*Gazi University Journal of Science*
*GU J Sci*
*24(1):45-49 (2011)*

GUJS
*www.gujs.org*

# Subdivision of the Spectra for Factorable Matrices on $c_0$.

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*Received:19/04/2010 Revised: 14/06/2010 Accepted:15/06/2010*

**ABSTRACT**

In a series of papers, B.E. Rhoades and M. Yildirim previously investigated the spectra and fine spectra for factorable matrices, considered as bounded operators over various sequence spaces. In the present paper approximation point spectrum, defect spectrum and compression spectrum of factorable matrices are investigated.

**Key words**: *Spectrum, fine spectrum, approximate point spectrum, defect spectrum, compression spectrum, factorable matrices.*

## 1. INTRODUCTION

Let $B(X)$ denote the linear space of all bounded linear operators on $X$. Given an operator $L \in B(X)$, the set

$$\rho(L) := \{\lambda \in K : \lambda I - L \text{ bijection}\} \quad (1)$$

is called the resolvent set of $L$ (where $K = C$ or $K = R$), its complement

$$\sigma(L) := K \setminus \rho(L) \quad (2)$$

the spectrum of $L$. By the closed graph theorem, the inverse operator

$$R(\lambda; L) := (\lambda I - L)^{-1} \quad (\lambda \in \rho(L)) \quad (3)$$

is always bounded; this operator is usually called resolvent operator of $L$ at $\lambda$.

### 1.1. Subdivision of the spectrum: The point spectrum, continuous spectrum and residual spectrum

Let $X$ be a Banach space over $K$ and $L \in B(X)$. Recall that a number $\lambda \in K$ is called eigenvalue of $L$ if the equation

$$Lx = \lambda x \quad (4)$$

has a nontrivial solution $x \in X$. Any such $x$ is then called eigenvector, and the set of all eigenvectors is a subspace of $X$ called eigenspace.

Throughout the following, we will call the set of eigenvalues

$$\sigma_p(L) := \{\lambda \in K : Lx = \lambda x \text{ for some } x \neq 0\} \quad (5)$$

We say that $\lambda \in K$ belongs to the continuous spectrum $\sigma_c(L)$ of $L$ if the resolvent operator (3) is defined on a dense subspace of $X$ and is unbounded. Furthermore, we say that $\lambda \in K$ belongs to the residual spectrum

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$\sigma_r(L)$ of $L$ if the resolvent operator (3) exists, but its domain of definition (i.e. the range $R(\lambda I - L)$ of $(\lambda I - L)$ is not densein $X$ ; in this case $R(\lambda; L)$ may be bounded or unbounded. Together with the point spectrum (5), these two subspectra form a disjoint subdivision

$$\sigma(L) = \sigma_p(L) \cup \sigma_r(L) \cup \sigma_c(L) \tag{6}$$

of the spectrum of $L$ .

## 1.2.The approximate point spectrum, defect spectrum and compression spectrum

Given a bounded linear operator $L$ in a Banach space $X$ , we call a sequence $(x_k)_k$ in $X$ a Weyl sequence for $L$ if $\|x_k\| = 1$ and $\|Lx_k\| \to 0$ as $k \to \infty$ .

In what follows, we call the set

$$\sigma_{ap}(L) := \{\lambda \in \mathrm{K} : \text{there is a Weyl sequence for } \lambda I - L\} \tag{7}$$

the approximate point spectrum of $L$ . Moreover, the subspectrum

$$\sigma_\delta(L) := \{\lambda \in \mathrm{K} : \lambda I - L \text{ is not surjective}\} \tag{8}$$

is called defect spectrum of $L$ .

The two subspectra (7) and (8) form a (not necessarily disjoint) subdivision

$$\sigma(L) = \sigma_{ap}(L) \cup \sigma_\delta(L) \tag{9}$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(L) := \left\{\lambda \in \mathrm{K} : \overline{R(\lambda I - L)} \neq \mathrm{X}\right\} \tag{10}$$

which is often called compression spectrum in the literature and which gives rise to another (not necessarily disjoint) decomposition

$$\sigma(L) = \sigma_{ap}(L) \cup \sigma_{co}(L) \tag{11}$$

of the spectrum. Clearly, $\sigma_p(L) \subseteq \sigma_{ap}(L)$ and $\sigma_{co}(L) \subseteq \sigma_\delta(L)$. Moreover, comparing these subspectra with those in (6) we note that

$$\sigma_r(L) = \sigma_{co}(L) \setminus \sigma_p(L) \tag{12}$$

and

$$\sigma_c(L) = \sigma(L) \setminus \left[\sigma_p(L) \cup \sigma_{co}(L)\right] \tag{13}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

