Gazi University Journal of Science
GU J Sci
25(1):97-106 (2012)

ORIGINAL ARTICLE

Better Error Estimation for a Certain Family of Summation Integral Type Operators

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Received: 28.04.2011 Revised: 08.06.2011 Accepted:05.08.2011

ABSTRACT
In the present paper, we propose a modification for a certain family of summation integral type operators. We study direct results on the modified operators. It is also observed that our modified operators have better estimates over the original operators.
Key words and phrases: Better error estimation, Peetre.s $K-$functional, Lipschitz condition, Bézier variant, Bounded variation function.

1. INTRODUCTION

Let f be a local integrable function on the interval $[0,\infty)$. In [8], a certain family of summation integral type operators is defined by

$$G_{n,c}(f,x)=n\sum_{k=1}^{\infty}p_{n,k}(x,c)\int_{0}^{\infty}p_{n+c,k-1}(t,c)f(t)dt+p_{n,0}(x,c)f(0) \qquad (1)$$

where

$$p_{n,k}(x,c)=\frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x),$$

(i) if $\phi_{n,c}(x)=e^{-nx}$ for $c=0$, then the operators are reduced to the Phillips operators introduced in [6,7] and (ii) if $\phi_{n,c}(x)=(1+cx)^{-n/c}$ for $c\in\mathbb{N}-\{0\}$, then the operators are reduced to the perators studied in [3].

Ispir and Yuksel [5] introduced the Bézier variant of these operators for approximation of functions of bounded variation. Yuksel and Ispir [9] studied weighted approximation of these operators. Later, Gupta and Ivan [4] studied the rate of simultaneous approximation for the Bézier variant of these operators. It is known that these operators do not preserve the linear functions. Recently, Duman, Ozarslan and Della Vecchia [2] introduced a modifed Szász- Mirakjan- Kantorovich type operators which preserve the linear functions. Our aim is to obtain better error estimates for a certain family of summation integral type operators.

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2. CONSTRUCTION OF THE OPERATORS

In this section, we modify the operators given by (1) such that the linear functions are preserved. To obtain linear positive operators which preserve the linear functions, for $n > c,$ we define $r_n^*(x) = \dfrac{n-c}{n}x.$ We replace x in the operators $G_{n,c}(f,x)$ by $r_n^*(x).$ Then we give the following modification of the $G_{n,c}(f,x)$:

$$B_{n,c}(f,x) = n\sum_{k=1}^{\infty} p_{n,k}\left(r_n^*(x),c\right)\int_0^{\infty} p_{n+c,k-1}(t,c)f(t)dt + p_{n,0}\left(r_n^*(x),c\right)f(0) \tag{2}$$

where for $c = 0,$ $p_{n,k}\left(r_n^*(x),0\right) = p_{n,k}(x,0)$ and for $c \in \mathbb{N} - \{0\},$

$$p_{n,k}\left(r_n^*(x),c\right) = \frac{\Gamma(n/c+k)\left(cr_n^*(x)\right)^k}{k!\Gamma(n/c)\left(1+cr_n^*(x)\right)^{n/c+k}}.$$

Here Γ denotes the Gamma function and the basis element $p_{n+c,k-1}(t,c)$ are defined in (1). By simple calculations, we obtain the following lemmas.

Lemma 2.1. For $x \in [0,\infty)$ and the test functions $e_i(x) = x^i,\ i = 0,1,2,$ we get

(i) $B_{n,c}(e_0,x) = 1,$

(ii) $B_{n,c}(e_1,x) = x,$

(iii) $B_{n,c}(e_2,x) = \dfrac{(n-c)(n+c)x^2 + 2nx}{n(n-2c)}.$

Proof. For $c = 0,$ since $p_{n,k-1}(t,0) = \dfrac{e^{-nx}(nt)^{k-1}}{(k-1)!},$ we easily obtain the integral

$$\int_0^{\infty} p_{n,k-1}(t,0)t^m dt = \frac{1}{n^{m+1}}\int_0^{\infty} e^{-u}\frac{u^{k+m-1}}{(k-1)!}du = \begin{cases} \dfrac{1}{n} & ,\ for\ m = 0 \\ \dfrac{k...(k+m-1)}{n^{m+1}} & ,\ for\ m \in \mathbb{N} - \{0\} \end{cases}. \tag{3}$$

