Copulas with Directional Dependence Property

Engin A. SUNGUR[1,♠], Salih ÇELEBİOĞLU[2]

[1]*Division of Science and Mathematics, Department of Statistics, University of Minnesota, Morris, Minnesota, USA*

[2]*Gazi University, Department of Statistics Ankara, Turkey*

ABSTRACT

In this article, we introduce a new way of generating copulas with the directional dependence property based on the logistic regression. The studied class of copulas leads to nonsymmetrical copula structures, therefore allowing for the development of various directional dependence models. The setup introduced in the research can be easily adapted to wide range of dependence modeling applications.

Key words Conditional Copulas, Directional Dependence, Logistic Regression, Principal Component Analysis.

1. INTRODUCTION AND MOTIVATION PROGRESSION OF RESEARCH

The main goal of this research is to create a class of copulas that exhibit the directional dependence property. Our investigation started with looking at the copulas of linear combinations of independently distributed uniform random variables. Let us denote such transformed random pair with (Y_1, Y_2), and the independent uniform pair with (U_1, U_2), i.e. $Y_1 = \alpha_{11}U_1 + \alpha_{12}U_2$ and $Y_2 = \alpha_{21}U_1 + \alpha_{22}U_2$, where $\alpha_{11} > \alpha_{12}$, $\alpha_{21} \leq -\alpha_{22}$, $\alpha_{11}, \alpha_{12} > 0, \alpha_{21} \geq 0, \alpha_{22} \leq 0$. Since

$$F_{Y_1 Y_2}(y_1, y_2) = P\{Y_1 \leq y_1, Y_2 \leq y_2\} = \int_0^1 P\left\{U_1 \leq \frac{y_1 - \alpha_{12}U_2}{\alpha_{11}}, U_1 \leq \frac{y_1 - \alpha_{22}U_2}{\alpha_{21}} \mid U_2 = u_2\right\} du_2$$

$$= \int_0^1 P\left\{U_1 \leq \min\left\{\frac{y_1 - \alpha_{12}u_2}{\alpha_{11}}, \frac{y_2 - \alpha_{22}u_2}{\alpha_{21}}\right\}\right\} du_2,$$

the copula of the transformed pair will have the form

$$C_{Y_1, Y_2}(v_1, v_2) = \int_0^1 P\left\{U \leq \min\left\{\frac{F_{Y_1}^{-1}(v_1) - \alpha_{12}u}{\alpha_{11}}, \frac{F_{Y_2}^{-1}(v_2) - \alpha_{22}u}{\alpha_{21}}\right\}\right\} du, \tag{1}$$

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♠Corresponding author, e-mail: sungurea@morris.umn.edu

where $F_{Y_i}^{-1}$ is the quantile function of $Y_i, i = 1,2$. By defining appropriate two place functions

$$k(v_1, u) = \left[F_{Y_1}^{-1}(v_1) - \alpha_{12}u\right]\big/\alpha_{11},$$

and $l(v_1, u) = \left[F_{Y_2}^{-1}(v_2) - \alpha_{22}u\right]\big/\alpha_{21}$,

we end up with a class of copula which had the general form

$$C(v_1, v_2) = \int_0^1 D\{k(v_1, u), l(v_2, u)\} du. \qquad (2)$$

Note that as long as the functions k and l are non-decreasing in their first arguments, the resulting function will be 2-increasing. Also, the structure of the class allows us to produce non-symmetric copulas, i.e., $C(v_1, v_2) \neq C(v_2, v_1)$ for some $v_1, v_2 \in [0,1]$, therefore leading to different copula regression functions.

Within this progression of our research we looked for some "natural" selections of functions k and l, and ended up with the following conditional copulas:

$$k(v_1, u) = P\{V_1 \leq v_1 \mid U = u\} = \frac{\partial C_1(u, v_1)}{\partial u} = C_1(v_1 \mid u),$$

$$l(v_2, u) = P\{V_2 \leq v_2 \mid U = u\} = \frac{\partial C_2(u, v_2)}{\partial u} = C_2(v_2 \mid u).$$

For these choices two-place function D is the conditional copula of V_1 and V_2 given U and the copula will have the form

$$C(v_1, v_2) = P(V_1 \leq v_1, V_2 \leq v_2) = \int_0^1 P(V_1 \leq v_1, V_2 \leq v_2 \mid U = u) du = \int_0^1 D\{C_1(v_1 \mid u), C_2(v_2 \mid u)\} du. \qquad (3)$$

It is trivial to show that the resulting two place function is grounded, 2-increasing and $C(1, v_2) = v_2$ and $C(v_1, 1) = v_1$. Note that $\partial C_i(u, v_i)\big/\partial u$ exists for almost all v_i, it lies between 0 and 1, and furthermore the functions $v_i \mapsto \partial C_i(u, v_i)\big/\partial u$ are defined and nondecreasing almost everywhere on $[0,1]$ (Nelsen [1], p. 11).

In fact, generating copulas of the form given in (2) are studied extensively with the objective of generating three

dimensional copulas with compatible marginals, see Dall'Aglio [2] and Quesada-Molina and Rodriguez-Lallena [3]. We are going to use the results obtained in this area heavily in the rest of the paper.

