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Research Article

Nilpotent Product of Parafree Lie Algebras and A Basis of This Product

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Abstract

In this work, n-th nilpotent and minimal nilpotent products of parafree Lie algebras are examined and it is shown that these products are parafree. Also, a base set is obtained for the n-th nilpotent products of two abelian Lie algebras that are nilpotent of class n.

Keywords: Parafree Lie algebras, Free Lie Algebras, Hall Bases, Nilpotent product

Paraserbest Lie Cebirlerinin Nilpotent Çarpımı ve Bu Çarpımın Bir Bazı

Öz

Bu çalışmada paraserbest Lie cebirlerinin n-inci nilpotent ve minimal nilpotent çarpımları incelenmiş ve bu çarpımların paraserbest olduğu gösterilmiştir. Ayrıca iki paraserbest abelyen ve n-inci sınıftan nilpotent Lie cebirinin n-inci nilpotent çarpımları için bir baz kümesi elde edilmiştir.

Anahtar Kelimeler: Paraserbest Lie Cebirleri, Serbest Lie Cebirleri, Hall bazları, Nilpotent Çarpım

1. Introduction

A parafree group is residually nilpotent and has the same lower central sequence as a given free group. These groups have been introduced by Baumslag (1967). Baumslag (1969) has obtained some important results about these groups. Parafree Lie algebras firstly arose from the work of Baur (1978). He has translated the definition of parafree groups to parafree Lie algebras. In literature, there are not many studies on parafree Lie algebras. Ekici (2013); Velioğlu (2013) have studied some important basic results on parafree Lie algebras and they have defined the metabelian product of parafree Lie algebras and have investigated verbal subalgebra and 2-symmetric words of this product (2019). Velioğlu (2019) has studied some residual properties of the soluble product of parafree Lie algebras The nilpotent products of groups were introduced by Golovin (1950) as examples of regular products of groups. Akdoğan (2014) has defined these products for Lie algebras. In this work, we deal with the nilpotent products of parafree Lie algebras and obtain a base set for the n-th nilpotent product of

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two abelian Lie algebras that are nilpotent of class n.

2. Preliminaries

This section contains necessary preliminary notation and results. We use standard notation. Let $(L_\alpha)_{\alpha\in I}$ be a family of Lie algebras. Throughout this paper, $\prod_\alpha * L_\alpha$ denotes the free product of the family of Lie algebras $(L_\alpha)_{\alpha\in I}$.

Definition 2.1. Let L be a Lie algebra. The Lie subalgebra

$$[\mathrm{L},\mathrm{L}] = \mathrm{sp}\{[\mathrm{x},\mathrm{y}] : \mathrm{x},\mathrm{y} \in \mathrm{L}\}$$

is called commutator subalgebra of L and denoted by L'.

Definition 2.2. Let L be a Lie algebra over a field k. The lower central series of L

$$L = \gamma_1(\mathrm{L}) \supseteq \gamma_2(\mathrm{L}) \supseteq .... \supseteq \gamma_n(\mathrm{L}) \supseteq ...$$

is defined inductively by $\gamma_2(\mathrm{L})=[\mathrm{L}, \mathrm{L}]$, …, $\gamma_{n+1}(\mathrm{L}) =[\gamma_n(\mathrm{L}), \mathrm{L}]$, $n\geq 1$.

The term $\gamma_k(\mathrm{L})$ is called the k-th term of the lower central series of L and sometimes denoted by L_k.

If n is the smallest integer satisfying $\gamma_n(\mathrm{L}) = 0$, then L is called nilpotent of class n. If L is nilpotent of class 2, then L is called abelian.

Definition 2.3. A Lie algebra L is called residually nilpotent if

$$\bigcap_{n=1}^{\infty}\gamma_n(L) = \{0\}.$$

We associate with the lower central series of L its lower central sequence:

$$\mathrm{L}/\gamma_2(\mathrm{L}), \mathrm{L}/\gamma_3(\mathrm{L}), ...$$

Definition 2.4. Let $F = F(X)$ be a free Lie algebra freely generated by a set X. A Lie algebra L is called parafree over X if,

i) L is residually nilpotent,

ii) The natural homomorphism $\phi: F \to L$ determines isomorphisms

$$\bar{\phi}_i: \ \mathrm{F}/\gamma_i(\mathrm{F}) \to \mathrm{L}/\gamma_i(\mathrm{L})$$

for every $i \geq 1$.

It is clear that, for every $n \geq 1$,

$$\mathrm{F}/\gamma_n(\mathrm{F}) \cong \mathrm{L}/\gamma_n(\mathrm{L}).$$

We express that as 'L has the same lower central sequence as the free Lie algebra F'.