For $c \in \mathbb{N} - \{0\},$ since $p_{n+c,k-1}(t,c) = \dfrac{\Gamma(n/c+k)(ct)^{k-1}}{(k-1)!\Gamma(n/c+1)(1+ct)^{n/c+k}},$ by a direct calculation we get

$$\int_0^{\infty} p_{n+c,k-1}(t,c)t^m dt = \frac{\Gamma(n/c+k)}{c^{m+1}(k-1)!\Gamma(n/c+1)}\int_0^{\infty}\frac{u^{k+m-1}}{(1+u)^{n/c+k}}du = \begin{cases} \dfrac{1}{n} & ,\ for\ m = 0 \\ \dfrac{k...(k+m-1)}{n...(n-mc)} & ,\ for\ m \in \mathbb{N} - \{0\} \end{cases}. \tag{4}$$

Therefore, combining the formulas (3) and (4), for every $c \in \mathbb{N}$ we can write

$$\int_0^{\infty} p_{n+c,k-1}(t,c)t^m dt = \begin{cases} \dfrac{1}{n} & ,\ for\ m = 0 \\ \dfrac{k...(k+m-1)}{n...(n-mc)} & ,\ for\ m \in \mathbb{N} - \{0\} \end{cases}. \tag{5}$$

From (5), the proof of (i) is obvious. By using (5) we write

$$B_{n,c}(e_1,x)=\sum_{k=1}^{\infty}\frac{k}{n-c}\,p_{n,k}\left(r_n^*(x),c\right). \tag{6}$$

In (6), writing $c=0,$ we have

$$B_{n,0}(e_1,x)=\frac{1}{n}\sum_{k=1}^{\infty}\frac{e^{-nx}(nx)^k}{(k-1)!}=x \tag{7}$$

and similarly, for $c\in\mathbb{N}-\{0\},$ we obtain

$$B_{n,c}(e_1,x)=\frac{n/c}{n-c}\sum_{k=1}^{\infty}\frac{\Gamma(n/c+k)\left(cr_n^*(x)\right)^k}{(k-1)!\Gamma(n/c+1)\left(1+cr_n^*(x)\right)^{n/c+k}}=x. \tag{8}$$

(7) and (8) completes the proof of (ii). Using the formula (5), we write

$$B_{n,c}(e_2,x)=\sum_{k=1}^{\infty}\frac{k(k+1)}{(n-c)(n-2c)}\,p_{n,k}\left(r_n^*(x),c\right). \tag{9}$$

In (9), using the equality

$$k(k+1)=k(k-1)+2k, \tag{10}$$

for $c=0,$ we have

$$\begin{aligned}B_{n,0}(e_2,x)&=\frac{1}{n^2}\sum_{k=2}^{\infty}\frac{e^{-nx}(nx)^k}{(k-2)!}+\frac{2}{n^2}\sum_{k=1}^{\infty}\frac{e^{-nx}(nx)^k}{(k-1)!}\\&=x^2+\frac{2}{n}x.\end{aligned} \tag{11}$$

In (9), using the equality (10), for $c\in\mathbb{N}-\{0\},$ we get

$$\begin{aligned}B_{n,c}(e_2,x)&=\frac{n/c(n/c+1)}{(n-c)(n-2c)}\sum_{k=2}^{\infty}\frac{\Gamma(n/c+k)\left(cr_n^*(x)\right)^k}{(k-2)!\Gamma(n/c+2)\left(1+cr_n^*(x)\right)^{n/c+k}}\\&\quad+\frac{2n/c}{(n-c)(n-2c)}\sum_{k=1}^{\infty}\frac{\Gamma(n/c+k)\left(cr_n^*(x)\right)^k}{(k-1)!\Gamma(n/c+1)\left(1+cr_n^*(x)\right)^{n/c+k}}\\&=\frac{(n-c)(n+c)}{n(n-2c)}x^2+\frac{2}{n-2c}x.\end{aligned} \tag{12}$$

(11) and (12) gives proof of (iii).

