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Operator (α, m)-convex functions and applications for synchronous and asynchronous functions

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Abstract
In this study, firstly the definition of operator (α, m)-convex function is defined. Secondly, a new lemma is given. Then, new theorems are obtained in terms of this lemma. Finally, they are applied for synchronous and asynchronous functions.

Keywords: *The Hermite-Hadamard inequality, (α, m)-convex functions, synchronous and asynchronous functions, Hilbert space, selfadjoint operator, operator (α, m)-convex functions.*

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1. Introduction

The following inequality holds for any convex function f define on $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_0^1 f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. The inequality (1.1) is known in the literature as the Hermite-Hadamard's inequality. The Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

First, we review the operator order in $B(H)$ and the continuous functional calculus for a bounded selfadjoint operator. For selfadjoint operators $A, B \in B(H)$ we write, for every $x \in H$,

$$A \leq B \quad (\text{or } B \geq A) \quad \text{if} \quad \langle Ax, x\rangle \leq \langle Bx, x\rangle \quad (\text{or } \langle Bx, x\rangle \geq \langle Ax, x\rangle)$$

we call it the operator order. Let A be a selfadjoint linear operator on a complex Hilbert space $(H, < \cdot, \cdot >)$ and $C(Sp(A))$ the C^* -algebra of all continuous complex-valued functions on the spectrum A. The Gelfand map establishes a $*$-isometrically isomorphism Φ between $C(Sp(A))$ and the C^* -algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows [1]: For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

1. $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
2. $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^*) = \Phi(f)^*$;
3. $\|\Phi(f)\| = \|f\| := sup_{t\in Sp(A)}|f(t)|$;
4. $\Phi(f_0) = 1$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

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If f is a continuous complex-valued functions on $C(Sp(A)$, the element $\Phi(f)$ of $C^*(A)$ is denoted by $f(A)$ and we call it the continuous functional calculus for a bounded selfadjoint operator A.

If A is bounded selfadjoint operator and f is real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in Sp(A)$, then $f(A) \leq g(A)$ in the operator order $B(H)$. A real valued continuous function f on an interval I is said to be operator convex (operator concave) if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

in the operator order in $B(H)$, for all $\lambda \in [0,1]$ and for every bounded self-adjoint operator A and B in $B(H)$ whose spectra are contained in I.

We denoted by $B(H)^+$ the set of all positive operators in $B(H)$ and K is subset of $B(H)^+$.
Erdaş [5], [11] et al. and Salaş [6] et al. studied at operator $m, (\alpha, m)$-convex and operator p-convex functions.

Also Rooin et al. [7] studied about operator m-convex functions. They generalized the celebrated Jensen inequality for continuous m-convex functions and Hilbert space operators and then used suitable weight functions to give some weighted refinements etc.

In the second chapter, we use the similar technique with Ghazanfari et al. [8],[9].

Definition 1.1. [2] The function $f : [a, b] \to \mathbb{R}$ is said to be (α, m)-convex, where $(\alpha, m) \in [0,1]^2$, if for every $x, y \in [a, b]$ and $t \in [0,1]$ we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

Note that $(\alpha, m) \in \{(0,0), (\alpha, 0), (1,0), (1, m), (1,1), (\alpha, 1)\}$ one obtains the following classes of functions respectively, increasing, α-starshaped, starshaped, m-convex, convex and α-convex
Denote by $K_m(b)$ the set of the (α, m)-convex functions on [a,b] for which $f(0) \leq 0$.

We note that Beta and Gamma functions [8] are defined respectively, as follows

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, x > y$$

and

$$\Gamma(x, y) = \int_0^\infty e^{-t}t^{x-1}, x > 0$$

Theorem 1.1. *[8] Let $f : I \to \mathbb{R}$ be operator s_1-convex and $g : I \to \mathbb{R}$ be operator s_2-convex function on the interval I for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, the inequality*

$$\begin{aligned} &\int_0^1 \langle f(tA + (1-t)B)x, x\rangle \langle g(tA + (1-t)B)x, x\rangle dt \\ \leq\ & \frac{1}{s_1 + s_2 + 1} M(A, B)(x) + \beta(s_1 + 1, s_2 + 1)N(A, B)(x) \end{aligned} \tag{1.2}$$

holds for any $x \in H$ with $\| x \| = 1$.

