{\|u(x,0)\|^2+2\|u_x(x,0)\|^2+2\|u_{xx}(x,0)\|^2+\|u_{xxx}(x,0)\|^2\right\}\leq\|u(x,t)\|^2+2\|u_x(x,t)\|^2+2\|u_{xx}(x,t)\|^2+\|u_{xxx}(x,t)\|^2 \tag{3.4}$$

where $\eta\leq 0$. This inequality gives a lower bound for the energy.

Now we will derive an upper bound for the energy. From (3.1) we have

$$\frac{d}{dt}\left\{\|u\|^2+2\|u_x\|^2+2\|u_{xx}\|^2+\|u_{xxx}\|^2\right\}\leq\left|1-\frac{\beta+\alpha}{2}\right|\max|u|\|u\|^2+2\left\{\left|\frac{1-5\alpha-2\beta}{4}\right|\max|u_{xx}|+\left(\frac{\beta}{2}+\frac{1}{4}\right)\max|u_{xxx}|\right\}\|u_x\|^2$$

$$+2\left|\frac{1-5\alpha-2\beta}{4}\right|\max|u_{xx}|\|u_{xx}\|^2+\left\{\left|1-\frac{\beta+\alpha}{2}\right|\max|u|+\left(\beta+\frac{1}{2}\right)\max|u_{xxx}|\right\}\|u_{xxx}\|^2 \tag{3.5}$$

Taking

$$\mu=\max\left\{\left|1-\frac{\beta+\alpha}{2}\right|\max|u|,\ \left|\frac{1-5\alpha-2\beta}{4}\right|\max|u_{xx}|+\left(\frac{\beta}{2}+\frac{1}{4}\right)\max|u_{xxx}|,\left|\frac{1-5\alpha-2\beta}{4}\right|\max|u_{xx}|,\right.$$
$$\left.\left|1-\frac{\beta+\alpha}{2}\right|\max|u|+\left(\beta+\frac{1}{2}\right)\max|u_{xxx}|\right\}$$

and

$$Y(t)=\|u\|^2+2\|u_x\|^2+2\|u_{xx}\|^2+\|u_{xxx}\|^2,$$

we have

$$\frac{d}{dt}Y(t)-\mu\ \mathrm{Y}(t)\leq 0 \tag{3.6}$$

Then integrating the inequality (3.6) from 0 to T we arrive at

$$\|u(x,T)\|^2+2\|u_x(x,T)\|^2+2\|u_{xx}(x,T)\|^2+\|u_{xxx}(x,T)\|^2\leq e^{\mu T}\left\{\|u(x,0)\|^2+2\|u_x(x,0)\|^2+2\|u_{xx}(x,0)\|^2+\|u_{xxx}(x,0)\|^2\right\} \tag{3.7}$$

where $\mu\geq 0$. (3.7) gives an upper bound for the energy in every finite interval $[0,T]$.

We may combine the above results as in the following theorem.

*Theorem 3***.** The energy corresponding to the solutions of the initial value problem (2.1)-(2.2) in $C_0^{4,1}(\mathbb{R}\times\mathbb{R}^+)$ satisfy

$$e^{\eta T}\left\{\|u(x,0)\|^2+2\|u_x(x,0)\|^2+2\|u_{xx}(x,0)\|^2+\|u_{xxx}(x,0)\|^2\right\}\le\|u(x,T)\|^2+2\|u_x(x,T)\|^2+2\|u_{xx}(x,T)\|^2+\|u_{xxx}(x,T)\|^2$$

$$\le e^{\mu T}\left\{\|u(x,0)\|^2+2\|u_x(x,0)\|^2+2\|u_{xx}(x,0)\|^2+\|u_{xxx}(x,0)\|^2\right\}$$

For fixed T where $\alpha,\ \beta>1$ are constants.

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