Obviously, for $s\in\mathbb{N},$ we choose the natural constants $a_0,\ldots,a_s$ as such that the equality

$$\prod_{j=0}^{s}(k+j)=\sum_{i=0}^{s}a_i\prod_{j=0}^{s-i}(k-j) \tag{13}$$

holds. From (13), for $m\in\mathbb{N}-\{0\},$ we can obtain the formula

$$B_{n,c}(t^m,x)=\sum_{i=0}^{m-1}a_i\frac{(n-c)^{m-i}n\ldots(n+(m-1-i)c)}{n^{m-i}(n-c)\ldots(n-mc)}x^{m-i}. \tag{14}$$

Notice that the operators $B_{n,c}(f,x)$ preserve the linear functions, that is, for $h(t)=at+b,a,b\in\mathbb{R},$ we get $B_{n,c}(h,x)=h(x).$

Lemma 2.2. For $x\in[0,\infty),n,m\in\mathbb{N}-\{0\}$ and with the notation $\varphi_x(t)=t-x$ for $t\in[0,\infty)$ we have

(i) $B_{n,c}(\varphi_x,x)=0,$

(ii) $B_{n,c}(\varphi_x^2,x)=\dfrac{c(2n-c)x^2+2nx}{n(n-2c)}$,

(iii) $B_{n,c}(\varphi_x^m,x)=O\left(n^{-[\![(m+1)/2]\!]}\right)$.

Proof. From linearity of $B_{n,c}(f,x)$ operators we write the equality $B_{n,c}(\varphi_x,x)=B_{n,c}(t,x)-xB_{n,c}(1,x)$. Using Lemma 2.1, we obtain the proof of (i). If we write $B_{n,c}(\varphi_x^2,x)=B_{n,c}(t^2,x)-2xB_{n,c}(t,x)+x^2B_{n,c}(1,x)$ then, using Lemma 2.1, we get $B_{n,c}(\varphi_x^2,x)=\left(\dfrac{(n-c)(n+c)}{n(n-2c)}-1\right)x^2+\dfrac{2x}{n-2c}$, which completes the proof of (ii). For the proof of (iii), using the equality (14), we obtain

$$\begin{aligned}B_{n,c}(\varphi_x^m,x)&=\sum_{j=0}^{m}(-1)^j\binom{m}{j}x^jB_{n,c}(t^{m-j},x)\\&=O\left(n^{-[\![(m+1)/2]\!]}\right).\end{aligned}$$

Let $C_B[0,\infty)$ be the space of all real valued continuous bounded functions on $[0,\infty)$, equipped with the norm $\|f\|=\sup_{x\in[0,\infty)}|f(x)|$. The Peetre's $K-$ functional is defined by

$$K_2(f,\delta)=\inf\left\{\|f-g\|+\delta\|g''\|:g\in W_\infty^2\right\},\delta>0$$

where $W_\infty^2=\left\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\right\}$. By [1], there exists a positive constant C such that

$$K_2(f,\delta)\le C\omega_2(f,\sqrt{\delta}) \tag{15}$$

where $\omega_2(f,\sqrt{\delta})=\sup_{0<h\le\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)-f(x)|$.

