Gazi University Journal of Science
GU J Sci
24(2):241-248 (2011)

www.gujs.org

A New Method for Evaulation of Bivariate Compound Poisson Distribution for Aggregate Claims

Gamze ÖZEL[1♠]

[1] Hacettepe University, Faculty of Science, Department of Statistics, 06800, Beytepe, Ankara, TURKEY

Received: 25.05.2010 Accepted: 14.10.2010

ABSTRACT

In this paper, a bivariate compound Poisson model is proposed for calculating the aggregate claims distribution in a discrete framework and the probabilistic characteristics of this model, such as the joint probability function, joint probability generating function, correlation coefficient and covariance are derived. Then, an algorithm is prepared in Oracle database to obtain the probabilities quickly. By means of prepared algorithm some numerical examples are also given to illustrate the usage of the bivariate compound Poisson model.

Key Words: *Bivariate compound Poisson distribution, correlation coefficient, joint probability generating function, insurance, claim severity.*

1. INTRODUCTION

A central problem in risk theory is the modeling of the probability distribution for the total claims. The total claims distribution and its components, the frequency and severity distributions are used to compute ruin probabilities and to provide other information of interest to decision makers [1].

For most organizations, total claims arise from exposure to multiple perils, each of which typically can produce more than one type of claim. However, there are practical situations for which this assumption is not appropriate. For example, weather conditions can affect the frequency of both fires and automobile accidents. Ignoring such dependencies can lead to serious underestimates in loss statistics used for decision making [2]. The purpose of this article is to present a methodology for dealing with this problem through the use of single claim frequency and bivariate claims severity distributions.

The risk theory describes the computation of bivariate aggregate distributions. Sundt [3] extended Panjer recursions to multiple dimesions. Homer and Clark [4] described bivarite examples using two-dimesional discrete Fourier transforms. Walhin [5] obtained an application of two-dimensional Panjer recursions. Like their univariate counterparts, these methods work best when the expected claim counts are small due to computer memory constraints [6].

In this paper a methodology for the evaluation of the bivariate compound Poisson distribution is obtained when the claim count distribution is the Poisson distribution and claim severities are discrete. The paper is organized as follows. In Section 2, the univariate compound Poisson distribution and its key properties are given. In Section 3, the joint probability function, correlation coefficient and covariance of the bivariate compound Poisson distribution are derived and an algorithm is prepared in Oracle database to compute probabilities quickly. This

♠Corresponding author, e-mail: gamzeozl@hacettepe.edu.tr

algorithm can be obtained from author upon request. In Section 4, some numerical examples are given by means of prepared algorithms in Oracle database. Some concluding remarks are given in Section 5.

2. UNIVARIATE COMPOUND POISSON MODEL

Let N denote the random variable representing the number of claims occuring in an insurance portfolio within a given period of time., i.e. the non-negative, integer-valued random variable counting the number of claims occuring in an insurance portfolio. Let $\{X_i, i = 1, 2, ...\}$ denote the severity random variables, i.e. the random variables representing the individual claim amount. It is assumed that they are discrete, positive, mutually independent and independent of the counting variable N. Then, the aggregate claims amount is given by the following variable

$$S = \sum_{i=1}^{N} X_i. \tag{1}$$

If N is Poisson distributed random variable with parameter λ, then the total number of claims X follows a (discrete) compound Poisson distribution. This model is used to describe the aggregate claims for a single line or book of business [6].

In Equation (1), $E(X)$ and $V(X)$ are the common mean and variance of the claim severities $\{X_i, i = 1, 2, ...\}$, then the expected value of aggregate claims is the product of the expected of claim severity and the expected number of claims

$$E(S) = \lambda E(X), \tag{2}$$

while the variance of aggregate claims is the sum of two components where first is attributed to the variability of claim severity and the other to the variability of the number of claims

$$V(S) = \lambda V(X) + \lambda E(X)^2. \tag{3}$$

If X_i, $i = 1, 2, ...,$ are discrete random variables with probability $P(X_i = j) = p_j$, $j = 0, 1, 2, ...$ in Equation (1), then the (defective) probability function of S is given by

$$p_S(s) = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} P(X_1 + X_2 + ... + X_n = s / N = n), s = 0,1,2,... \tag{4}$$

However, the explicit evaluation of probability $p_S(s)$ in Eq. (4) is mostly impossible because of the complicated nature of convolutions [7]. It occurs in underflow problems which are not always easy to overcome and which therefore further restrict their applicability [3]. Thus, they can be applied only in some practical circumtances or in an approximated way.

Sundt [8] derived the following recursion for the compound Poisson distribution

$$p_S(0) = e^{-\lambda[1-P(X=0)]},$$

$$p_S(s) = \lambda \sum_{j=1}^{s} \frac{j}{s} p_X(j) p_S(s-j), \quad s = 1,2,... \tag{5}$$

More generally, for large n and x, Eq. (5) may be difficult to use because of the high order of convolutions involved, which is for the same reason which had motivated recursive evaluation of Eq. (4).