**Proposition 1** ([6], Proposition 1.3). The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations:

(a) $\sigma(L^*) = \sigma(L)$,

(b) $\sigma_c(L^*) \subseteq \sigma_{ap}(L)$,

(c) $\sigma_{ap}(L^*) = \sigma_\delta(L)$,

(d) $\sigma_\delta(L^*) = \sigma_{ap}(L)$,

(e) $\sigma_p(L^*) = \sigma_{co}(L)$,

(f) $\sigma_{co}(L^*) \supseteq \sigma_p(L)$,

(g) $\sigma(L) = \sigma_{ap}(L) \cup \sigma_p(L^*) = \sigma_p(L) \cup \sigma_{ap}(L^*)$.

## 1.3.Goldberg's Classification of Spectrum

If $X$ is a Banach space, $B(X)$ denotes the collection of all bounded linear operators on $X$ and $T \in B(X)$, then there are three possibilities for $R(T)$, the range of $T$ :

(I) $R(T) = X$,

(II) $\overline{R(T)} = X$, but $R(T) \neq X$,

(III) $\overline{R(T)} \neq X$.

and three possibilities for $T^{-1}$ :

(1) $T^{-1}$ exists and continuous,

(2) $T^{-1}$ exists but discontinuous,

(3) $T^{-1}$ does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3$. If an operator is in state $III_2$ for example, then $\overline{R(T)} \neq X$ and $T^{-1}$ exist but is discontinuous (see [13]).

If $\lambda$ is a complex number such that $T = \lambda I - L \in I_1$ or $T = \lambda I - L \in II_1$ then $\lambda \in \rho(L, X)$. All scalar values of $\lambda$ not in $\rho(L, X)$ comprise the spectrum of $L$ . The further classification of $\sigma(L, X)$ gives rise to the fine spectrum of $L$ .That is, $\sigma(L, X)$ can be divided into the subsets $I_2\sigma(L, X) = \emptyset, I_3\sigma(L, X)$ $II_2\sigma(L, X), II_3\sigma(L, X), III_1\sigma(L, X), III_2\sigma(L, X), III_3\sigma(L, X)$. For example, if $T = \lambda I - L$ is in a given state, $III_2$ (say), then we write $\lambda \in III_2\sigma(L, X)$.

By the definitions given above, we can write following table

| Table 1 | | 1<br>$R(\lambda;L)$ exists and is bounded | 2<br>$R(\lambda;L)$ exists and is unbounded | 3<br>$R(\lambda;L)$ does not exists |
|---|---|---|---|---|
| *I* | $R(\lambda I-L)=X$ | $\lambda\in\rho(L)$ | – | $\lambda\in\sigma_p(L)$<br>$\lambda\in\sigma_{ap}(L)$ |
| *II* | $\overline{R(\lambda I-L)}=X$ | $\lambda\in\rho(L)$ | $\lambda\in\sigma_c(L)$<br>$\lambda\in\sigma_{ap}(L)$<br>$\lambda\in\sigma_\delta(L)$ | $\lambda\in\sigma_p(L)$<br>$\lambda\in\sigma_{ap}(L)$<br>$\lambda\in\sigma_\delta(L)$ |
| *III* | $\overline{R(\lambda I-L)}\neq X$ | $\lambda\in\sigma_r(L)$<br>$\lambda\in\sigma_\delta(L)$<br>$\lambda\in\sigma_{co}(L)$ | $\lambda\in\sigma_r(L)$<br>$\lambda\in\sigma_{ap}(L)$<br>$\lambda\in\sigma_\delta(L)$<br>$\lambda\in\sigma_{co}(L)$ | $\lambda\in\sigma_p(L)$<br>$\lambda\in\sigma_{ap}(L)$<br>$\lambda\in\sigma_\delta(L)$<br>$\lambda\in\sigma_{co}(L)$ |

Let $c_0; c; \ell^p; bv; bv_0$ denote the space of all null sequences; convergent sequences; sequences such that $\sum_k |x_k| < \infty$; sequences such that $\sum_k |x_{k+1} - x_k| < \infty$; $bv_0 := bv \cap c_0$. respectively.