Theorem 1.2. *[8] Let $f : I \to \mathbb{R}$ be operator s_1-convex and $g : I \to \mathbb{R}$ be operator s_2-convex function on the interval I for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, the inequality*

$$\begin{aligned} &2^{s_1+s_2-1}\Big\langle f\Big(\frac{A+B}{2}\Big)g\Big(\frac{A+B}{2}\Big)x, x\Big\rangle \\ \leq\ & \int_0^1 \langle f(tA + (1-t)B)x, x\rangle \langle g(tA + (1-t)B)x, x\rangle dt \\ +\ & \beta(s_1 + 1, s_2 + 1)M(A, B)(x) + \frac{1}{s_1 + s_2 + 1}N(A, B)(x) \end{aligned} \tag{1.3}$$

holds for any $x \in H$ with $\| x \| = 1$, where

$$M = M(A, B)(x) = \langle f(A)x, x\rangle \langle g(A)x, x\rangle + \langle f(B)x, x\rangle \langle g(B)x, x\rangle \tag{1.4}$$

$$N = N(A, B)(x) = \langle f(A)x, x\rangle \langle g(B)x, x\rangle + \langle f(B)x, x\rangle \langle g(A)x, x\rangle \tag{1.5}$$

Theorem 1.3. *[3] Let $A, B \in B(H)^+$. Then $AB + BA$ is positive if and only if*

$$f(A+B) \leq f(A) + f(B)$$

for all non-negative operator functions f on $[0, \infty)$.

Theorem 1.4. *[4] Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for all selfadjoint operators A and B with spectra in I, we have the inequality*

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \\ &\leq \int_0^1 f\Big((1-t)A + tB\Big)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right] \\ &\leq \frac{f(A)+f(B)}{2} \end{aligned}$$

2. MAIN RESULT

2.1 Some New Hermite-Hadamard type Inequalities via Operator (α, m)-convex Functions

Definition 2.1. For $(\alpha, m) \in [0, 1]^2$, the function $f : I \to \mathbb{R}$ is said to be operator (α, m)-convex, if for every bounded selfadjoint operators A, B, whose spectra are contained in I and $t \in [0, 1]$,

$$f(tA + m(1-t)B) \leq t^\alpha f(A) + m(1-t^\alpha)f(B)$$

inequality holds.

Lemma 2.1. *If f is operator (α, m)-convex on $[0, \infty)$ and nondecreasing function for operator in K, $\frac{1}{m}\langle Ax, x\rangle$, $\frac{1}{m}\langle Bx, x\rangle \subset I$, $m \in (0, 1]$ and $\alpha \in [0, 1]$ then $f(A)$ is positive for every $A \in K$.*

Proof. Since $A \in K$ and f operator (α, m)-convex function, we have

$$\begin{aligned} f(A) &= f\left(\frac{tA + m(1-t)B + m\Big((1-t)\frac{A}{m} + tA\Big)}{m+1}\right) \\ &\leq f\Big(tA + m(1-t)A + m\Big((1-t)\frac{A}{m} + tA\Big)\Big) \\ &\leq t^\alpha f(A) + m(1-t^\alpha)f(A) + (1-t^\alpha)f(A) + mt^\alpha f(A) \\ f(A) &\leq f(A)(m+1) \\ 0 &\leq mf(A) \end{aligned}$$

This implies that $f(A) \geq 0$

Lemma 2.2. *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a continuous function on the interval I. Then for every two positive operators $A, B \in K \subseteq B(H)^+$ with spectra in I the function f is operator (α, m)-convex for operators in*

$$[A, B] := \{(1-t)A + mtB : t \in [0, 1]\}$$

if and only if the function

$$\varphi_{x,A,B} : [0, 1] \to \mathbb{R}$$

defined by

$$\varphi_{x,A,B}(t) =< f((1-t)A + mtB)x, x >$$

is (α, m)-convex on $(0, 1]$ for every $x \in H$ with $\|x\| = 1$.

Proof. Let f be operator (α,m)-convex for operators in [A,B], then for any $t_1,t_2\in[0,1]$ and $\lambda,\gamma\geq 0$, $m,\lambda,\gamma\in(0,1]$ with $\lambda+m\gamma=1$ we have

$$\begin{aligned}
&\varphi_{x,A,B}(\lambda t_1+m\gamma t_2)\\
=\ &\Big\langle f\Big((1-(\lambda t_1+m\gamma t_2)A)+m(\lambda t_1+m\gamma t_2)B\Big)x,x\Big\rangle\\
=\ &\Big\langle f\Big(\lambda A+m\gamma A-\lambda t_1A-m\gamma t_2A+m\lambda t_1B+m^2\gamma t_2B\Big)x,x\Big\rangle\\
=\ &\Big\langle f\Big(\lambda[(1-t_1)A+mt_1B]+m\gamma[(1-t_2)A+mt_2B]\Big)x,x\Big\rangle\\
=\ &\Big\langle f\Big(\lambda[(1-t_1)A+mt_1B]+m(1-\lambda)\big[(1-t_2)\frac{A}{m}+mt_2\frac{B}{m}\big]\Big)x,x\Big\rangle\\
\leq\ &\lambda^\alpha\Big\langle\Big([(1-t_1)A+mt_1B]+m(1-\lambda^\alpha)\big[(1-t_2)\frac{A}{m}+mt_2\frac{B}{m}\big]\Big)x,x\Big\rangle\\
\leq\ &\lambda^\alpha\varphi_{x,A,B}(t_1)+m(1-\lambda^\alpha)\varphi_{x,A,B}(t_2)
\end{aligned}$$