A very useful recent review of literature on the probability function of S is Ozel and Inal [9] using the probability generating function

$$g_S(z) = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} [g_X(z)]^n$$

$$= e^{\lambda(p_0 + p_1 z + ... - 1)}$$

$$= e^{-\lambda(1-p_0)} e^{\lambda(p_1 z + p_2 z^2 + ...)}$$

$$= e^{-\lambda(1-p_0)} e^{\lambda_1 z + \lambda_2 z^2 + ...} \tag{6}$$

where $\lambda_j = \lambda p_j$, $j = 0, 1, 2, ...$ and the common probability generating function of the claims severity X_i, $i = 1, 2, ...,$ is

$$g_X(z) = p_0 + p_1 z + p_2 z^2 + ...$$

The probability function of S is derived by [9] as follows

$$P(S=0) = e^{-\lambda(1-p_0)},$$

$$P(S=1) = e^{-\lambda(1-p_0)} \frac{\lambda_1}{1!},$$

$$P(S=2) = e^{-\lambda(1-p_0)} \left[\frac{\lambda_1^2}{2!} + \frac{\lambda_2}{1!} \right],$$

$$P(S=3) = e^{-\lambda(1-p_0)} \left[\frac{\lambda_1^3}{3!} + \frac{\lambda_1 \lambda_2}{1!1!} + \frac{\lambda_3}{1!} \right],$$

$$P(S=4) = e^{-\lambda(1-p_0)} \left[\frac{\lambda_1^4}{4!} + \frac{\lambda_1^2 \lambda_2}{2!1!} + \frac{\lambda_1 \lambda_3}{1!1!} + \frac{\lambda_2^2}{2!} + \frac{\lambda_4}{1!} \right],$$

$$P(S=5) = e^{-\lambda(1-p_0)} \left[\frac{\lambda_1^5}{5!} + \frac{\lambda_1^3 \lambda_2}{3!1!} + \frac{\lambda_1^2 \lambda_3}{2!1!} + \frac{\lambda_1 \lambda_2^2}{2!1!} + \frac{\lambda_1 \lambda_4}{1!1!} + \frac{\lambda_2 \lambda_3}{1!1!} + \frac{\lambda_5}{1!} \right],$$

$$\vdots \tag{7}$$

According to above probabilities for $s = 1, 2, ...$, the right-hand side terms depend on how s can be partitioned into different forms using integers $1, 2, ..., m$. For example, if $s = 5$, it is partitioned in 7 ways and all the partitions of 5 are {1,1,1,1,1}, {1,1,1,2}, {1,2,2}, {1,1,3}, {2,3}, {1,4}, {5}. Let us point out that an application of Equation (7) is given [10] where N is the number of main shocks, X_i, $i = 1, 2, ...,$ is the number of aftershocks of *i*th main shock and S is the total number of aftershocks. Furthermore, a special form of Equation (7) is derived by [11] when X_i, $i = 1, 2, ...,$ are geometric distributed random variables.

3. BIVARIATE COMPOUND POISSON MODEL

Some probability distributions that may be appropriate to characterize the single claim count distribution and bivariate severity components of the model are presented in this section. Over the past two decade there has been an increasing interest in bivariate discrete probability distributions and many forms of these distributions have been studied [12, 13]. The bivariate Poisson distribution has been constructed by [14] using three independent Poisson variates $\lambda_1, \lambda_2, \lambda_3 > 0$ and the joint probability function $p_{n_1,n_2} = P(N_1 = n_1, N_2 = n_2)$ is given by

$$p_{n_1,n_2} = e^{-(\lambda_1+\lambda_2+\lambda_3)} \sum_{k=0}^{\min(n_1,n_2)} \frac{\lambda_1^{n_1-k}\lambda_2^{n_2-k}\lambda_3^{k}}{(n_1-k)!(n_2-k)!k!}, \qquad (8)$$

Bivarite compound distributions can be used to model a book of business which contains a single claim count distribution and bivariate claims severities [15]. In most actuarial literature related to risk theory, the assumption of independence between classes of business in an insurance book of business is made. In practice, however, there are situations in which this assumption is not verified. In the case of a catastrophe such as an earthquake for example, the damages covered by homeowners and private passenger automobile insurance can not be considered independent [2]. In this situation bivariate compound distributions can be used to model aggregate loss claim amount. Bivariate compound Poisson distribution is useful for when the claim count is Poisson distributed and the claim size distribution is bivariate. The bivariate compound Poisson distribution can be defined as follows:

Let the claim size pairs X_i, Y_i, $i = 1, 2, ...,$ are seperately i.i.d., independent from each other and also independent from the number of claims N. Then, the aggregate loss claims amount are given by

$$\left(S_1 = \sum_{i=1}^{N} X_i, S_2 = \sum_{i=1}^{N} Y_i \right) \qquad (9)$$

If N is a Poisson distributed random variable, then the aggregate claims for a single line or book of business S_1 and S_2 follow a bivariate compound Poisson distribution [6]. However, explicit joint probability function of the bivariate compound Poisson model has not been obtained yet [16].