An infinite matrix $A$ is said to be conservative if it is a selfmap of $c$, the space of convergent sequences. Necessary and sufficient conditions for $A$ to be conservative are the well-known Kojima-Schur conditions; i.e.,

(i) $\|A\| := \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$;

(ii) $\lim_n a_{nk} =: \alpha_k$, exists for each $k$, and

(iii) $t := \lim_n \sum a_{nk} < \infty$ exists.

Associated with each conservative matrix $A$ is a function $\chi$ defined by $\chi(A) = t - \sum \alpha_k$. If $\chi(A) \neq 0$, $A$ is called coregular, and, if $\chi(A) = 0$ then $A$ is called conull. A matrix $A = (a_{nk})$ is said to be regular if $\lim_A x = \lim x$ for each $x \in c$. If $\alpha_k = 0$ for each $k$ and $t = 1$ in (iii), then the operator $A$ is called regular.

The spectrum and fine spectrum of several operators on some sequence spaces have been investigated recently. For example: [1]-[5], [7], [8] and [11]. Now we define factorable matrix as follows.

A lower triangular matrix $A$ is said to be factorable if $a_{nk} = a_n b_k$ for all $0 \le k \le n$.

The choices $a_n = 1/(n+1)$ and each $b_k = 1, a_n = (n+1)^{-p}$ $(p > 1)$ and each $b_k = 1, a_n = a_n$ and each $b_k = 1$, and $a_n = P_n$, $b_k = p_k$, where $\{p_k\}$ is a nonnegative sequence with $p_0 > 0$, $P_n := \sum_{k=0}^{n} p_k$, generate $C$ (the Cesáro matrix of order one), the p-Cesáro matrices and terraced matrices defined by Rhaly, and the weighted mean matrices, respectively.

B. E. Rhoades and M. Yildirim have calculated spectrum and fine spectrum of factorable matrices on $c$, $\ell^p$ and $c_0$ in [23], [24] and [25]. It is the purpose of this paper to determine the approximate point spectra, defect spectra and compression spectra of factorable matrices over $c_0$. As corollaries we obtain the known corresending results for weighted mean matrices, teraced matrices and $C$.

In previous work B. E. Rhoades determined the fine spectra of certain classes of weighted mean matrices, considered as bounded linear operators over $c$, $c_0$, $\ell^p$, and $bv_0$ (See, e.g., [10], [20], [21], [22].) M. Yildirim has considered spectral questions for certain classes of Rhaly matrices (See, e.g. [15], [19], [26], [27], [28], [29], [30]). The Spectrum of $C$, on various spaces, has been computed in [9], [12], [14], [16], [17], [18], [31]. For many of our results we shall consider factorable matrices which belong to $F := \{A : A$ is a factorable lower triangular matrix with nonnegative entries and $0 \le a_n b_n \le 1$ diagonal entries and with at most a finite number of zeros on the main diagonal$\}$. Define $\gamma = \lim a_n b_n$, $c_n = a_n b_n$, $E := \left\{ \lambda = c_n : 0 \le \lambda \le \frac{\gamma}{2-\gamma},\ n \ge 0 \right\}$ and $S := \overline{\{c_n : n \ge 0\}}$.

**Theorem 1.** *Let $A \in F$ be regular such that $\gamma = \lim c_n$ exists and is less than 1 and $c_n \ge \gamma$ for all sufficiently large $n$ ,then*

$$\sigma_{ap}(A, c_0) = \left\{ \lambda : \left| \lambda - \frac{1}{2-\gamma} \right| = \frac{1-\gamma}{2-\gamma} \right\} \cup E.$$