Theorem 2.3. Let $f\in C_B[0,\infty)$. Then there exists a positive constant C such that for every $x\in[0,\infty)$ and for $n>2c$, we have

$$|B_{n,c}(f,x)-f(x)|\le C\omega_2\left(f,\sqrt{\frac{c(2n-c)x^2+2nx}{n(n-2c)}}\right).$$

Proof. Let $g\in W_\infty^2$ and $x\in[0,\infty)$. Using Taylor's expansion

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-u)g''(u)du,$$

Lemma 2.1 and Lemma 2.2, we have

$$B_{n,c}(g,x)-g(x)=B_{n,c}\left(\int_x^t(t-u)g''(u)du,x\right).$$

Using the inequality

$$\left|\int_x^t(t-x)g''(u)du\right|\le\|g''\|(t-x)^2$$

and from Lemma 2.2, we write

$$\begin{aligned}|B_{n,c}(g,x)-g(x)|&\le B_{n,c}\left((t-x)^2,x\right)\|g''\|\\&=\frac{c(2n-c)x^2+2nx}{n(n-2c)}\|g''\|.\end{aligned}$$

Hence, in view of Lemma 2.1,

$$\left|B_{n,c}(f,x)-f(x)\right|\le\left|B_{n,c}(f-g,x)-(f-g)(x)\right|+\left|B_{n,c}(g,x)-g(x)\right|$$
$$=2\|f-g\|+\frac{c(2n-c)x^2+2nx}{n(n-2c)}\|g''\|.$$

Now taking infimum over $g\in W_\infty^2$ on the right side of the above inequality and using the inequality (15) we get the desired result.

3. BETTER ERROR ESTIMATION

In this section, we compute the rate of convergence of the operators $B_{n,c}(f,x)$ given by (2). Then, we show that the operators $B_{n,c}(f,x)$ have a better error estimation than the operators $G_{n,c}(f,x)$ given by (1).

Theorem 3.1. For every $f\in C_B[0,\infty), x\in[0,\infty)$ and $n>2c,$ we have

$$\left|B_{n,c}(f,x)-f(x)\right|\le 2\omega\left(f,\eta_x\right)$$

where $\eta_x=\sqrt{\dfrac{c(2n-c)x^2+2nx}{n(n-2c)}}$ and $\omega\left(f,\eta_x\right)$ is the first modulus of continuity of f.

Proof. Let $f\in C_B[0,\infty)$ and $x\in[0,\infty).$ Using linearity and monotonicity of $B_{n,c}(f,x)$ we easily get, for every $\delta>0$ and $n>2c,$ that

$$\left|B_{n,c}(f,x)-f(x)\right|\le 2\omega(f,\delta)\left\{1+\frac{1}{\delta}\sqrt{B_{n,c}\left(\varphi_x^2,x\right)}\right\}.$$

Now applying Lemma 2.2 and choosing $\delta=\eta_x$ the proof is completed.

Remark 3.1. For the operator $G_{n,c}(f,x)$ given by (1) we may write that, for every $f\in C_B[0,\infty),\ x\in[0,\infty)$ and $n>2c,$

$$\left|G_{n,c}(f,x)-f(x)\right|\le 2\omega(f,\nu_x)$$

where $\nu_x=\sqrt{\dfrac{2c(n+c)x^2+2nx}{(n-c)(n-2c)}}.$

Now we claim that the error estimation in Theorem 3.1 is better than that of Remark 3.1 provided $f\in C_B[0,\infty),\ x\in[0,\infty)$ and $n>2c.$ Indeed, considering the equality

$\dfrac{n+c}{n-c}=\dfrac{n-c/2}{n-c}+\dfrac{3c/2}{n-c},$ we get $\eta_x\le\nu_x$, which corrects our claim.

Now we can compute the rate of convergence of the operators $B_{n,c}(f,x)$ by means of the elements of the Lipschitz class $Lip_M(\beta), 0<\beta\le 1.$ As usual, we say that $f\in C_B[0,\infty)$ belongs to $Lip_M(\beta)$ if the inequality

$$\left|f(t)-f(x)\right|\le M\left|t-x\right|^\beta \tag{16}$$

holds for all $x,t\in[0,\infty).$

Theorem 3.2. If $f\in Lip_M(\beta), x\in[0,\infty)$ and $n>2c,$ then we have

$$\left|B_{n,c}(f,x)-f(x)\right|\le M\left\{\frac{c(2n-c)x^2+2nx}{n(n-2c)}\right\}^{\beta/2}.$$