In this section joint probability function of S_1 and S_2 is obtained. Let N is a Poisson distributed random variable with parameter λ and let X_i, Y_i, $i = 1, 2, ...$ be i.i.d. discrete random variables with the probabilities $P(X_i = j) = p_j$, $j = 0, 1, 2, ..., m$ and $P(Y_i = k) = q_k$, $k = 0, 1, 2, ..., r$. Then, the joint probability function of S_1 and S_2 is given by the following expression

$$p_{S_1,S_2}(s_1, s_2) = P(S_1 = s_1, S_2 = s_2)$$
$$= P\left(\sum_{i=1}^{N} X_i = s_1, \sum_{i=1}^{N} Y_i = s_2 \right)$$
$$= \sum_{n}^{\infty} P\left(\sum_{i=1}^{n} X_i = s_1, \sum_{i=1}^{n} Y_i = s_2 \right) P(N = n)$$

$$= P(N = 0) + P(N = 1)P(X_1 = s_1, Y_1 = s_2) + P(N = 2)P(X_1 + X_2 = s_1, Y_1 + Y_2 = s_2) + P(N = 3)P(X_1 + X_2 + X_3 = s_1, Y_1 + Y_2 + Y_3 = s_2) + ...$$

Since the random variables X_i, Y_i, $i = 1, 2, ...$ are independent, we have

$$= P(N = 0) + P(N = 1)P(X_1 = s_1)P(Y_1 = s_2) + P(N = 2)P(X_1 + X_2 = s_1)\,P(Y_1 + Y_2 = s_2) + P(N = 3)P(X_1 + X_2 + X_3 = s_1)P(Y_1 + Y_2 + Y_3 = s_2) + ... \qquad (10)$$

The joint probability function, given in Equation (10), contains a summation from 0 to ∞ and it is not suitable to obtain probabilities easily. Thus, this formula is very time-consuming and it can be applied only in some practical circumtances or in approximated way [16].

We first compute the joint probability generating function S_1 and S_2 as follows

$$g_{S_1,S_2}(z_1, z_2) = \sum_{s_1}^{\infty}\sum_{s_2}^{\infty} P\left(\sum_{i=1}^{N} X_i = s_1, \sum_{i=1}^{N} Y_i = s_2 \right) z_1^{s_1} z_2^{s_2}$$
$$= \sum_{s_1}^{\infty}\sum_{s_2}^{\infty}\sum_{n=0}^{\infty} P\left(\sum_{i=1}^{n} X_i = s_1, \sum_{i=1}^{n} Y_i = s_2 \right) P(N = n) z_1^{s_1} z_2^{s_2}$$
$$= \sum_{s_1}^{\infty}\sum_{s_2}^{\infty} [P(S_1 = s_1, S_2 = s_2)/P(N = 0)] P(N = 0) z_1^{s_1} z_2^{s_2}$$
$$+ \sum_{s_1}^{\infty}\sum_{s_2}^{\infty}\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} X_i = s_1, \sum_{i=1}^{n} Y_i = s_2 \right) P(N = n) z_1^{s_1} z_2^{s_2}$$
$$= [P(S_1 = 0, S_2 = 0)/P(N = 0)] P(N = 0)$$
$$+ \sum_{x}^{\infty}\sum_{y}^{\infty}\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} X_i = s_1, \sum_{i=1}^{n} Y_i = s_2 \right) P(N = n) z_1^{s_1} z_2^{s_2}$$

Since the random variables X_i, Y_i, $i = 1, 2, ...$ are independent, we have

$$= P(N = 0) + P(N = 1)\sum_{s_1}^{\infty}\sum_{s_2}^{\infty} P(X_1 = s_1)P(Y_1 = s_2) z_1^{s_1} z_2^{s_2}$$
$$+ P(N = 2)\sum_{s_1}^{\infty}\sum_{s_2}^{\infty} P(X_1 + X_2 = s_1)\,P(Y_1 + Y_2 = s_2) z_1^{s_1} z_2^{s_2}$$
$$+ P(N = 3)\sum_{x}^{\infty}\sum_{y}^{\infty} P(X_1 + X_2 + X_3 = s_1)P(Y_1 + Y_2 + Y_3 = s_2) z_1^{s_1} z_2^{s_2} + ...$$

$$= P(N = 0) + P(N = 1) g_{X_1}(z_1) g_{Y_1}(z_2) + P(N = 2) g_{X_1+X_2}(z_1)\, g_{Y_1+Y_2}(z_2) + P(N = 3) g_{X_1+X_2+X_3}(z_1) g_{Y_1+Y_2+Y_3}(z_2) + ...$$

$$= P(N=0) + P(N=1) g_X(z_1) g_Y(z_2) + P(N=2) [g_X(z_1)]^2 [g_Y(z_2)]^2 + \ldots \tag{11}$$

where $g_X(z_1)$, $g_Y(z_2)$ are the common probability generating functions of the random variables X_i, Y_i, $i = 1, 2, \ldots,$ respectively.