***Proof**. If $A \in F$ be regular such that $\gamma = \lim c_n$ exists and is less than 1 and $c_n \ge \gamma$ for all sufficiently large $n$ ,then, $I_3\sigma(A, c_0) = \emptyset$ and $III_2\sigma(A, c_0) = \emptyset$ follow from [25] Corollary 2.1, Corollary 3.1 and Theorem 3.2.-3.5. Since $\sigma_{ap}(A, c_0) = \sigma(A, c_0) \setminus III_1\sigma(A, c_0)$,*

$$\begin{aligned}
\sigma_{ap}(A, c_0) &= \left[ \left\{ \lambda : \left| \lambda - \frac{1}{2-\gamma} \right| \le \frac{1-\gamma}{2-\gamma} \right\} \cup S \right] \\
&\setminus \left[ \left\{ \lambda : \left| \lambda - \frac{1}{2-\gamma} \right| < \frac{1-\gamma}{2-\gamma} \right\} \setminus S \right] \\
&\cup \left\{ \lambda = c_n : \frac{\gamma}{2-\gamma} < \lambda < 1 \right\} \\
&= \left\{ \lambda : \left| \lambda - \frac{1}{2-\gamma} \right| = \frac{1-\gamma}{2-\gamma} \right\} \cup E,
\end{aligned}$$

*is obvious from [25] Corollary 2.1, Corollary 3.1 and Theorem 3.2.* ¤

**Theorem 2**. *Let* $A \in F$ *be regular such that* $\gamma = \lim c_n$ *exists and is less than 1 and* $c_n \geq \gamma$ *for all sufficiently large n ,then*

$$\sigma_\delta(A,c_0) = \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| \leq \frac{1-\gamma}{2-\gamma}\right\} \cup S.$$

***Proof***. *If* $A \in F$ *be regular such that* $\gamma = \lim c_n$ *exists and is less than 1 and* $c_n \geq \gamma$ *sufficiently large n ,then,* $I_3\sigma(A,c_0) = \emptyset$ *and* $III_2\sigma(A,c_0) = \emptyset$ *follow from [25] Corollary 2.1, Corollary 3.1 and Theorem 3.2.-3.5. Since* $\sigma_\delta(A,c_0) = \sigma(A,c_0) \setminus I_3\sigma(A,c_0)$, *and* $I_3\sigma(A,c_0) = \emptyset$. *the equality*

$$\sigma_\delta(A,c_0) = \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| \leq \frac{1-\gamma}{2-\gamma}\right\} \cup S.$$

*is true.* ¤

**Theorem 3**. *Let* $A \in F$ *be regular such that* $\gamma = \lim c_n$ *exists and is less than 1 and* $c_n \geq \gamma$ *for all sufficiently large n ,then*

$$\sigma_{co}(A,c_0) = \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| < \frac{1-\gamma}{2-\gamma}\right\} \cup S.$$

***Proof***. *If* $A \in F$ *be regular such that* $\gamma = \lim c_n$ *exists and is less than 1 and* $c_n \geq \gamma$ *for all sufficiently large n ,then,* $I_3\sigma(A,c_0) = \emptyset$ *and* $III_2\sigma(A,c_0) = \emptyset$ *follow from [25] Corollary 2.1, Corollary 3.1 and Theorem 3.2.-3.5. From table 1*

$$\sigma_{co}(A,c_0) = III_1\sigma(A,c_0) \cup III_2\sigma(A,c_0) \cup III_3\sigma(A,c_0)$$

*Since* $III_2\sigma(A,c_0) = \emptyset$ *then from [25] Corollary 2.1 Corollary 3.1 and Theorem 3.2-3.3, we get*

$$\begin{aligned}\sigma_{co}(A,c_0) &= \left[\left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| < \frac{1-\gamma}{2-\gamma}\right\} \cup S\right] \\ &\quad \cup \left\{\lambda = c_n : \frac{\gamma}{2-\gamma} < \lambda < 1\right\} \cup \left\{\lambda = c_n : 0 \leq \lambda \leq \frac{\gamma}{2-\gamma}\right\} \\ &= \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| < \frac{1-\gamma}{2-\gamma}\right\} \cup S.\end{aligned}$$

¤

The following corollaries can be obtained by Proposition 1.

**Corollary 1**. The following equalities are true;

*(a)* $$\sigma_{ap}(A^*, \ell^1) = \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| \leq \frac{1-\gamma}{2-\gamma}\right\} \cup S,$$

*(b)* $$\sigma_\delta(A^*, \ell^1) = \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| = \frac{1-\gamma}{2-\gamma}\right\} \cup E,$$

*(c)* $$\sigma_p(A^*, \ell^1) = \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| < \frac{1-\gamma}{2-\gamma}\right\} \cup S,$$

*where* $A^*$ *denotes adjoint of* $A$.

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