Definition 2.5. Let L be a Lie algebra and B be a subset of L such that

$$L \equiv \bar{L}\big(mod\ \gamma_2(L)\big)$$

and

$$B \equiv \bar{B}\big(mod\ \gamma_2(L)\big).$$

If $\bar{B}$ freely generates $\bar{L}$, then it is said that 'B freely generates L modulo $\gamma_2(L)$'.

If L is a parafree Lie algebra, then the subset B of L is called parabasis set of L .

For proof of the following propositions, see Velioğlu (2013).

Proposition 2.1. The quotient algebra of a parafree Lie algebra is parafree.

Proposition 2.2. The direct sum of two parafree Lie algebras is parafree.

3. The Main Results

In this section, we investigate nilpotent and minimal nilpotent products of parafree Lie algebras. Later, we obtain a basis set for the n-th nilpotent product of two abelian Lie algebras that are nilpotent of class n.

Definition 3.1. Let $(L_\alpha)_{\alpha\in I}$ be a family of Lie algebras and $\oplus L_\alpha$ be the direct sum of this family. Define an epimorphism as

$$\chi: \prod_\alpha * L_\alpha \to \oplus L_\alpha,$$
$$\chi e = e,$$

where $e \in L_\alpha$. The kernel of the epimorphism χ is called the cartesian subalgebra of the free product of $(L_\alpha)_{\alpha\in I}$.

Definition 3.2. Let L_1 and L_2 be two Lie algebras and D be the cartesian subalgebra of the free product of L_1 and L_2. The n-nilpotent product of L_1 and L_2, denoted by $L_1 *_{nil}^{n} L_2$, defined to be the algebra

$$L_1 *_{nil}^{n} L_2 = {L_1 * L_2}/{\gamma_n(L_1 * L_2) \cap D}$$

Proposition 3.1. Let P_1 and P_1 be two parafree Lie algebras that have the same lower central sequence as two free Lie algebras that are freely generated by sets X and Y, respectively. Then $P_1 * P_2$ is a parafree Lie algebra that has the same lower central sequence as a free Lie algebra generated by $X \cup Y$.

Proof. For the proof, see Baur (1978).

Lemma 3.1. Let P_1 and P_2 be two parafree Lie algebras then the n-nilpotent product of P_1 and P_2 is parafree.

Proof. Consider parafree Lie algebras P_1 and P_2. By Proposition 3.1. the free product $P = P_1 * P_2$ is parafree. Now let

$$u \in \cap_{n=1}^{\infty} \gamma_n \left({P}/{\gamma_n(P) \cap D}\right).$$

Then for every n,

$$u \in \gamma_n \left({P}/{(\gamma_n(P) \cap D)}\right) = {\gamma_n(P) + (\gamma_n(P) \cap D)}/{(\gamma_n(P) \cap D)}.$$

Suppose that $u = a + (\gamma_n(P) \cap D)$, where

$$a \in \gamma_n(P) + (\gamma_n(P) \cap D).$$

Therefore

$$a \in \bigcap_{n=1}^{\infty} \gamma_n(P) + (\gamma_n(P) \cap D).$$

Since P is residually nilpotent, then $a \in (\gamma_n(P) \cap D)$ and so $u = (\gamma_n(P) \cap D)$. Thus we have

$$\cap_{n=1}^{\infty} \gamma_n \left({P}/{(\gamma_n(P) \cap D)}\right) = 0.$$

It means that $P_1 *_{nil}^{n} P_2$ is residually nilpotent. It remains to show that ${P}/{(\gamma_n(P) \cap D)}$ has the same lower central sequence as a free Lie algebra. Now consider

$$ {(\gamma_n(P) + (\gamma_n(P) \cap D))}/{(\gamma_n(P) \cap D)} \cong {\gamma_n(P)}/{(\gamma_n(P) \cap D)}.$$

Therefore

$$\left({P}/{(\gamma_n(P) \cap D)}\right) \Big/ \gamma_n \left({P}/{\gamma_n(P) \cap D)}\right) \cong$$

$$\left({P}/{(\gamma_n(P) \cap D)}\right) \Big/ \left(\gamma_n(P) + (\gamma_n(P) \cap D)\right) \Big/ (\gamma_n(P) \cap D)$$

$$\cong {P}/{(\gamma_n(P) \cap D)} \Big/ \left({\gamma_n(P)}/{(\gamma_n(P) \cap D)}\right) \cong {P}/{\gamma_n(P)}.$$

Since P is a parafree Lie algebra, then there exists a free Lie algebra F such that

$${P}/{\gamma_n(P)} \cong {F}/{\gamma_n(F)}.$$

So we have

$$\left({P}/{(\gamma_n(P) \cap D)}\right) \Big/ \gamma_n \left({P}/{(\gamma_n(P) \cap D)}\right) \cong {F}/{\gamma_n(F)}.$$

Therefore $P_1 *_{nil}^{n} P_2$ is parafree. ■

Definition 3.3. Let L be a Lie algebra. The minimal central series of L determined by a subalgebra A is the decreasing series of ideals whose terms are given by recursive formulas:

$$ {}_0A = \langle A \rangle, \ldots, \ {}_lA = \left[G, {}_{l-1}A\right] $$

where $l \geq 1$ and $\langle A \rangle$ is ideal of L generated by the subalgebra A. For every $k \geq 1$, ${}_kA$ is the k-th term of the minimal central series of L determined by A and it is called k-th minimal central term of L determined by A.