Since N has a Poisson distribution with parameter λ, using Equation (11), it is more convenient to deal with the joint probability generating function

$$g_{S_1,S_2}(z_1,z_2) = g_N(g_X(z_1) g_Y(z_2))$$

$$= e^{\lambda[g_X(z_1) g_Y(z_2) - 1]}$$

$$= e^{-\lambda} e^{\lambda[g_X(z_1) g_Y(z_2)]}$$

$$= e^{-\lambda} e^{\lambda(p_0 + p_1 z_1 + \ldots + p_m z_1^m)(q_0 + q_1 z_2 + \ldots + q_r z_2^r)}$$

$$= e^{-\lambda} e^{\lambda(p_0 q_0 + p_0 q_1 z_2 + \ldots + p_0 q_r z_2^r + p_1 q_0 z_1 + p_1 q_1 z_1 z_2)}$$

$$e^{\lambda(p_1 q_r z_1 z_2^r + p_m q_0 z_1^m + \ldots + p_m q_r z_1^m z_2^r)} \tag{12}$$

where the probability generating function of N is

$$g_N(z) = \sum_{i=0}^{\infty} P(N=i) z^i = e^{\lambda(z-1)} \tag{13}$$

The joint probability generating function in Equation (12) can be differentiated any number of times with respect to s_1 and s_2 and evaluated at $(0,0)$ yielding

$$P(S_1 = 0, S_2 = 0) = g_{S_1,S_2}(0,0),$$

$$P(S_1 = s_1, S_2 = s_2) = \frac{\dfrac{\partial^{s_1+s_2} g_{S_1,S_2}(z_1,z_2)}{\partial z_1^{s_1} \partial z_2^{s_2}}\Big|_{z_1=z_2=0}}{s_1! s_2!},$$

$$s_1, s_2 = 0, 1, 2, \ldots \tag{14}$$

Differentiating the joint probability generating function given by Equation (12) and substituting in Equation (14)

$$P(S_1 = 0, S_2 = 0) = e^{-\lambda} e^{\lambda p_0 q_0} = e^{-\lambda(1-p_0 q_0)},$$

$$P(S_1 = 0, S_2 = 1) = e^{-\lambda(1-p_0 q_0)} \left[q_1 \left(\lambda \frac{p_0}{1!} \right) \right],$$

$$P(S_1 = 0, S_2 = 2) = e^{-\lambda(1-p_0 q_0)} \left[q_1^2 \left(\lambda^2 \frac{p_0^2}{2!} \right) + q_2 \left(\lambda \frac{p_0}{1!} \right) \right],$$

$$P(S_1 = 0, S_2 = 3) = e^{-\lambda(1-p_0 q_0)} \left[q_1^3 \left(\lambda^3 \frac{p_0^3}{3!} \right) + q_1 q_2 \left(\lambda^2 \frac{p_0^2}{2!} \right) + q_3 \left(\lambda \frac{p_0}{1!} \right) \right],$$

$$P(S_1 = 1, S_2 = 0) = e^{-\lambda(1-p_0 q_0)} \left[p_1 \left(\lambda \frac{q_0}{1!} \right) \right],$$

$$P(S_1 = 2, S_2 = 0) = e^{-\lambda(1-p_0 q_0)} \left[p_1^2 \left(\lambda^2 \frac{q_0^2}{2!} \right) + p_2 \left(\lambda \frac{q_0}{1!} \right) \right],$$

$$P(S_1 = 3, S_2 = 0) = e^{-\lambda(1-p_0 q_0)} \left[p_1^3 \left(\lambda^3 \frac{q_0^3}{3!} \right) + p_1 p_2 \left(\lambda^2 \frac{q_0^2}{2!} \right) + p_3 \left(\lambda \frac{q_0}{1!} \right) \right],$$

$$P(S_1 = 1, S_2 = 1) = e^{-\lambda(1-p_0 q_0)} \left[p_1 q_1 \left(\lambda^2 \frac{p_0}{1!} \frac{q_0}{1!} + \lambda \right) \right],$$

$$P(S_1 = 1, S_2 = 1) = e^{-\lambda(1-p_0 q_0)} \left[p_1 q_1 \left(\lambda^2 \frac{p_0}{1!} \frac{q_0}{1!} + \lambda \right) \right],$$