Proof. Since $f\in Lip_M(\beta)$ and $x\in[0,\infty),$ using the inequality (16) and then applying the Hölder inequality with $p=\dfrac{2}{\beta}$ and $q=\dfrac{2}{2-\beta},$ we get

$$\left|B_{n,c}(f,x)-f(x)\right| \le \left|B_{n,c}\left(\left|f(t)-f(x)\right|,x\right)\right|$$
$$\le MB_{n,c}\left(\left|t-x\right|^{\beta},x\right)$$
$$\le M\left\{B_{n,c}\left(\varphi_x^2,x\right)\right\}^{\beta/2}$$
$$\le M\left\{\frac{c(2n-c)x^2+2nx}{n(n-2c)}\right\}^{\beta/2}.$$

Remark 3.2. Following the proof of Theorem 3.2, using $G_{n,c}\left(\varphi_x^2,x\right)$ instead of $B_{n,c}\left(\varphi_x^2,x\right)$, we get following result:

$$\left|G_{n,c}(f,x)-f(x)\right| \le M\left\{\frac{2c(n+c)x^2+2nx}{(n-c)(n-2c)}\right\}^{\beta/2}.$$

We conclude the paper with the study of the rate of convergence for bounded variation functions. To this result, we consider the Bézier variant of $B_{n,c}(f,x)$.

4. BEZIER VARIANT OF THE OPERATORS $B_{n,c,\alpha}(f,x)$

We define the Bézier variant of the operators $B_{n,c,\alpha}(f,x)$ given by (2) as follows:

$$B_{n,c,\alpha}(f,x)=n\sum_{k=1}^{\infty}Q_{n,k}^{(\alpha)}\left(r_n^*(x),c\right)\int_0^{\infty}p_{n+c,k-1}(t,c)f(t)dt+Q_{n,0}^{(\alpha)}\left(r_n^*(x),c\right)f(0) \tag{17}$$

where $Q_{n,k}^{(\alpha)}\left(r_n^*(x),c\right)=J_{n,k}^{\alpha}\left(r_n^*(x),c\right)-J_{n,k+1}^{\alpha}\left(r_n^*(x),c\right)$ and $J_{n,k}\left(r_n^*(x),c\right)=\sum_{j=k}^{\infty}p_{n,j}\left(r_n^*(x),c\right)$.

Alternatively we may rewrite the operators $B_{n,c,\alpha}(f,x)$ given by (17) as

$$B_{n,c,\alpha}(f,x)=\int_0^{\infty}K_{n,c,\alpha}(x,t)f(t)dt$$

where

$$K_{n,c,\alpha}(x,t)=n\sum_{k=1}^{\infty}Q_{n,k}^{(\alpha)}\left(r_n^*(x),c\right)p_{n+c,k-1}(t,c)+Q_{n,0}^{(\alpha)}\left(r_n^*(x),c\right)\theta(t) \tag{18}$$

and $\theta(t)$ is the Dirac delta function.

Lemma 4.1 ([8]). For all $x\in(0,\infty), n>c$ and $k\in\mathbb{N}$ we have the inequality

$$p_{n,k}\left(r_n^*(x),c\right)\le\sqrt{\frac{1+c((n-c)/n)x}{2e(n-c)x}}.$$

Lemma 4.2. Let $x\in(0,\infty)$ and $K_{n,c,\alpha}(x,t)$ be defined by (18). Then, for $\lambda>2$ and for sufficiently large n, we have

(i) $\beta_{n,c,\alpha}(x,y)=\int_0^{y}K_{n,c,\alpha}(x,t)dt\le\frac{\alpha\lambda x(1+x)}{n(x-y)^2}$ *for* $0\le y<x$,

(ii) $1-\beta_{n,c,\alpha}(x,z)=\int_z^{\infty}K_{n,c,\alpha}(x,t)dt\le\frac{\alpha\lambda x(1+cx)}{n(z-x)^2}$ *for* $x<z<\infty$.