For the random triple (V_1, V_2, U) considered in equation (3), we can assume that U has a Bernoulli distribution instead of Uniform distribution on the unit interval. That will lead us to

$$C(v_1, v_2) = \sum_{u=0}^1 D\big(P(V_1 \leq v_1 \mid U = u), P(V_2 \leq v_2 \mid U = u)\big) P(U = u). \qquad (4)$$

The conditional distribution of U given $V_i = v_i$ can be modeled by using various link functions such as logit, probit, and complementary log-log. In this research we

will consider the logistic regression set up and use the model

$$\pi(v_i, \beta_{i,0}, \beta_{i,1}) = P\{U = 1 \mid V_i = v_i\} = \frac{\exp\{\beta_{i,0} + \beta_{i,1}v_i\}}{1 + \exp\{\beta_{i,0} + \beta_{i,1}v_i\}} = \frac{1}{1 + \exp\{-\beta_{i,0} - \beta_{i,1}v_i\}}.$$

Now if we let f_i and g be the conditional probability density function of V_i given U, and probability density function of V_i, respectively

$$f_i(v_i \mid U = 1) = \frac{P(U = 1 \mid V_i = v_i)g(v_i)}{P(U = 1)} = \frac{\pi(v_i, \beta_{i,0}, \beta_{i,1})}{P(U = 1)}$$

and

$$P\big(V_i \le v_i | U=1\big) = \frac{\int_0^{v_i} \pi(w,\beta_{i,0},\beta_{i,1})dw}{P(U=1)} = \frac{\Pi(v_1,\beta_{i,0},\beta_{i,1})}{P(U=1)},$$

$$\Pi(v_i,\beta_{i,0},\beta_{i,1}) = \frac{1}{\beta_{i,1}}\log\left|\frac{1+\exp\{\beta_{i,0}+\beta_{i,1}v_i\}}{1+\exp\{\beta_{i,0}\}}\right|.$$

Since

$$P\{U=1\} = \int_0^1 P\{U=1 | V_i=v_i\}dv_i = \Pi(1,\beta_{i,0},\beta_{i,1}),\ i=1,2,\ \text{i.e.,}\ \Pi(1,\beta_{1,0},\beta_{1,1}) = \Pi(1,\beta_{2,0},\beta_{2,1}),$$

we need restrictions on the model parameters. To get around of this restriction, we will fix the success probability for the Bernoulli random variable at *p,* and determine the $\beta_{i,0}$ in terms of $\beta_{i,1}$.

$$\Pi(1,\beta_{i,0},\beta_{i,1}) = p \Rightarrow \beta_{i,0} = \log\left(\frac{1-e^{\beta_{i,1}p}}{e^{\beta_{i,1}p}-e^{\beta_{i,1}}}\right).$$

Therefore,

$$\Pi(v_i,\beta_{i,0},\beta_{i,1}) = \frac{1}{\beta_{i,1}}\log\left[\frac{e^{\beta_{i,1}p}-e^{\beta_{i,1}}+e^{\beta_{i,1}v_i}-e^{\beta_{i,1}p}e^{\beta_{i,1}v_i}}{1-e^{\beta_{i,1}}}\right] = \Pi_r(v_i,\beta_{i,0},\beta_{i,1}).$$

The resulting copula class will have the form

$$D\left(\frac{\Pi_r(v_1,\beta_{1,1},p)}{p},K,\frac{\Pi_r(v_k,\beta_{k,1},p)}{p}\right)p + D\left(\frac{v_1-\Pi_r(v_1,\beta_{1,1},p)}{1-p},K,\frac{v_k-\Pi_r(v_k,\beta_{k,1},p)}{1-p}\right)(1-p). \qquad (5)$$

In the case of conditional independence, i.e., $D(v_1,\ldots,v_k | u) = \prod_{i=1}^{k} P\{V_i \le v_i | u\}$

$$C(v_1,K,v_k) = \frac{\prod_{i=1}^{k}\Pi_r(v_i,\beta_{i,1},p)}{p^{k-1}} + \frac{\prod_{i=1}^{k}[v_i-\Pi_r(v_i,\beta_{i,1},p)]}{(1-p)^{k-1}}$$

Lemma 1: Let

$$\Pi_r(v_i,\beta_{i,1},p) = \frac{1}{\beta_{i,1}}\log\left[\frac{e^{\beta_{i,1}p}-e^{\beta_{i,1}}+e^{\beta_{i,1}v_i}-e^{\beta_{i,1}p}e^{\beta_{i,1}v_i}}{1-e^{\beta_{i,1}}}\right].$$

(a) $\Pi_r(v_i,1,p) = \log\left[\dfrac{e^p-e^1+e^{v_i}-e^pe^{v_i}}{1-e^1}\right].$

(i) $\Pi_r(v_i,\beta_{i,1},1) = v_i,\ \Pi_r(v_i,\beta_{i,1},0) = 0,\ \Pi_r(1,\beta_{i,1},p) = p,\ \Pi_r(0,\beta_{i,1},p) = 0.$

(ii) $C(v_1,K,v_k) = D(v_1,K,v_k)$ for $p=0,1$.