$$P(S_1 = 1, S_2 = 2) = e^{-\lambda(1-p_0 q_0)} \left[p_1 q_1^2 \left(\lambda^3 \frac{p_0^2}{2!} \frac{q_0}{1!} + \lambda^2 \frac{p_0}{1!} \right) + p_1 q_2 \left(\lambda^2 \frac{p_0}{1!} \frac{q_0}{1!} + \lambda \right) \right],$$

$$P(S_1 = 1, S_2 = 3) = e^{-\lambda(1-p_0 q_0)} \left[p_1 q_1^3 \left(\lambda^4 \frac{p_0^3}{3!} \frac{q_0}{1!} + \lambda^3 \frac{p_0^2}{2!} \right) + p_1 q_1 q_2 \left(\lambda^3 \frac{p_0^2}{2!} \frac{q_0}{1!} + \lambda^2 \frac{p_0}{1!} \right) + p_1 q_3 \left(\lambda^2 \frac{p_0}{1!} \frac{q_0}{1!} + \lambda \right) \right],$$

$$P(S_1 = 2, S_2 = 1) = e^{-\lambda(1-p_0 q_0)} \left[p_1^2 q_1 \left(\lambda^3 \frac{p_0}{1!} \frac{q_0^2}{2!} + \lambda^2 \frac{q_0}{1!} \right) + p_2 q_1 \left(\lambda^2 \frac{p_0}{1!} \frac{q_0}{1!} + \lambda \right) \right],$$

$$P(S_1 = 2, S_2 = 2) = e^{-\lambda(1-p_0 q_0)} \left[p_1^2 q_1^2 \left(\lambda^4 \frac{p_0^2}{2!} \frac{q_0^2}{2!} + \lambda^3 \frac{p_0}{1!} \frac{q_0}{1!} + \lambda^2 \right) + p_1^2 q_2 \left(\lambda^3 \frac{p_0}{1!} \frac{q_0^2}{2!} + \lambda^2 \frac{q_0}{1!} \right) + p_2 q_1^2 \left(\lambda^3 \frac{p_0^2}{2!} \frac{q_0}{1!} + \lambda^2 \frac{p_0}{1!} \right) + p_2 q_2 \left(\lambda^2 \frac{p_0}{1!} \frac{q_0}{1!} + \lambda \right) \right],$$

$$\vdots \tag{15}$$

According to above probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 1, 2, 3, \ldots,$ the right-hand side terms depend on how s_1 and s_2 can be partitioned into different forms using integers $1, 2, \ldots$ For example, if $(s_1 = 1, s_2 = 3)$, they are partitioned in three ways and all the partitions of $(s_1 = 1, s_2 = 3)$ are $\{[(1),(1,1,1)],[(1),(1,2)],[(1),(3)]\}$. Furthermore, the denominator of right-hand side terms, p_0 and q_0, are suitable to these partitions. Using these properties, an algorithm is prepared in Oracle database for the joint probability function of the bivariate compound Poisson distribution. The major advantage of this algorithm is computation time. One can obtain the probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 1, 2, 3, \ldots,$ within 0.05s on a 32 bit machine using the algorithm.

Note that if the random variables X_i, Y_i, $i = 1, 2, \ldots,$ have infinite values, the joint probability function given in Equation (15) can also be used. Since the probabilities

$P(X_i = j) = p_j$, $j = 0,1,2,...$, and $P(Y_i = k) = q_k$, $k = 0,1,2,...$, decrease with increasing values of j and k. For this reason, j and k can be taken as finite values.

In the study of properties of random variables, especially discrete random variables, an important role is played by the moment characteristics of distributions. As far as we know, correlation coefficient and covariance of the bivariate compound Poisson distribution have never been investigated before [16]. Now consider the correlation coefficient and covariance of the random variables S_1 and S_2. We start with finding the expected value of product S_1S_2. The probability generating function $g_{S_1,S_2}(z_1,z_2)$ makes it possible to compute $E(S_1S_2)$ by differentiating

$$E[S_1{}^r S_2{}^s] = \frac{\frac{\partial^{r+s} g_{S_1,S_2}(z_1,z_2)}{\partial z_1^r \partial z_2^s}\Big|_{z_1=z_2=1}}{r!s!}, \quad r,s = 0,1,2,... \tag{16}$$