showing that $\varphi_{x,A,B}$ is a (α,m)-convex function on $[0,1]$ Let now $\varphi_{x,A,B}$ be (α,m)-convex on $[0,1]$, we show that f is operator (α,m)-convex for operators in $[A,B]$. For every $C:=(1-t_1)A+mt_1B$ and $D:=(1-t_2)A+mt_2B$, we have

$$\begin{aligned}
&\langle f((1-\lambda)C+m\lambda D)x,x\rangle\\
=\ &\langle f((1-\lambda)[(1-t_1)A+mt_1B]+m\lambda[(1-t_2)A+mt_2B])x,x\rangle\\
=\ &\langle f(A-t_1A+mt_1B-\lambda A+\lambda t_1A\\
&-\lambda mt_1B+m\lambda A-m\lambda t_2A+m^2\lambda t_2B)x,x\rangle\\
=\ &\langle f(A(1-t_1)-\lambda A(1-t_1)+m\lambda A(1-t_2)+mt_1B-\lambda mt_1B+m^2\lambda t_2B)x,x\rangle\\
=\ &\langle f(-\lambda((1-t_1)A+mt_1B)+A(1-t_1)+mt_1B\\
&+m\lambda(A(1-t_2)+mt_2B))x,x\rangle\\
=\ &\langle f((1-\lambda)((1-t_1)A+mt_1B)+m\lambda((1-t_2)A+mt_2B))x,x\rangle\\
\leq\ &(1-\lambda^\alpha)\langle f(C)x,x\rangle+m\lambda^\alpha\langle f(D)x,x\rangle
\end{aligned}$$

Theorem 2.1. *Let $f:I\to\mathbb{R}$ be on operator (α,m)-convex and nondecreasing function on the interval $I\subseteq[0,\infty)$ for operators in $K\subseteq B(H)^+$. Then for all positive operators $A,B\in K\subseteq B(H)^+$ with spectra in I, $\frac{1}{m}\langle Ax,x\rangle$, $\frac{1}{m}\langle Bx,x\rangle\subset I$ and $m\in(0,1]$, $\alpha\in[0,1]$ then we have the inequality*

$$\begin{aligned}
\Big\langle f\Big(\frac{A+mB}{2}\Big)x,x\Big\rangle\ &\leq\ \frac{1}{2}\int_0^1\Big[\Big\langle f(tA+m(1-t)B)+mf((1-t)\frac{A}{m}+tB)x,x\Big\rangle\Big]dt\\
&\leq\ \frac{1}{2}\Big[\frac{\langle f(A)x,x\rangle+m\langle f(B)x,x\rangle}{\alpha+1}\\
&\quad+\frac{m\Big(\langle f(\frac{A}{m})x,x\rangle+\langle f(B)x,x\rangle\Big)}{\alpha+1}\Big]
\end{aligned}$$

Proof. For $x\in H$ with $\|x\|=1$ and $t\in[0,1]$, we have $<[tA+m(1-t)B]x,x>=t<Ax,x>+m(1-t)<Bx,x>\in I$. For $\langle Ax,x\rangle\in Sp(A)\subseteq I$ and $\langle Bx,x\rangle\in Sp(B)\subseteq I$, continuity of f and imply that the following operator-valued integral exist

$$\int_0^1 f(tA+(1-t)B)dt.$$

Since f is operator (α,m)-convex, therefore for $t\in[0,1]$ and $A,B\in K$, we have the belove inequality

$$f(tA+m(1-t)B)\leq t^\alpha f(A)+m(1-t^\alpha)f(B).$$

And then we get following inequalities,

$$\begin{aligned}\Big\langle f\Big(\tfrac{A+mB}{2}\Big)x,x\Big\rangle &= \Big\langle f\Big(\frac{tA+m(1-t)B+m((1-t)\frac{A}{m}+tB)}{2}\Big)x,x\Big\rangle \\ &\leq \frac{1}{2}\Big[\Big\langle f(tA+m(1-t)B)x,x\Big\rangle + m\Big\langle f((1-t)\frac{A}{m}+tB)x,x\Big\rangle\Big]\end{aligned}$$

$$\Big\langle f\Big(\frac{A+mB}{2}\Big)x,x\Big\rangle \leq \frac{1}{2}\int_0^1 \Big[\Big\langle f(tA+m(1-t)B)x,x\Big\rangle + m\Big\langle f((1-t)\frac{A}{m}+tB)x,x\Big\rangle\Big]$$