(iii) $\pi_r(v_i,\beta_{i,1},p) = \dfrac{\partial\Pi_r(v_i,\beta_{i,1},p)}{\partial v_i} = \dfrac{e^{\beta_{i,1}v_i}-e^{\beta_{i,1}p}e^{\beta_{i,1}v_i}}{e^{\beta_{i,1}p}-e^{\beta_{i,1}}+e^{\beta_{i,1}v_i}-e^{\beta_{i,1}p}e^{\beta_{i,1}v_i}}.$

(iv) Conditional distribution function and density function of $V_i | U=u$ are

$$F_{V_i|U=u}(v_i|u) = \frac{(1-u)v_i-(-1)^u\Pi_r(v_i,\beta_{i,1},p)}{(1-u)-(-1)^u p}\ \text{and}$$

$$f_{V_i|U=u}(v_i|u) = \frac{(1-u)-(-1)^u\pi_r(v_i,\beta_{i,1},p)}{(1-u)-(-1)^u p},$$

respectively.

(v) $f_{V_i|U=1}(v_i|1)$ is nondecreasing iff $b_{i,1} > 0$ and nonincreasing iff $b_{i,1} < 0$.

(vi) $f_{V_i|U=1}(v_i|1)$ is nondecreasing (nonincreasing) iff $f_{V_i|U=0}(v_i|0)$ is nonincreasing (nondecreasing). Therefore, for $b_{i,1} > 0$, V_i will tend to higher values when $U=1$ and lower values when $U=0$. The effect of this can be easily seen from the copula density plots.

The class of copula which is constructed through the logistic regression model is very rich. It allows one to work with both symmetric and asymmetric dependence structures and the conditional copula D could be a member of any class of copulas. The symmetry behavior of the class has been determined by the model parameters. The symmetry has been achieved when $\beta_{1,1} = \beta_{2,1}$. Therefore, these parameters can provide "hidden" information on the directional dependence. The variable U will be called the direction variable and it can be defined in many different ways in application. As an example, one may want to study the relationship between wife and husband's ages at death and consider a Bernoulli direction variable which identifies who die first. In medical research, the dependence between two vital variables during a surgery can be modeled by considering outcome of the surgery as direction variable.

In the following sections we will look at the properties and characteristics of these two classes of copulas and discuss their possible applications on directional dependence modeling.

2. DIRECTIONAL COPULAS THROUGH CONDITIONING

Both of the classes of copulas that we have introduced are based on conditioning the random pair V_1, V_2 on U. In one case U is continuous in the other case it is discrete random variable. The second class takes the construction

process one step further and uses a model for U given V_i. We assumed that U is either uniformly distributed over the unit interval or a Bernoulli random variable.

Following Sungur [4], we will say that the uniformly distributed pair (V_1, V_2) is directionally dependent in their joint behavior if the form of the regression functions for $V_2 | V_1 = v_1$ and $V_1 | V_2 = v_2$ differ, i.e., $E[V_2 | V_1 = v_1] = r_{V_2|V_1}(w) \neq r_{V_1|V_2}(w)$. The well-known class of copula that has this property is the Rodríguez-Lallena and Úbeda-Flores [5] class, i.e., $C(v_1, v_2) = v_1 v_2 + f(v_1)g(v_2)$, and copulas with cubic sections (see Nelsen [6], and p.80 of [1]), i.e., $C(v_1, v_2) = v_1 v_2 + v_1(1-v_1)[\alpha(v_2)(1-v_1) + \beta(v_2)v_1]$. Another way of generating asymmetric copulas has been discussed in Genest et al. [7], and Edward Frees and Emiliano Valdez [8]. In their construction, they form a non-exchangeable class of bivariate copulas of the form

$$C_{\kappa,\lambda}(u,v) = u^{1-\kappa}v^{1-\lambda}C(u^\kappa, v^\lambda), \ 0 < \kappa, \lambda < 1.$$

2.1. BERNOULLI DIRECTION VARIABLE

The parameters of the copula that we have introduced in (5) can be grouped into three: (i) directional dependence parameters, $\beta_{1,1}$ and $\beta_{2,1}$, that are originated from the logistic regression, (ii) dependence parameter, θ, that is the parameter for the conditional copula D, and (iii) the "mixture" parameter, p, the parameter of the Bernoulli direction variable. The value of the p can be i. selected based on prior knowledge, ii. pre-determined through the design of experiment, or iii. estimated together with the other parameters of the introduced copula model. To study the properties of this class of copulas we need the following Lemma:

Lemma 1: Let

$$\Pi_r(v_i, \beta_{i,1}, p) = \frac{1}{\beta_{i,1}}\log\left[\frac{e^{\beta_{i,1}p} - e^{\beta_{i,1}} + e^{\beta_{i,1}v_i} - e^{\beta_{i,1}p}e^{\beta_{i,1}v_i}}{1 - e^{\beta_{i,1}}}\right]$$

and

$$\pi_r(v_i, \beta_{i,1}, p) = \frac{\partial\Pi_r(v_i, \beta_{i,1}, p)}{\partial v_i} = \frac{e^{\beta_{i,1}v_1}\left(1 - e^{\beta_{i,1}p}\right)}{e^{\beta_{i,1}v_1}\left(1 - e^{\beta_{i,1}p}\right) + e^{\beta_{i,1}p} - e^{\beta_{i,1}}}.$$

(i) $\Pi_r(v_i, \beta_{i,1}, 1) = v_i$, $\Pi_r(v_i, \beta_{i,1}, 0) = 0$, $\Pi_r(1, \beta_{i,1}, p) = p$, $\Pi_r(0, \beta_{i,1}, p) = 0$.