Differentiating the joint probability generating function given by Equation (12), we have

$$\frac{\partial^2 g_{S_1,S_2}(z_1,z_2)}{\partial z_1 \partial z_2} = e^{\lambda[(p_0+p_1z_1+...+p_m z_1^m)(q_0+q_1z_2+...+q_r z_2^r)-1]}$$
$$[\lambda(p_1 + 2p_2z_1 + ... + mp_m z_1^{m-1})(q_1 + 2q_2z_2 + ... + rq_r z_2^{r-1})]$$
$$\{\lambda[(p_0 + p_1z_1 + ... + p_m z_1^m)(q_0 + q_1z_2 + ... + q_r z_2^r) + 1]\}$$

and substituting in Equation (16), $E(S_1S_2)$ has the form

$$E(S_1S_2) = \frac{\partial^{r+s} g_{S_1,S_2}(z_1,z_2)}{\partial z_1^r \partial z_2^s}\Big|_{z_1=z_2=1}$$

$$= \lambda(\lambda+1)[(p_1 + 2p_2 + ... + mp_m)(q_1 + 2q_2 + ... + rq_r)]$$

$$= \lambda(\lambda+1)\left(\sum_{j=1}^{m} jp_j\right)\left(\sum_{k=1}^{r} kq_k\right). \tag{17}$$

Since $E(X) = \left(\sum_{j=1}^{m} jp_j\right)$ and $E(Y) = \left(\sum_{k=1}^{r} kq_k\right)$, the right-hand side of Equation (17) takes the form

$$E(S_1S_2) = \lambda(\lambda+1)E(X)E(Y) \tag{18}$$

The covariance of S_1 and S_2 is obtained using Equations (2) and (18)

$$Cov(S_1,S_2) = E(S_1S_2) - E(S_1)E(S_2)$$
$$= \lambda(\lambda+1)E(X)E(Y) - [\lambda E(X)][\lambda E(Y)]$$
$$= \lambda E(X)E(Y). \tag{19}$$

Let σ_{S_1} and σ_{S_2} be standart deviations of the random variables S_1 and S_2, then the correlation coefficient of S_1 and S_2 is obtained from Equations (2) and (19) as follows

$$\rho = Cor(S_1,S_2) = \frac{Cov(S_1,S_2)}{\sigma_{S_1}\sigma_{S_2}}$$
$$= \frac{\lambda E(X)E(Y)}{\sqrt{\lambda[V(X)+[E(X)]^2]\lambda[V(Y)+[E(Y)]^2]}}$$
$$= \frac{E(X)E(Y)}{\sqrt{[V(X)+[E(X)]^2][V(Y)+[E(Y)]^2]}}.$$

This implies that $\rho = 0$ is a necessary condition for S_1 and S_2 to be independent. Also, $\rho = 1$ if and only if S_1 and S_2 are linearly dependent.

4. NUMERICAL EXAMPLES

The purpose of this section is to provide numerical illustrations of the methodology discussed above by using several discrete distribution for the random variables X_i, Y_i, $i = 1,2,...$ The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0,1,2,...$ are presented in Table 1 which are calculated from Eq. (15). In these calculations, the claim amount variables X_i, $i = 1,2,...,$ have a geometric distribution with parameter $\theta_1 = 0.18$ and Y_i, $i = 1,2,...,$ have a geometric distribution with parameter $\theta_2 = 0.13$; the claim count variable N has a Poisson distribution with parameter $\lambda = 0.2$.

Table 2 presents $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0,1,2,...$ where the claim amount X_i, $i = 1,2,3,...,$ are Poisson distributed with parameter $\mu_1 = 5$ and Y_i, $i = 1,2,...,$ are Poisson distributed with parameter $\mu_2 = 2$; the claim count N is a Poisson random variable with parameter $\lambda = 0.9$.

The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0,1,2,...$ is computed from Equation (15) and presented in Table 3 where the claim amount variables X_i, $i = 1,2,3,...,$ are binomial distributed with parameters $(m_1 = 15, \alpha_1 = 0.15)$ and Y_i, $i = 1,2,...,$ are binomial distributed with parameters $(m_2 = 10, \alpha_2 = 0.4)$; the claim count N is a Poisson random variable with parameter $\lambda = 0.4$.

The expected values and variance of the random variables N, S_1, S_2; expected value of the product S_1S_2; covariance and corelation coefficient of S_1, S_2 are given in Table 4. The claim count N is a Poisson random variable with several values of λ, the claim amount variables X_i, $i = 1,2,3,...,$ are geometric distributed with parameter θ_1 and Y_i, $i = 1,2,...,$ are geometric distributed with parameter θ_2; X_i, $i = 1,2,3,...,$ are Poisson distributed with parameter μ_1 and Y_i, $i = 1,2,...,$ are Poisson distributed with parameter μ_2;

X_i, $i = 1, 2, 3, \ldots$, are binomial distributed with parameters (m_1, α_1) and Y_i, $i = 1, 2, \ldots$, are binomial distributed with parameters (m_2, α_2).