Proof. In view of Lemma 2.2 and using inequality $\left|a^{\alpha}-b^{\alpha}\right|\le\alpha\left|a-b\right|,\ 0\le a,b\le1$ and $\alpha\ge1$, we have

$$\beta_{n,c,\alpha}(x,y)=\int_0^y \frac{(x-t)^2}{(x-y)^2}K_{n,c,\alpha}(x,t)dt$$
$$\leq \frac{\alpha}{(x-y)^2}\,B_{n,c}\left(\varphi_x^2,x\right),$$

which gives (i). The proof of (ii) is similar.

Theorem 4.3. Let f be a bounded variation function on every finite subinterval of $[0,\infty)$ and let satisfy the growth condition $f(t)=O(1+t^m),t\to\infty.$ If $x\in(0,\infty),\alpha\geq 1,m\in\mathbb{N}$ and $\lambda>2$ are given, then there exists a constant $C(f,\alpha,m,x)$ such that for n sufficiently large we have

$$\left|B_{n,c,\alpha}(f,x)-\left[\frac{1}{\alpha+1}f(x+)+\frac{\alpha}{\alpha+1}f(x-)\right]\right|\leq 4\alpha\sqrt{\frac{1+c((n-c)/n)x}{2e(n-c)x}}\left|f(x+)-f(x-)\right|$$
$$+\frac{6\alpha\lambda(1+cx)}{nx}\sum_{k=1}^{n}V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)+C(f,\alpha,m,x)\alpha 2^m\frac{(1+x)^m}{x^m}O(n^{-m}),$$

where

$$g_x(t)=\begin{cases}f(t)-f(x-),0\leq t<x\\ 0 \quad ,t=x\\ f(t)-f(x+),t>x\end{cases},$$

$V_a^b(g_x)$ is the total variation of g_x on $[a,b]$.

Proof. For any bounded variation function $f,$ it is known that

$$f(t)=\frac{1}{\alpha+1}f(x+)+\frac{\alpha}{\alpha+1}f(x-)+\frac{f(x+)-f(x-)}{2}\left(sgn_x(t)+\frac{\alpha-1}{\alpha+1}\right)$$
$$+g_x(t)+\delta_x(t)\left[f(t)-\frac{f(x+)+f(x-)}{2}\right]$$

where

$$sgn_x(t)=\begin{cases}-1,0\leq t<x\\ 0 \ ,t=x\\ 1 \ ,t>x\end{cases} \quad \text{and} \quad \delta_x(t)=\begin{cases}1, \ t=x\\ 0, \ t\neq x\end{cases}.$$

It follows that

$$\left|B_{n,c,\alpha}(f,x)-\left[\frac{1}{\alpha+1}f(x+)+\frac{\alpha}{\alpha+1}f(x-)\right]\right|$$
$$\leq\frac{1}{2}\left|f(x+)-f(x-)\right|\left|B_{n,c,\alpha}(sign_x,x)+\frac{\alpha-1}{\alpha+1}\right| \tag{19}$$
$$+\left|f(x)-\frac{f(x+)+f(x-)}{2}\right|\left|B_{n,c,\alpha}(\delta_x,x)\right|+\left|B_{n,c,\alpha}(g_x,x)\right|$$

For the operators $B_{n,c,\alpha}(f,x),$ it is obvious that $B_{n,c,\alpha}(\delta_x,x)=0.$ Considering the equality $n\int_x^\infty p_{n+c,k}(t,c)dt=\sum_{j=0}^{k}p_{n,k}(x,c),$ we may write $B_{n,c,\alpha}(sign_x,x)$ as

$$B_{n,c,\alpha}(sign_x,x)=-1+2n\sum_{k=1}^{\infty}Q_{n,k}^{(\alpha)}\left(r_n^*(x),c\right)\int_x^{\infty}p_{n+c,k-1}(t,c)dt$$
$$=-1+2\sum_{k=1}^{\infty}Q_{n,k}^{(\alpha)}\left(r_n^*(x),c\right)\sum_{j=0}^{k-1}p_{n,j}(x,c)$$
$$=-1+2\sum_{j=0}^{\infty}p_{n,j}(x,c)J_{n,j+1}^{\alpha}\left(r_n^*(x),c\right).$$