and,

$$\begin{aligned}& \frac{1}{2}\int_0^1 \Big[\Big\langle f(tA+m(1-t)B)x,x\Big\rangle + m\Big\langle f((1-t)\frac{A}{m}+tB)x,x\Big\rangle\Big]dt \\ \leq\ & \frac{1}{2}\int_0^1 \Big[t^{\alpha}\Big\langle f(A)x,x\Big\rangle + m(1-t^{\alpha})\Big\langle f(B)x,x\Big\rangle \\ & + m(1-t^{\alpha})\Big\langle f(\frac{A}{m})x,x\Big\rangle + mt^{\alpha}\Big\langle f(B)x,x\Big\rangle\Big]dt\end{aligned}$$

integrating of inequalities on [0,1], we get the inequality.

Remark 2.1. If $m = \alpha = 1$ is taken in the above theorem, Hermite-Hadamard inequality is obtained.

2.2 The Hermite-Hadamard type Inequalites for the Product two Operator (α, m)-convex Functions

Let $f : I \to \mathbb{R}$ be operator (α_1, m_1) and $g : I \to \mathbb{R}$ operator (α_2, m_2) function on the interval I. Then for all positive operators A and B on a Hilbert space H with spectra in I.

Theorem 2.2. *For $m_1, m_2 \in (0,1]$ and $\alpha_1, \alpha_2 \in [0,1]$, let $f : I \to \mathbb{R}$ be operator (α_1, m_1)-convex and $g : I \to \mathbb{R}$ operator (α_2, m_2)-convex and nondecreasing function on the interval I for operators in K. Then, $A, B \in K$ with spectra in I, the inequality*

$$\begin{aligned}& \int_0^1 \Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big]dt \\ \leq\ & \Big(\frac{\mathbb{K}}{\alpha_1+\alpha_2+1}\Big) + \Big(\frac{m_2\alpha_2 L}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+1)}\Big) \\ +\ & \Big(\frac{m_1\alpha_1 R}{(\alpha_2+1)(\alpha_1+\alpha_2+1)}\Big) \\ +\ & \Big(Sm_1m_2\Big(1-\frac{1}{\alpha_1+1}-\frac{1}{\alpha_2+1}+\frac{1}{\alpha_1+\alpha_2+1}\Big)\Big)\end{aligned}$$

exist. Where

$$\mathbb{K} := \mathbb{K}(A)(x) = \langle f(A)x,x\rangle\langle g(A)x,x\rangle \tag{2.1}$$
$$L := L(A,B)(x) = \langle f(A)x,x\rangle\langle g(B)x,x\rangle \tag{2.2}$$
$$R := R(A,B)(x) = \langle f(B)x,x\rangle\langle g(A)x,x\rangle \tag{2.3}$$
$$S := S(B)(x) = \langle f(B)x,x\rangle\langle g(B)x,x\rangle. \tag{2.4}$$

Proof. For $x \in H$, $\|x\| = 1$ and $t \in [0,1]$ we have

$$\Big\langle [tA+m(1-t)B]x,x\Big\rangle = t\langle Ax,x\rangle + m(1-t)\langle Bx,x\rangle \in I.$$

Since $\langle Ax,x\rangle \in Sp(A) \subseteq I$ and $\langle Bx,x\rangle \in Sp(B) \subseteq I$,

$$\int_0^1 f(tA+m_1(1-t)B)dt, \qquad \int_0^1 g(tA+m_2(1-t)B)dt$$

and

$$\int_0^1 (fg)(tA + m(1-t)B)dt$$

integrals exist.

f, g operator (α_1, m_1)-convex and (α_2, m_2)-convex, respectively, every for $t \in [0,1]$

$$\begin{aligned} \langle f(tA + m_1(1-t)B)x, x\rangle &\leq t^{\alpha_1}\langle f(A)x, x\rangle + m_1(1-t^{\alpha_1})\langle f(B)x, x\rangle \\ \langle g(tA + m_2(1-t)B)x, x\rangle &\leq t^{\alpha_2}\langle g(A)x, x\rangle + m_2(1-t^{\alpha_2})\langle g(B)x, x\rangle \end{aligned}$$

$$\begin{aligned} &\Big(\langle f(tA + m_1(1-t)B)x, x\rangle\Big)\Big(\langle g(tA + m_2(1-t)B)x, x\rangle\Big) \\ &\leq t^{\alpha_1+\alpha_2}\langle f(A)x, x\rangle\langle g(A)x, x\rangle + t^{\alpha_1} m_2(1-t^{\alpha_2})\langle f(A)x, x\rangle\langle g(B)x, x\rangle \\ &+ t^{\alpha_2} m_1(1-t^{\alpha_1})\langle f(B)x, x\rangle\langle g(A)x, x\rangle \\ &+ m_1 m_2(1-t^{\alpha_1})(1-t^{\alpha_2})\langle f(B)x, x\rangle\langle g(B)x, x\rangle \end{aligned}$$