(ii) $\lim_{\beta_{i,1} \to 0}\Pi_r(v_i, \beta_{i,1}, p) = pv_i$.

(iii) Conditional distribution function and density function of $V_i|U = u$ are

$$F_{V_i|U=u}(v_i|u) = \frac{(1-u)v_i - (-1)^u \Pi_r(v_i, \beta_{i,1}, p)}{(1-u) - (-1)^u p},$$

$$f_{V_i|U=u}(v_i|u) = \frac{(1-u) - (-1)^u \pi_r(v_i, \beta_{i,1}, p)}{(1-u) - (-1)^u p},$$

respectively.

Note that $f_{V_i|U=1}(v_i|1)$ is nondecreasing iff $\beta_{i,1} > 0$ and nonincreasing iff $\beta_{i,1} < 0$. Also, $f_{V_i|U=1}(v_i|1)$ is nondecreasing (nonincreasing) iff $f_{V_i|U=0}(v_i|0)$ is nonincreasing (nondecreasing). Therefore, for $\beta_{i,1} > 0$, V_i will tend to have higher values when $U=1$ and lower values when $U=0$.

Theorem 1. For the class of copulas introduced in equation (5), the conditional copula structure will be preserved, i.e., $C(v_1, v_2) = D(v_1, v_2)$ if $\beta_{1,1} = \beta_{2,2} = 0$ or $p = 0, 1$.

The Pearson's correlation for the members of the class, ρ_C, can be decomposed into two parts

$$\rho_C = 12 \int_0^1 \int_0^1 [C(v_1, v_2) - D(v_1, v_2)]dudv + \rho_D = \delta_{C,D} + \rho_D$$

,

where ρ_D is the Pearson's correlation for the copula D. Provided that D is symmetric, all of the directional dependence information will be hidden in $\delta_{C,D}$ which can be viewed as a measure of distance between the two corresponding copulas.

Let us assume that the conditional copula D is a member of the Farlie-Gumbel-Morgenstern class, i.e., $D(v_1, v_2) = v_1 v_2 [1 + \theta((1-v_1)(1-v_2)]$. The copula density functions generated for this class are given in Figure 1.

(a) $\beta_{1,1} = 10, \beta_{2,1} = 30, p = 0.5, \theta = 0$.

(b) $\beta_{1,1} = 10, \beta_{2,1} = 30, p = 0.5, \theta = 0.9$.

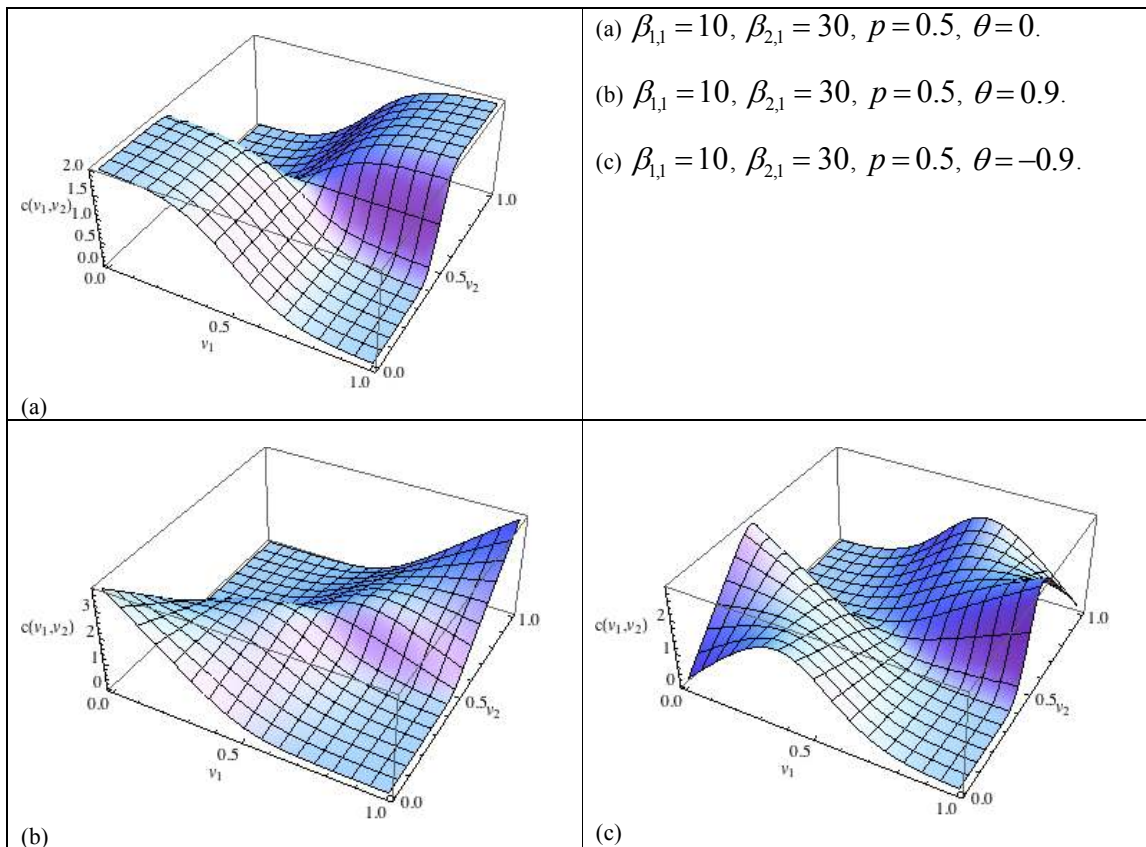(c) $\beta_{1,1} = 10, \beta_{2,1} = 30, p = 0.5, \theta = -0.9$.