Table 1. The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$ with the parameters $\lambda = 0.2$, $\theta_1 = 0.18$, $\theta_2 = 0.13$.

s_2	s_1									
	0	1	2	3	4	5	6	7	8	9
0	0.82246	0.00309	0.00001	0.00211	0.00175	0.00039	0.00008	0.00006	0.0000090	0.0000042
1	0.00326	0.00270	0.00224	0.00185	0.00154	0.00038	0.00006	0.00005	0.0000070	0.0000032
2	0.00001	0.00236	0.00196	0.00163	0.00134	0.00037	0.00005	0.00003	0.0000051	0.0000022
3	0.00248	0.00206	0.00172	0.00143	0.00118	0.00036	0.00004	0.00002	0.0000034	0.0000012
4	0.00216	0.00180	0.00149	0.00125	0.00001	0.00034	0.00003	0.00002	0.0000030	0.0000010
5	0.00021	0.00021	0.00053	0.00046	0.00020	0.00031	0.00002	0.00001	0.0000025	0.0000008
6	0.00019	0.00019	0.00041	0.00043	0.00013	0.00022	0.00001	0.00001	0.0000020	0.0000007
7	0.00015	0.00017	0.00035	0.00041	0.00013	0.00010	0.00001	0.00001	0.0000016	0.0000006
8	0.00013	0.00016	0.00026	0.00038	0.00012	0.00009	0.00001	0.00001	0.0000014	0.0000005
9	0.00009	0.00014	0.00018	0.00029	0.00010	0.00008	0.00001	0.00001	0.0000012	0.0000004

Table 2. The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$ with the parameters $\lambda = 0.9$, $\mu_1 = 5$, $\mu_2 = 2$.

s_2	s_1										
	0	1	2	3	4	5	6	7	8	9	10
0	0.4069	0.00074	0.00001	0.00443	0.00665	0.00039	0.00008	0.00006	0.000009	0.0000042	0.0000037
1	0.0007	0.00148	0.00443	0.00887	0.01333	0.00038	0.00006	0.00005	0.000007	0.0000032	0.0000021
2	0.0000	0.00148	0.00444	0.00890	0.01337	0.00037	0.00005	0.00003	0.000005	0.0000022	0.0000011
3	0.0004	0.00099	0.00297	0.00597	0.00900	0.00036	0.00004	0.00002	0.000003	0.0000012	0.0000009
4	0.0002	0.00050	0.00149	0.00301	0.00002	0.00034	0.00003	0.00002	0.000003	0.0000010	0.0000008
5	0.0002	0.00021	0.00053	0.00046	0.00020	0.00031	0.00002	0.00001	0.000002	0.0000008	0.0000007
6	0.0001	0.00019	0.00041	0.00043	0.00013	0.00022	0.00001	0.00001	0.000002	0.0000007	0.0000006
7	0.0001	0.00017	0.00035	0.00041	0.00012	0.00010	0.00001	0.00001	0.000001	0.0000006	0.0000005
8	0.0001	0.00016	0.00026	0.00038	0.00011	0.00009	0.00001	0.00001	0.000001	0.0000005	0.0000004
9	0.000	0.00014	0.00018	0.00029	0.00009	0.00008	0.00001	0.00001	0.000001	0.0000004	0.0000003
10	0.00007	0.000132	0.00008	0.00019	0.00007	0.00001	0.000006	0.000005	0.000001	0.0000034	0.0000002

Table 3. The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, ...$ with the parameters $\lambda = 0.4$, $(m_1 = 15, \alpha_1 = 0.15)$, $(m_2 = 10, \alpha_2 = 0.4)$.

s_2	s_1							
	0	1	2	3	4	5	6	7
0	0.67071	0.00058	0.00000	0.00017	0.00005	0.00039	0.00008	0.00006
1	0.00263	0.00390	0.00270	0.00116	0.00035	0.00038	0.00006	0.00005
2	0.00005	0.01172	0.00813	0.00351	0.00105	0.00037	0.00005	0.00003
3	0.01405	0.02090	0.01455	0.00632	0.00190	0.00036	0.00004	0.00002
4	0.01645	0.02457	0.01710	0.00749	0.00005	0.00034	0.00003	0.00002
5	0.00021	0.00021	0.00053	0.00046	0.00020	0.00031	0.00002	0.00001
6	0.00019	0.00019	0.00041	0.00043	0.00013	0.00022	0.00001	0.00001
7	0.00015	0.00017	0.00035	0.00041	0.00013	0.00010	0.00001	0.00001

Table 4. Expected values, variances, covariance and correlation coefficients of the random variables.