confirmation. Integrating of inequalities over [0,1] we get the following inequality,

$$\begin{aligned} &\int_0^1 \Big[\langle f(tA + m_1(1-t)B)x, x\rangle\langle g(tA + m_2(1-t)B)x, x\rangle\Big]dt \\ &\leq \Big(\frac{\mathbb{K}}{\alpha_1+\alpha_2+1}\Big) + \Big(\frac{m_2\alpha_2 L}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+1)}\Big) \\ &+ \Big(\frac{m_1\alpha_1 R}{(\alpha_2+1)(\alpha_1+\alpha_2+1)}\Big) \\ &+ \Big(S m_1 m_2\Big(1 - \frac{1}{\alpha_1+1} - \frac{1}{\alpha_2+1} + \frac{1}{\alpha_1+\alpha_2+1}\Big)\Big) \end{aligned}$$

Remark 2.2. If we take $m_1, m_2, \alpha_1, \alpha_2 = 1$ in Theorem(2.2) and $s_1, s_2 = 1$ in (1.2) inequalities is same.

Theorem 2.3. *$f : I \to \mathbb{R}^+$ operator (α_1, m_1)-convex function and $g : I \to \mathbb{R}^+$ operator (α_2, m_2)-convex function. $Sp(A), Sp(B) \subseteq I$, For every $A, B \in K$ operators, $\frac{1}{m}\langle Ax, x\rangle$, $\frac{1}{m}\langle Bx, x\rangle \subset I$, $m_1, m_2 \in (0,1]$ and $\alpha_1, \alpha_2 \in [0,1]$*

$$\begin{aligned} &\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x, x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x, x\Big\rangle \\ \leq\ &\frac{1}{2}\int_0^1 \Big[\langle f(tA + m_1(1-t)B)x, x\rangle\langle g(tA + m_2(1-t)B)x, x\rangle\Big] \\ &+\frac{1}{4}\Bigg[\frac{2\mathbb{K}\alpha_2}{(\alpha_1+1)(\alpha_1+\alpha_2+1)} \\ &+L\Big(\frac{2m_2}{\alpha_1+\alpha_2+1} + m_2 - \frac{m_2}{\alpha_2+1} - \frac{m_2}{\alpha_1+1}\Big) \\ &++R\Big(\frac{2m_1}{\alpha_1+\alpha_2+1} + m_1 - \frac{m_1}{\alpha_1+1} - \frac{m_1}{\alpha_2+1}\Big) \\ &+\frac{2m_1m_2\alpha_1 S}{(\alpha_1+1)(\alpha_1+\alpha_2+1)}\Bigg] \end{aligned}$$

exist. Here $\mathbb{K}$, L, R and S are given by (2.1)-(2.4)

Proof. $f : I \to \mathbb{R}^+$ operator (α_1, m_1)-convex function and $g : I \to \mathbb{R}^+$ operator (α_2, m_2)-convex function. For every $t \in I$, $x \in H$ and $||x|| = 1$

$$\begin{aligned} \Big\langle f\Big(\tfrac{A+m_1B}{2}\Big)x, x\Big\rangle &= \Big\langle f\Big(\frac{tA + m_1(1-t)B + m_1\Big((1-t)\frac{A}{m_1} + tB\Big)}{2}\Big)x, x\Big\rangle \\ &\leq \frac{1}{2}\Big[\Big\langle f(tA + m_1(1-t)B)x, x\Big\rangle + \Big\langle f\Big(m_1((1-t)\frac{A}{m_1} + tB)\Big)x, x\Big\rangle\Big] \end{aligned}$$

$$\begin{aligned}\Big\langle g\Big(\tfrac{A+m_2B}{2}\Big)x,x\Big\rangle &= \Big\langle g\Big(\frac{tA+m_2(1-t)B+m_2\Big((1-t)\frac{A}{m_2}+tB\Big)}{2}\Big)x,x\Big\rangle\\ &\leq \frac{1}{2}\Big[\Big\langle g(tA+m_2(1-t)B)x,x\Big\rangle+\Big\langle g\Big(m_2((1-t)\frac{A}{m_2}+tB)\Big)x,x\Big\rangle\Big]\end{aligned}$$

$$\begin{aligned}&\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle\\ \leq\ & \frac{1}{4}\Bigg[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\\ &+m_2\langle f(tA+m_1(1-t)B)x,x\rangle\langle g((1-t)\frac{A}{m_2}+tB)x,x\rangle\\ &+m_1\langle f((1-t)\frac{A}{m_1}+tB)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\\ &+m_1m_2\langle f((1-t)\frac{A}{m_1}+tB)x,x\rangle\langle g((1-t)\frac{A}{m_2}+tB)x,x\rangle\Bigg]\end{aligned}$$