Figure 1. Copula density functions for the three members of the class given in (5) under Farlie-Gumbel-Morgenstern conditional dependence structure.

If we look at the copula regression functions we can easily see the directional dependence structure. Figure 2, provides the copula regression functions for the

conditional independence case for various values of p. The copula regression functions are calculated by using

$$r_{V_i|V_j}\left(v_j\right)=1-\int\limits_0^1 \frac{\partial C(v_1,v_2)}{\partial v_j}dv_i.$$

For the conditional independence case

$$r_{V_i|V_j}\left(v_j\right)=1+\frac{p\left\{\int\limits_0^1 \Pi_r\left(v_i,\beta_{i,1},p\right)dv_2-\frac{1}{2}\right\}}{p(1-p)}-\frac{\left\{\int\limits_0^1 \Pi_r\left(v_i,\beta_{i,1},p\right)dv_2-\frac{p}{2}\right\}}{p(1-p)}\pi_r\left(v_j,\beta_{j,1},p\right)$$

where $\pi_r\left(v_i,\beta_{i,1},p\right)=\dfrac{e^{\beta_{i,1}v_i}\left(1-e^{\beta_{i,1}p}\right)}{e^{\beta_{i,1}v_i}\left(1-e^{\beta_{i,1}p}\right)+e^{\beta_{i,1}p}-e^{\beta_{i,1}}}.$
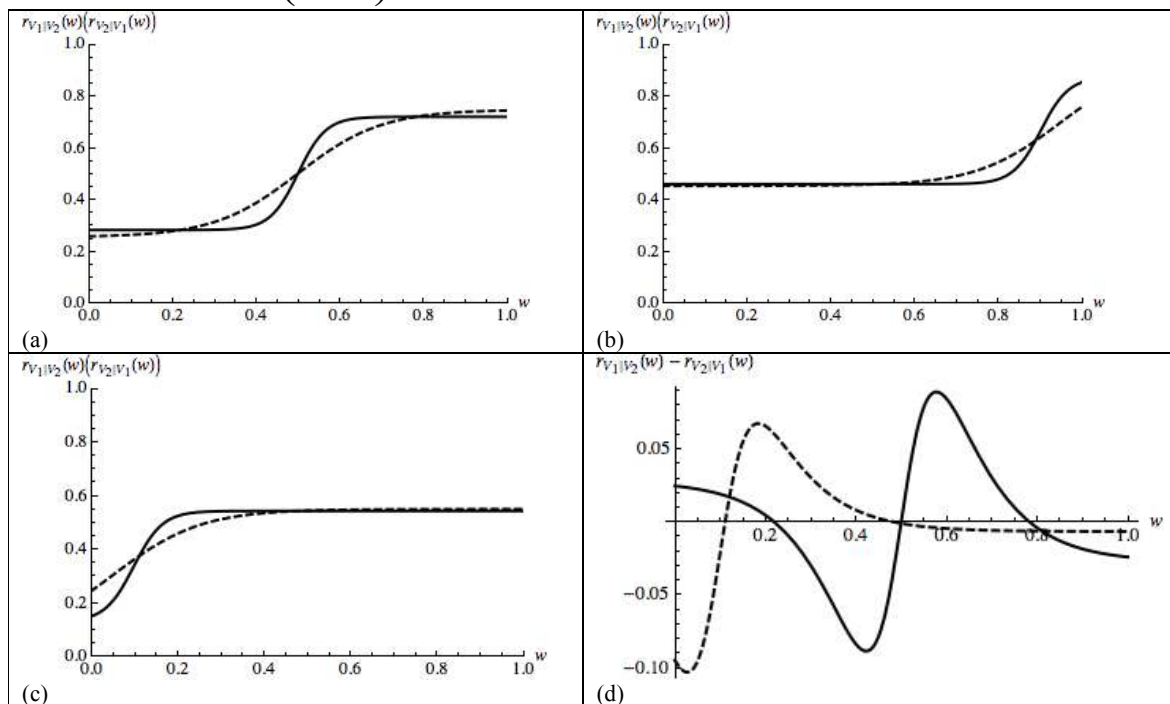

Figure 2. The plots of the copula regression functions [(a) $\beta_{1,1}=10$, $\beta_{2,1}=30$, $p=0.5$, $\theta=0$, (b) $\beta_{1,1}=10$, $\beta_{2,1}=30$, $p=0.1$, $\theta=0$, (c) $\beta_{1,1}=10$, $\beta_{2,1}=30$, $p=0.9$, $\theta=0$] and the plot of difference between copula regression functions for $\beta_{1,1}=10$, $\beta_{2,1}=30$, $p=0.5$, $\theta=0$ (solid line) and $p=0.9$ (dashed line).