The lower central series of L is an example of a minimal central series of L.

Definition 3.4. Let L_1 and L_2 be two Lie algebras, $L = L_1 * L_2$ and $min_k[L_1, L_2]_L$ denotes the k-th the minimal central term of L determined by $[L_1, L_2]$. The k-th minimal nilpotent product of L_1 and L_2 determined by $[L_1, L_2]$ is defined as

$$ L_1 \ *_{nil}^{[L_1, L_2]} L_2 = {}^{L}\!\big/_{min_k[L_1, L_2]_L}. $$

Lemma 3.1. Let P_1 and P_2 be two parafree Lie algebras then $P_1 \ *_{nil}^{[P_1, P_2]} P_2$ is parafree.

Proof. Suppose that k=0, then

$$ min_0[P_1, P_2]_P = [P_1, P_2]. $$

Therefore, we have

$$ P_1 \ *_{nil}^{[P_1, P_2]} P_2 = {}^{P_1 \ * \ P_2}\!\big/_{[P_1, P_2]} \cong {}^{P_1 \ * \ P_2}\!\big/_{D} \cong P_1 \oplus P_2. $$

So by Proposition 2.1. the 0-th minimal nilpotent product of P_1 and P_2 defined by $[P_1, P_2]$ is parafree.

Now , let $k > 0$, then

$$ min_k[P_1, P_2]_P = \underbrace{[P, [P, [\ldots [P}_{k-times}, [P_1, P_2]] \ldots]]] $$

That is the k-th nilpotent commutator subalgebra. Denote that algebra by $[P_1, P_2]^{(k)}$.

Hence

$$ P_1 \ *_{nil}^{[P_1, P_2]} P_2 = {}^{P_1 \ * \ P_2}\!\big/_{[P_1, P_2]^{(k)}}. $$

By the Proposition 3.1 $P_1 \ * \ P_2$ is parafree so by Proposition 2.2. $P_1 \ *_{nil}^{[P_1, P_2]} P_2$ is parafree. ■

Note that if P_1 and P_2 are parafree abelian Lie algebra that are nilpotent of class n, then

$$ D \cap \gamma_n(P_1 * P_2) = \gamma_n(P_1 * P_2) $$

and so

$$ P_1 *_{nil}^{n} P_2 = {}^{P_1 * P_2}\!\big/_{\gamma_n(P_1 * P_2)}. $$

Now we want to obtain a base set for $P_1 *_{nil}^{n} P_2$.

Consider sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Let $P_1(resp. P_2)$ be a parafree Lie algebra that has same lower central sequence as the free Lie algebra $F(X)(resp. F(Y))$. On the other hand let $P_1 * P_2 = L$ and $\mathrm{F}(X \cup Y) = \mathrm{F}$. By the Proposition 3.1,

$$ {}^{\mathrm{L}}\!\big/_{\gamma_n(\mathrm{L})} \cong {}^{\mathrm{F}}\!\big/_{\gamma_n(\mathrm{F})}. $$

We index the set X ∪ Y as

$$ y_1 = x_{n+1}, y_2 = x_{n+2}, \ldots, y_n = x_{2n} $$

with

$$ x_1 > x_2 > \cdots > x_{2n}. $$

Consider the bijection

$$ \hat{\varphi}: X \cup Y \to B $$

$$ \mathrm{x_i} \ \to \ \overline{\mathrm{x}}_{\mathrm{i}} $$

where B is a generating set of L. We extend the bijection $\hat{\varphi}$ to the epimorphism

$$\varphi: F \rightarrow L$$

bilinearly. Then

$\hat{\varphi}(X \cup Y)= \{\hat{\varphi}(x_1), \hat{\varphi}(x_2),\ldots,\hat{\varphi}(x_{2n})\}$

$=\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2n}\}=B.$

Now consider Hall sets of the free Lie algebra F.