Claim Distribution	E(N)	$E(S_1)$	$E(S_2)$	$E(S_1S_2)$	$V(S_1)$	$V(S_2)$	$Cov(S_1S_2)$	ρ_{S_1,S_2}
Geometric								
(a) $\theta_1 = 0.3$, $\theta_2 = 0.4$	$\lambda = 0.5$	0.150	0.200	0.090	0.150	0.200	0.060	0.346
(b) $\theta_1 = 0.1$, $\theta_2 = 0.6$	$\lambda = 1.5$	0.150	0.900	0.225	0.150	0.900	0.090	0.244
(c) $\theta_1 = 0.9$, $\theta_2 = 0.3$	$\lambda = 2.0$	1.800	0.600	1.620	1.800	0.600	0.540	0.519
Poisson								
(a) $\mu_1 = 0.25$, $\mu_2 = 0.45$	$\lambda = 0.5$	0.125	0.225	0.084	0.156	0.326	0.056	0.249
(b) $\mu_1 = 0.50$, $\mu_2 = 0.65$	$\lambda = 1.5$	0.750	0.975	1.219	1.125	1.609	0.488	0.362
(c) $\mu_1 = 0.75$, $\mu_2 = 0.35$	$\lambda = 2.0$	1.500	0.700	1.575	2.625	0.945	0.525	0.333
Binomial								
(a) $(m_1 = 10, \alpha_1 = 0.30)$ $(m_2 = 20, \alpha_2 = 0.60)$	$\lambda = 0.5$	1.500	4.000	18.000	5.550	34.400	12.000	0.868
(b) $(m_1 = 25, \alpha_1 = 0.10)$ $(m_2 = 35, \alpha_2 = 0.60)$	$\lambda = 1.5$	3.750	31.500	196.875	12.750	674.100	78.750	0.849
(c) $(m_1 = 30, \alpha_1 = 0.90)$ $(m_2 = 40, \alpha_2 = 0.30)$	$\lambda = 2.0$	54.000	24.000	1944.000	1463.400	304.800	648.000	0.970

5. CONCLUSION

In this study, a correlated bivariate version of the univariate compound Poisson distribution is defined and studied. For this aim, the joint probability function, some important parameters as mean, covariance and correlation of the bivariate compound Poisson distribution are derived. The proposed algorithm gives a simple and efficient way for $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, ...$ We conclude with the comment that the joint probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, ...$ can be computed easily if $P(X_i = j) = p_j$, $j = 0, 1, 2, ..., m$ and $P(Y_i = k) = q_k$, $k = 0, 1, 2, ..., r$ are known. The joint probability function of the bivariate compound Poisson model which is obtained in this study can be a good tool for the probabilistic fitness for bivariate aggregate claims.

REFERENCES

[1] Cummins, D.J., Wiltbank, L.J., "Estimating the total claims distribution using multivariate frequency and severity distributions", ***Journal of Risk and Insurance***, 50: 377–403 (1983).

[2] Cossette, H., Gaillardetz, P., Marceau, É., Rioux, J., "On two dependent individual risk models", ***Insurance: Mathematics and Economics,*** 30: 153–166 (2002).

[3] Sundt, B., "On some extensions of Panjer's class of counting distributions", ***Astin Bulletin,*** 22: 61-80 (1992).

[4] Homer, D.L., Clark, D.R., "Insurance applications of bivariate distributions", ***CAS Forum***, 274-307 (2003).

[5] Walhin, J., "On the Optimality of Multiline Excess of Loss Covers", ***CAS Forum***, 231-243 (2003).

[6] Bruno, M.G., Camerini, E., Manna, A., Tomassetti, A., "A new method for evaluating the distribution of aggregate claims", ***Applied Mathematics and Computation***, 176: 488-505 (2006).

[7] Rolski, T., Schmidli, H., Schmidt, V., Teugels, J., "Stochastic Processes for Insurance and Finance", ***John Wiley and Sons***, (1999).

[8] Sundt, B., "On some extensions of Panjer's class of counting distributions", ***ASTIN Bulletin***, 22: 61-80 (1992).

[9] Ozel, G., Inal, C., "The Probability Function of the Compound Poisson Distribution Using Integer Partitions and Ferrer's Graph", ***Bulletin of Statistics and Economics***, 2 (1): 75-87 (2008).

[10] Ozel, G., Inal, C., "The probability function of the compound Poisson process and an application to aftershock sequences", ***Environmetrics,*** 19: 79-85 (2008).

[11] Ozel, G., Inal, C., "The probability function of a geometric Poisson distribution", ***Journal of Statistical Computation and Simulation***, 80: 479-487 (2010).

[12] Kocherlakota, S., Kocherlakota, K., "Bivariate Discrete Distributions", ***Marcel Decker***, New York, (1992).

[13] Johnson, N.L., Kotz, S., Balakrishnan, N., "Discrete Multivariate Distributions", ***Wiley***, New York, (1997).

[14] Holgate, P., "Estimation for the bivariate Poisson distribution", ***Biometrika,*** 51: 241–245 (1964).

[15] Ambagaspitiya, R., "Compound bivariate Lagrangian Poisson distributions", ***Insurance: Mathematics & Economics,*** 23 (1): 21-31 (1998).

[16] Homer, D.L., "Aggregating bivariate claim severities with numerical Fourier inversion", ***CAS Forum***, (2006).