Following Sungur [4], we will assess the strength of the directional dependence by using

$$\rho_{V_1\rightarrow V_2}=\frac{E\left\{\left[E(V_2|V_1)-E(V_2)\right]^2\right\}}{E\left\{\left[V_2-E(V_2)\right]^2\right\}}\text{ and }\rho_{V_1\leftarrow V_2}=\frac{E\left\{\left[E(V_1|V_2)-E(V_1)\right]^2\right\}}{E\left\{\left[V_1-E(V_1)\right]^2\right\}}.$$

These two directional dependence measures can be interpreted as the proportion of the variance (total variation) of V_i that can be explained by the regression of V_i on V_j. The behaviors of these two directional

dependence measures are given for the conditional independence case in Figure 3. The directional dependence gets stronger as one moves toward $p=0.5$ for fixed $\beta_{1,1}$ and $\beta_{2,1}$.
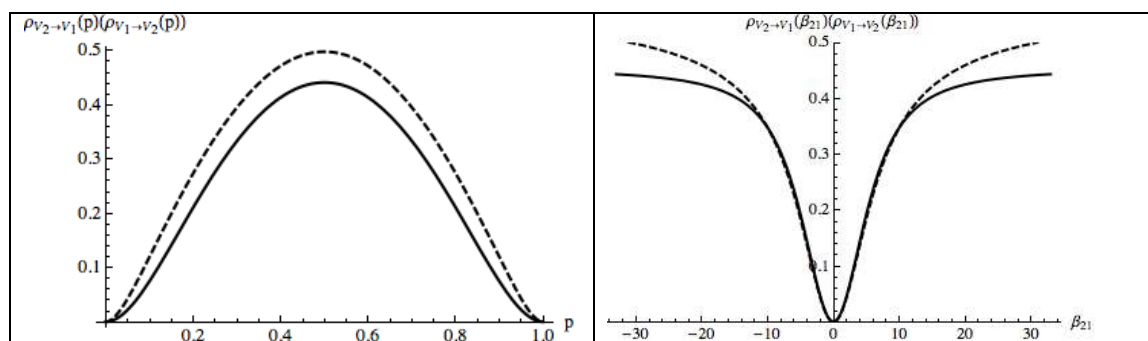
Figure 3. The plots of the directional dependence measure as a function of p and $\beta_{2,1}$ (remaining parameters are fixed at $\beta_{1,1} = 10$, $\beta_{2,1} = 30$ and $p = 0.5$).

2.2. UNIFORM DIRECTION VARIABLE

In this section, we will study the class of functions with a form given in Equation 3. Suppose that (V_1, V_2, U) is a random triple, where conditional on U, the dependence of V_1 and V_2 is explained by the two-dimensional conditional distribution D which leads to the unconditional copula given in (3). The asymmetric behavior on this unconditional copula could originate either from the one-dimensional conditional copulas or two-dimensional conditional copula or both. The selection of functions triple (D, C_1, C_2) is restricted with the compatibility conditions, i.e.,

$$\int_0^u D\{C_1(v_1 \mid u^*), C_2(v_2 \mid u^*)\}du^*$$ should be a three-dimensional copula.

Table 1 summarizes the key results that are related with the directional dependence modeling for the copulas given in (5). We have borrowed them from Dall'Aglio [2] and Quesada-Molina and Rodriguez-Lallena [3].

Table 1. Basic forms of copula generated by various conditional copula combinations.

$C_2(u,v_2) \rightarrow$ $C_1(u,v_1) \downarrow$	$C_2(u,v_2)$	$\max\{u+v_2-1,0\}$	uv_2	$\min\{u,v_2\}$
$C_1(u,v_1)$		$v_1 - C_1(v_1, 1-v_2)$		$C_1(v_1,v_2)$
$\max\{u+v_1-1,0\}$	$v_2 - C_2(1-v_1,v_2)$	$\min\{v_1,v_2\}$		
uv_1			$D(v_1,v_2)$	
$\min\{u,v_1\}$	$C_2(v_1,v_2)$			$\min\{v_1,v_2\}$

Within our setup, directional dependence property of the copula could be generated from the various sources; (i) directional dependence on the conditional copulas C_1 and C_2, (ii) structure of the bivariate distribution connecting conditional copulas, (iii) or both. In this section we will concentrate on (i). Similar to the concept of conditional independence, we will assume that conditioned on U, directional dependence disappears. In the application this can be faced when the two variables are perfectly dependent given the random "common explanatory" variable, such as random time, population etc. This setup opens new possibilities in time series modeling.

When the two conditional copulas are the same we end up with the copula of prefect dependence for (V_1, V_2). Another case is when V_1 and U are independent and V_2 and U are perfectly positively dependent. Then,

$$C(v_1, v_2) = \int_0^{v_2} \min\{v_1, 1\}du = v_1 v_2.$$

If one of them is stochastically dominant, i.e.,

$$\left(V_i \mid U = u\right) \overset{D}{\succ} \left(V_j \mid U = u\right) \Leftrightarrow \frac{\partial C_i(u,w)}{\partial u} \geq \frac{\partial C_j(u,w)}{\partial u}, \forall w \in [0,1],$$

then new class returns the non-dominant variable.