$H_1 = X \cup Y,$

$H_2 = \{xy: x, y \in H_1, \ x > y\},$

$\ldots$

$H_n = \{x = (ab)c: a, b, c, ab \in H_1 \cup \ldots \cup H_{n-1}, a > b, b \leq c, l(x) = n\}.$
$H=\bigcup_{n=1}^{\infty} H_n$ is Hall base of the free Lie algebra F. Since $\hat{\varphi}$ is the restriction of φ to $X \cup Y$, then

$$\varphi(H_1) = \hat{\varphi}(H_1) = B.$$

Denote $\varphi(H_1)$ by ${H_1}^*$. Now we define following sets

${H_1}^* = B,$

$\varphi(H_2) = {H_2}^* = \{\hat{\varphi}(x)\hat{\varphi}(y): \hat{\varphi}(x), \hat{\varphi}(y) \in {H_1}^*, \quad \hat{\varphi}(x) > \hat{\varphi}(y)\}$

$,\ldots,$

$\varphi(H_n) = {H_n}^* =$

$\{\bar{x} = (\hat{\varphi}(a)\hat{\varphi}(b))\hat{\varphi}(c): \hat{\varphi}(a)\hat{\varphi}(b),$

$\hat{\varphi}(a), \hat{\varphi}(b), \hat{\varphi}(c) \in {H_1}^* \cup \ldots \cup {H_{n-1}}^*$

$\hat{\varphi}(a) > \hat{\varphi}(b), \hat{\varphi}(b) \leq \hat{\varphi}(c), l(\bar{x}) = n\}.$

Since L is parafree, then there exist isomorphisms such that

$$\bar{\varphi}_n: {}^{F}/_{\gamma_n(F)} \rightarrow {}^{L}/_{\gamma_n(L)}$$

$$x + \gamma_n(F) \rightarrow \varphi(x) + \gamma_n(L).$$

Therefore for $n = 2$ and $i \in \{1,2, \ldots, 2n\}$, we have

$$\bar{\varphi}_2: {}^{F}/_{\gamma_2(F)} \rightarrow {}^{L}/_{\gamma_2(L)}$$

$$x + \gamma_2(F) \rightarrow \varphi(x) + \gamma_2(L).$$

Suppose that

$B^*=\{\varphi(x_1) + \gamma_2(L), \varphi(x_2) + \gamma_2(L),\ldots, \varphi(x_{2n}) + \gamma_2(L)\}$

$= \{\bar{\varphi}_2(x_1 + \gamma_2(F)),\bar{\varphi}_2(x_2 + \gamma_2(F)),\ldots,\bar{\varphi}_2(x_{2n} + \gamma_2(F))\}.$

Now we want to show that the set B^* is linearly independent. For $i \in \{1,2, \ldots, 2n\}$ and $\alpha_i \in K$, let

$$\sum_{i=1}^{2n} \alpha_i \bar{\varphi}_2(x_i + \gamma_2(F)) = \bar{0}.$$

Then we have

$$\sum_{i=1}^{2n} \bar{\varphi}_2(\alpha_i x_i + \gamma_2(F)) = \bar{0}$$

and

$$\bar{\varphi}_2(\sum_{i=1}^{2n} (\alpha_i x_i + \gamma_2(F))) = \bar{0}.$$

Since $\bar{\varphi}_2$ is an isomorphism then

$$\sum_{i=1}^{2n} (\alpha_i x_i + \gamma_2(F)) = \gamma_2(F).$$

If $\alpha_i \neq 0$, then $\alpha_i x_i \in \gamma_2(F)$. But it is imposible. Then $\alpha_i = 0$ and so B^* is lineerly independent. Since each $\bar{\varphi}_n$ is an isomorphism, we have the result for each n. That is, the set

$\{\varphi(x_1) + \gamma_n(L), \varphi(x_2) + \gamma_n(L),\ldots, \varphi(x_{2n}) + \gamma_n(L)\}$

$=\{\bar{\varphi}_n(x_1 + \gamma_n(F)), \bar{\varphi}_n(x_2 + \gamma_n(F)), \ldots, \bar{\varphi}_n(x_{2n} + \gamma_n(F))\}$

is lineerly independent. It is known that the set H is Hall basis for F, so the set $\{H_1 \cup H_2 \cup \ldots \cup H_{n-1}\}$ is a basis for ${}^{F}/_{\gamma_n(F)}$.

Now consider the isomorphism

$$\bar{\varphi}_n: \ {}^{F}/_{\gamma_n(F)} \ \to {}^{L}/_{\gamma_n(L)}.$$

Thus we get

${}^{L}/_{\gamma_n(L)} = \bar{\varphi}_n({}^{F}/_{\gamma_n(F)})$

$= \bar{\varphi}_n(\mathrm{sp}\{ H_1 \cup H_2 \cup \ldots \cup H_{n-1}\})$

$= \mathrm{span}\ \{ \bar{\varphi}_n(H_1 \cup H_2 \cup \ldots \cup H_{n-1}).$

So the set

$\bar{\Phi}_n(H_1 \cup H_2 \cup \ldots \cup H_{n-1})$

$= \{H_1^* \cup H_2^* \cup \ldots \cup H_{n-1}^*\}$

is a basis set for ${}^{L}/_{\gamma_n(L)}$.

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