$$\begin{aligned}&\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle\\ \leq\ & \frac{1}{4}\Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\\ &+\langle f((1-t)A+m_1tB)x,x\rangle\langle g((1-t)A+m_2tB)x,x\rangle\Big]\\ &+\frac{1}{4}\Big[\big(t^{\alpha_1}\langle f(A)x,x\rangle+m_1(1-t^{\alpha_1})\langle f(B)x,x\rangle\big)\\ &\times\big((1-t^{\alpha_2})\langle g(A)x,x\rangle+m_2t^{\alpha_2}\langle g(B)x,x\rangle\big)\\ &+\big((1-t^{\alpha_1})\langle f(A)x,x\rangle+m_1t^{\alpha_1}\langle f(B)x,x\rangle\big)\\ &\times\big(t^{\alpha_2}\langle g(A)x,x\rangle+m_2(1-t^{\alpha_2})\langle g(B)x,x\rangle\big)\Big]\end{aligned}$$

Integrating of above inequalities over [0,1] we get the inequality.

Remark 2.3. If we take $m_1, m_2, \alpha_1, \alpha_2 = 1$ in Theorem(2.3) and $s_1, s_2 = 1$ in (1.3) inequalities is same.

Theorem 2.4. *$f, g : I \subset \mathbb{R} \to \mathbb{R}^+$ (α_1, m_2), (α_2, m_2)-convex and non-negative functions respectively. Let A, B selfadjoint operators on interval I.*
Let $\frac{1}{m}\langle Ax, x\rangle, \frac{1}{m}\langle Bx, x\rangle \subset I$, $m_1, m_2 \in (0, 1]$ and $\alpha_1, \alpha_2 \in [0, 1]$. Then the following inequality holds.

$$\begin{aligned}&\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\int_0^1\langle g(tA+m_2(1-t)B)x,x\rangle dt\\ &+\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle\int_0^1\langle f(tA+m_1(1-t)B)x,x\rangle dt\\ \leq\ & \frac{1}{2}\int_0^1\Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big]\\ &+\frac{1}{4}\Bigg[\Big(\frac{1}{\alpha_1+1}+\frac{1}{\alpha_2+1}-\frac{2}{\alpha_1+\alpha_2+1}\Big)(\mathbb{K}+m_1m_2S)\\ &+\Big(\frac{2}{\alpha_1+\alpha_2+1}+1-\frac{1}{\alpha_1+1}-\frac{1}{\alpha_2+1}\Big)(m_2L+m_1R)\Bigg]\\ &+\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle\end{aligned}$$

Here $\mathbb{K}, L, R$ and S are given by (2.1)-(2.4).

Proof. Since f and g are operator m_1, m_2-convex, respectively, then for $m, t \in [0,1]$ observe that

$$\begin{aligned}\Big\langle f\Big(\tfrac{A+m_1B}{2}\Big)x,x\Big\rangle &= \Big\langle f\Bigg(\frac{tA+m_1(1-t)B+m_1\Big((1-t)\frac{A}{m_1}+tB\Big)}{2}\Bigg)x,x\Big\rangle \\ &\le \frac{1}{2}\Big[\Big\langle f(tA+m_1(1-t)B)x,x\Big\rangle+\Big\langle f((1-t)A+m_1tB)x,x\Big\rangle\Big]\end{aligned}$$

$$\begin{aligned}\Big\langle g\Big(\tfrac{A+m_2B}{2}\Big)x,x\Big\rangle &= \Big\langle g\Bigg(\frac{tA+m_2(1-t)B+m_2\Big((1-t)\frac{A}{m_2}+tB\Big)}{2}\Bigg)x,x\Big\rangle \\ &\le \frac{1}{2}\Big[\Big\langle g(tA+m_2(1-t)B)x,x\Big\rangle+\Big\langle g((1-t)A+m_2tB)x,x\Big\rangle\Big]\end{aligned}$$