One of the options for D that will work for any selection of C_1 and C_2 is the membership to the Fréchet class of copulas, i.e.,

$$D \in \zeta_F = \left\{ D_F(u,v) = \alpha \min\{u,v\} + (1-\alpha)\max\{u+v-1,0\}; 0 \leq \alpha \leq 1 \right\}.$$

Quesada-Molina and Rodriguez-Lallena [3] shows that any two-place function defined by

$$C(v_1,v_2) = \int_0^1 D_F \left\{ C_1(v_1 \mid u), C_2(v_2 \mid u) \right\} du$$

is a copula. The basic form of copula that is generated under this restriction will have the following form

$$C(v_1,v_2) = \alpha \left\{ \int_{B^+(u|v_1,v_2)} C_1(v_1 \mid u) du + \int_{\left(B^+(u|v_1,v_2)\right)^C} C_2(v_2 \mid u) du \right\}$$

$$+ (1-\alpha) \int_{A^-(u|v_1,v_2)} \left\{ C_1(v_1 \mid u) + C_2(v_2 \mid u) - 1 \right\} du \tag{6}$$

where

$$B^+(u \mid v_1,v_2) = \left\{ u : C_1(v_1 \mid u) \leq C_2(v_2 \mid u), u \in [0,1] \right\}$$

$$A^-(u \mid v_1,v_2) = \left\{ u : C_1(v_1 \mid u) + C_2(v_2 \mid u) - 1 \geq 0, u \in [0,1] \right\}.$$

$D \in \zeta_F$				
$C_2(u,v_2) \rightarrow$ $C_1(u,v_1) \downarrow$	$C_2(u,v_2)$	$\max\{u+v_2-1,0\}$	uv_2	$\min\{u,v_2\}$
$C_1(u,v_1)$	See equation 6	$v_1 - C_1(v_1,1-v_2)$		$C_1(v_1,v_2)$
$\max\{u+v_1-1,0\}$	$v_2 - C_2(1-v_1,v_2)$	$\min\{v_1,v_2\}$		
uv_1			$D_F(v_1,v_2)$	
$\min\{u,v_1\}$	$C_2(v_1,v_2)$			$\min\{v_1,v_2\}$

Table 2. Basic forms of copula generated by various conditional copula combinations for the $D \in \zeta_F$.

Example. *A simple directional dependence model.* Let us view the mixing random variable U as a directional dependence covariate. Such a covariate may behave independent of one of the key variables, say V_1, and dependent on the second variable through the FGM copula structure, i.e.,

$$C_1(u,v_1) = uv_1, C_2(u,v_2) = uv_2 + \theta uv_2(1-u)(1-v_2)$$

, In this case

$$C(v_1,v_2) = \int_0^1 P\left(U \leq \min\{v_1, v_2(1+\theta-\theta v_2) - 2\theta uv_2(1-v_2)\} \right) du$$

$$= \left(\frac{1}{2} - \frac{v_1 - v_2}{2\theta v_2(1-v_2)} \right) \left\{ (v_1 - v_2) - \theta v_2(1-v_2)\left(\frac{1}{2} + \frac{v_1 - v_2}{2\theta v_2(1-v_2)} \right) \right\} + v_2 ,$$

for $-\theta v_2(1-v_2) \leq v_1 - v_2 \leq \theta v_2(1-v_2), v_1, v_2 \in [0,1]$

$$= v_2, v_1 \geq v_2 + \theta v_2(1-v_2), v_1, v_2 \in [0,1]$$

$$= v_1, v_1 \leq v_2 - \theta v_2(1-v_2), v_1, v_2 \in [0,1]. \tag{7}$$

It is trivial, to show that the two-place function given in (7) is in fact a copula. The Pearson's correlation is $\rho_C = 1 - \theta^2/15$. The copula regression function for $V_1 \mid V_2 = v_2$ is:

$$r_{V_1 \mid V_2}(v_2) = 1 - \int_0^1 \frac{\partial C(v_1, v_2)}{\partial v_2} dv_1 = \frac{1}{3} v_2 \left(3 + \theta^2 \left(1 - 3v_2 + 2v_2^2\right)\right)$$

For the $V_2 \mid V_1 = v_1$, the form of copula regression function gets rather complicated. To present the result in a manageable complexity we will let $\theta > 0$,

$$v_2^L = \frac{1 + \theta - \sqrt{((1+\theta)^2 - 4\theta v_1)}}{2\theta} \text{ and } v_2^U = \frac{-1 + \theta + \sqrt{((1-\theta)^2 + 4\theta v_1)}}{2\theta}.$$

$$r_{V_2 \mid V_1}(v_1) = 1 - \int_0^1 \frac{\partial C(v_1, v_2)}{\partial v_1} dv_2$$

$$= 1 - \left[\frac{v_2^U - v_2^L}{2} + \frac{1}{2\theta} \log \left\{ \left(\frac{v_2^U - 1}{v_2^L - 1} \right)^{v_1 - 1} \Bigg/ \left(\frac{v_2^U}{v_2^L} \right)^{v_1} \right\} + v_2^L \right].$$

The behaviors of the copula regression functions in both directions are given in Figure 4.