Note: For $a, b, c, d \in \mathbb{R}$ if $a \le b$ and $c \le d$ inequality

$$ad + bc \le ac + bd$$

exist. If we use the above note, then we obtain followings,

$$\begin{aligned} & \Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle \\ & \times\frac{1}{2}\Big[\langle g(tA+m_2(1-t)B)x,x\rangle+\langle g((1-t)A+m_2tB)x,x\rangle\Big] \\ & +\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle \\ & \times\frac{1}{2}\Big[\langle f(tA+m_1(1-t)B)x,x\rangle+\langle f((1-t)A+m_1tB)x,x\rangle\Big] \\ \le\ & \Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle + \\ & \frac{1}{2}\Big[\langle f(tA+m_1(1-t)B)x,x\rangle+\langle f((1-t)A+m_1tB)x,x\rangle\Big] \\ & \times\frac{1}{2}\Big[\langle g(tA+m_2(1-t)B)x,x\rangle+\langle g((1-t)A+m_2tB)x,x\rangle\Big] \\ \le\ & \frac{1}{4}\Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle \\ & +\langle f((1-t)A+m_1tB)x,x\rangle\langle g((1-t)A+m_2tB)x,x\rangle\Big] \\ & +\frac{1}{4}\Big[(t^{\alpha_1}\langle f(A)x,x\rangle+m_1(1-t^{\alpha_1})\langle f(B)x,x\rangle) \\ & \times((1-t^{\alpha_2})\langle g(A)x,x\rangle+m_2t^{\alpha_2}\langle g(B)x,x\rangle) \\ & +((1-t^{\alpha_1})\langle f(A)x,x\rangle+m_1t^{\alpha_1}\langle f(B)x,x\rangle) \\ & \times(t^{\alpha_2}\langle g(A)x,x\rangle+m_2(1-t^{\alpha_2})\langle g(B)x,x\rangle)\Big] \\ & +\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle. \end{aligned}$$

So, the proof is completed.

Remark 2.4. If we take $\alpha_1, \alpha_2 = 1$, then we obtain the following inequality;

$$\begin{aligned} & \Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle \int_0^1 \langle g(tA+m_2(1-t)B)x,x\rangle dt \\ & +\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle \int_0^1 \langle f(tA+m_1(1-t)B)x,x\rangle dt \\ \leq\ & \frac{1}{2}\int_0^1 \Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big] \\ & +\frac{\mathbb{K}}{12}+\frac{m_2L}{6}+\frac{m_1R}{6}+\frac{m_1m_2S}{12} \\ & +\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle \end{aligned}$$

2.3 Applications for Synchronous and Asynchronous Functions

We say that the functions $f, g : [a, b] \to \mathbb{R}$ are synchronous (asynchronous) on the interval [a,b] if the satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0$$

for each $t, s \in [a, b]$ [10]. It is obvious that, if f, g are monotonic and have the same monotonicity on the interval [a,b], then they are synchronous on [a,b] while if they have opposite monotonicity, they are synchronous. The following result provides an inequality of Ćebyśev type for functions of selfadjoint operators.

Theorem 2.5. *[10] Let A be a selfadjoint operator with $Sp(A) \subset [m, M]$ for some real numbers $m < M$, if $f, g : [m, M] \to \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\langle f(A)g(A)x, x\rangle \geq (\leq) \langle f(A)x, x\rangle\langle g(A)x, x\rangle$$

for any $x \in H$ with $\| x \| = 1$.

As a simple consequence of the above Theorem(2.5) we imply that if f, g synchronous, then

$$N(A, B)(x) \leq M(A, B)(x) \leq P(A, B)(x)$$

for any $x \in H$ with $\| x \| = 1$. If f, g asynchronous, then reverse inequalities holds as follow

$$N(A, B)(x) \geq M(A, B)(x) \geq P(A, B)(x)$$

Where M and N are given by (1.4) and (1.5).

$$P := P(A, B)(x) = \Big\langle [f(A)g(A) + f(B)g(B)]x, x\Big\rangle \tag{2.5}$$

Theorem 2.6. *Let $f, g : [m, M] \to \mathbb{R}^+$ operator (α, m)-convex and A, B, $Sp(A) \bigcup Sp(B) \subset [m, M]$, $\frac{1}{m}\langle Ax, x\rangle$, $\frac{1}{m}\langle Bx, x\rangle \subset I$, $m_1, m_2 \in (0, 1]$ and $\alpha_1, \alpha_2 \in [0, 1]$ Then,*

1. *If f, g are synchronous and $f, g \geq 0$ then the inequality*

$$\begin{aligned} & \int_0^1 \Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big] dt \\ \leq\ & \Big(\frac{m_1m_2(\alpha_1+1)+m_2\alpha_2}{(\alpha_1+1)(\alpha_1+1)}\Big)P \end{aligned}$$

Remark 2.5. If we get $\alpha_1, \alpha_2 = 1$ then we obtain the following inequality

$$\begin{aligned} & \int_0^1 \Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big] dt \\ \leq\ & \Big(\frac{2m_1m_2+m_2}{6}\Big)P \end{aligned}$$