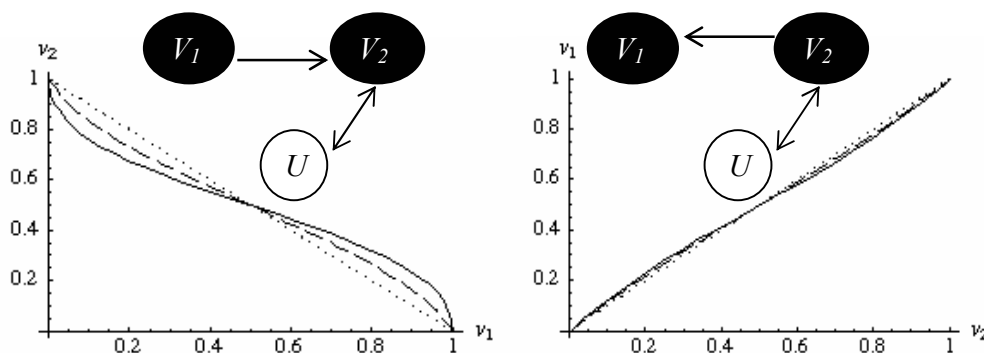Figure 4. The copula regression functions for $V_2 \mid V_1 = v_1$ (on the left), and $V_1 \mid V_2 = v_2$ (on the right) evaluated at $\theta = 0.01, 0.75, 0.99$ and represented dotted to solid curves respectively.

3. SOME USES OF THE CLASS IN STATISTICAL ANALYSIS

For the class of copula that we have introduced in Section 2.1 we can set up a classification process. Suppose that we have observed only (V_1, V_2) with the copula that is a member of this class. This setup will allow us to estimate $\beta_{1,1}, \beta_{2,1}, p$ and remaining dependence parameters that will come from the D. Assuming that we have two groups that will match with the levels of the Bernoulli variable U, two classification regions R_1 and R_2 will be

$$R_1 : P\{U = 1 \mid V_1 = v_1, V_2 = v_2\} \geq P\{U = 0 \mid V_1 = v_1, V_2 = v_2\}$$

$$R_2 : P\{U = 1 \mid V_1 = v_1, V_2 = v_2\} < P\{U = 0 \mid V_1 = v_1, V_2 = v_2\}.$$

By using

$$P\{U=u|V_1=v_1,V_2=v_2\}=\frac{f_{V_1,V_2|U}(v_1,v_2|u)P\{U=u\}}{c(v_1,v_2)}=\frac{d_u(v_1,v_2,\beta_{1,1},\beta_{2,1},p)P\{U=u\}}{c(v_1,v_2)}$$

where,

$$d_u(v_1,v_2,\beta_{1,1},\beta_{2,1},p)=\begin{cases}\dfrac{\partial^2}{\partial v_1\partial v_2}D\left(\dfrac{\Pi_r(v_1,\beta_{1,1},p)}{p},\dfrac{\Pi_r(v_2,\beta_{2,1},p)}{p}\right), & u=1\\[3mm]\dfrac{\partial^2}{\partial v_1\partial v_2}D\left(\dfrac{v_1-\Pi_r(v_1,\beta_{1,1},p)}{1-p},\dfrac{v_2-\Pi_r(v_2,\beta_{2,1},p)}{1-p}\right), & u=0\end{cases}$$

one ends up with the following regions

$$R_1:d_1(v_1,v_2,\beta_{1,1},\beta_{2,1},p)p\geq d_0(v_1,v_2,\beta_{1,1},\beta_{2,1},p)(1-p),$$

$$R_2:d_1(v_1,v_2,\beta_{1,1},\beta_{2,1},p)p< d_0(v_1,v_2,\beta_{1,1},\beta_{2,1},p)(1-p).$$

Under the special case of conditional independence, classification function turns out to be linear with the following form

$$\beta_{1,1}v_1+\beta_{2,1}v_2-\log\left\{\frac{p\left[e^{\beta_{1,1}p}-e^{\beta_{1,1}}\right]\left[e^{\beta_{2,1}p}-e^{\beta_{2,1}}\right]}{(1-p)\left(1-e^{\beta_{1,1}p}\right)\left(1-e^{\beta_{2,1}p}\right)}\right\}.$$

4. CONCLUDING REMARKS

The classes of copulas introduced in this paper possess directional dependence property and use directional variable in discrete and continuous forms giving researchers a chance to move away from the symmetric case and create more general dependence models. The dependence structure could change based on a third variable which might be discrete or continuous. Selection of a discrete or continuous direction variable will heavily depend on the researcher's interest. In discrete setup the Bernoulli direction variable has an advantage of using developed inferential tools for the logistic regression. Increasing the dimension of dependence covariates, $U,$ and the responses, $V_1,$ and $V_2,$ will be our top priority. The copula class given in (3) bases on the linear combinations of independent uniform variables has promising features to achieve this objective. Since the setup mimics the ideas in principal component analysis and logistic regression, we are hoping to introduce a fresh look to these techniques through copulas in our next paper.

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