2. *If f,g are synchronous and $f,g\geq 0$ then the inequality*

$$\begin{aligned}
&\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle\\
\leq\ & \frac{1}{2}\int_0^1\Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big]\\
&+\frac{1}{4}\left[\frac{2\alpha_2m_1m_2(\alpha_2+1)+2m_2(\alpha_2+1)(\alpha_1+1)}{(\alpha_1+1)(\alpha_2+1)(\alpha_1+\alpha_2+1)}\right]P
\end{aligned}$$

Remark 2.6. If we get $\alpha_1,\alpha_2=1$ then we obtain the following inequality

$$\begin{aligned}
&\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle\\
\leq\ & \frac{1}{2}\int_0^1\Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big]\\
&+\Big(\frac{m_1m_2+2m_2}{12}\Big)P
\end{aligned}$$

3. *If f,g are synchronous and $f,g\geq 0$ then the inequality*

$$\begin{aligned}
&\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\int_0^1\langle g(tA+m_2(1-t)B)x,x\rangle dt\\
&+\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle\int_0^1\langle f(tA+m_1(1-t)B)x,x\rangle dt\\
\leq\ & \frac{1}{2}\int_0^1\Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big]\\
&+\frac{1}{4}\left[\frac{2\alpha_2m_1m_2(\alpha_2+1)+2m_2(\alpha_2+1)(\alpha_1+1)}{(\alpha_1+1)(\alpha_2+1)(\alpha_1+\alpha_2+1)}\right]P\\
&+\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle
\end{aligned}$$

Remark 2.7. If we get $\alpha_1,\alpha_2=1$ then we obtain the following inequality

$$\begin{aligned}
&\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\int_0^1\langle g(tA+m_2(1-t)B)x,x\rangle dt\\
&+\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle\int_0^1\langle f(tA+m_1(1-t)B)x,x\rangle dt\\
\leq\ & \frac{1}{2}\int_0^1\Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big]\\
&+\Big(\frac{m_1m_2+2m_2}{12}\Big)P+\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle
\end{aligned}$$

Theorem 2.7. 1. *If f,g are asynchronous and $f,g\geq 0$ then the inequality*

$$\begin{aligned}
&\int_0^1\Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big]dt\\
\leq\ & \Big(\frac{m_1m_2(\alpha_1+1)+m_2\alpha_2}{(\alpha_1+1)(\alpha_1+1)}\Big)N
\end{aligned}$$

Remark 2.8. If we get $\alpha_1,\alpha_2=1$ then we obtain the following inequality

$$
\begin{aligned}
& \int_0^1 \Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big]dt \\
\leq\ & \Big(\frac{2m_1m_2+m_2}{6}\Big)N
\end{aligned}
$$

2. *If f, g are asynchronous and $f, g \geq 0$ then the inequality*

$$
\begin{aligned}
& \Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle \\
\leq\ & \frac{1}{2}\int_0^1 \Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big] \\
& +\frac{1}{4}\Bigg[\frac{2\alpha_2m_1m_2(\alpha_2+1)+2m_2(\alpha_2+1)(\alpha_1+1)}{(\alpha_1+1)(\alpha_2+1)(\alpha_1+\alpha_2+1)}\Bigg]N
\end{aligned}
$$

Remark 2.9. If we get $\alpha_1, \alpha_2 = 1$ then we obtain the following inequality

$$
\begin{aligned}
& \Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle \\
\leq\ & \frac{1}{2}\int_0^1 \Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big] \\
& +\Big(\frac{m_1m_2+2m_2}{12}\Big)N
\end{aligned}
$$

3. *If f, g are asynchronous and $f, g \geq 0$ then the inequality*

$$
\begin{aligned}
& \Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle \int_0^1 \langle g(tA+m_2(1-t)B)x,x\rangle dt \\
& +\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle \int_0^1 \langle f(tA+m_1(1-t)B)x,x\rangle dt \\
\leq\ & \frac{1}{2}\int_0^1 \Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big] \\
& +\frac{1}{4}\Bigg[\frac{2\alpha_2m_1m_2(\alpha_2+1)+2m_2(\alpha_2+1)(\alpha_1+1)}{(\alpha_1+1)(\alpha_2+1)(\alpha_1+\alpha_2+1)}\Bigg]N \\
& +\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle
\end{aligned}
$$

Remark 2.10. If we get $\alpha_1, \alpha_2 = 1$ then we obtain the following inequality

$$
\begin{aligned}
& \Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle \int_0^1 \langle g(tA+m_2(1-t)B)x,x\rangle dt \\
& +\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle \int_0^1 \langle f(tA+m_1(1-t)B)x,x\rangle dt \\
\leq\ & \frac{1}{2}\int_0^1 \Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big] \\
& +\Big(\frac{m_1m_2+2m_2}{12}\Big)N+\Big\langle f\Big(\frac{A+m_1B}{2}\Big)x,x\Big\rangle\Big\langle g\Big(\frac{A+m_2B}{2}\Big)x,x\Big\rangle.
\end{aligned}
$$

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