Hence, we get

$$B_{n,c,\alpha}(sgn_x,x)+\frac{\alpha-1}{\alpha+1}=2\sum_{j=0}^{\infty}p_{n,j}(x,c)J_{n,j+1}^{\alpha}\left(r_n^*(x),c\right)-\frac{2}{\alpha+1}\sum_{j=0}^{\infty}Q_{n,j}^{(\alpha+1)}\left(r_n^*(x),c\right)$$

since $\sum_{j=0}^{\infty}Q_{n,j}^{(\alpha+1)}\left(r_n^*(x),c\right)=1$.

By the mean value theorem, it follows

$$Q_{n,j}^{(\alpha+1)}\left(r_n^*(x),c\right)=J_{n,j}^{\alpha+1}\left(r_n^*(x),c\right)-J_{n,j+1}^{\alpha+1}\left(r_n^*(x),c\right)=(\alpha+1)p_{n,j}\left(r_n^*(x),c\right)\gamma_{n,j}^{\alpha}\left(r_n^*(x),c\right)$$

where $J_{n,j+1}\left(r_n^*(x),c\right)<\gamma_{n,j}\left(r_n^*(x),c\right)<J_{n,j}\left(r_n^*(x),c\right)$.

Hence, using Lemma 4.1, we obtain

$$\left|B_{n,c,\alpha}(sgn_x,x)+\frac{\alpha-1}{\alpha+1}\right|\le 2\sum_{j=0}^{\infty}\max\left\{p_{n,j}(x,c),p_{n,j}\left(r_n^*(x),c\right)\right\}\left|J_{n,j+1}^{\alpha}\left(r_n^*(x),c\right)-J_{n,j}^{\alpha}\left(r_n^*(x),c\right)\right|$$
$$\le 2\alpha\sum_{j=0}^{\infty}\max\left\{p_{n,j}(x,c),p_{n,j}\left(r_n^*(x),c\right)\right\}p_{n,j}\left(r_n^*(x),c\right) \tag{20}$$
$$\le 4\alpha\sqrt{\frac{1+c((n-c)/n)x}{2e(n-c)x}}.$$

Using Lemma 4.2, we can estimate $B_{n,c,\alpha}(g_x,x)$. By Lebesgue- Stieltjes integral representation, we have

$$B_{n,c,\alpha}(g_x,x)=\left(\int_0^{x-x/\sqrt{n}}+\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}+\int_{x+x/\sqrt{n}}^{\infty}\right)K_{n,c,\alpha}(x,t)g_x(t)dt \tag{21}$$
$$=E_1+E_2+E_3.$$

Firstly we estimate E_2. For $t\in[x-x/\sqrt{n},x+x/\sqrt{n}]$ we have

$$\left|E_2\right|\le\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}\left|g_x(t)\right|K_{n,c,\alpha}(x,t)dt.$$

Since $\left|g_x(t)\right|\le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x)$ and $0\le\int_a^b K_{n,c,\alpha}(x,t)dt\le 1$, for $(a,b)\subset[0,\infty)$, we conclude

$$\left|E_2\right|\le\frac{1}{n}\sum_{k=1}^{n}V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x). \tag{22}$$

Next we estimate E_1. Writing $y=x-x/\sqrt{n}$ and using Lebesgue- Stieltjes integration by parts, we have

$$E_1=\int_0^y g_x(t)d_t\left(\beta_{n,c,\alpha}(x,t)\right)=g_x(y)\beta_{n,c,\alpha}(y,x)+\int_0^y\beta_{n,c,\alpha}(x,t)d_t\left(-g_x(t)\right).$$

$-V_t^x(g_x)$ is a nondecreasing function of t. Since $\left|g_x(y)\right|\le V_y^x(g_x)$ we conclude that

$$|E_1| \le V_y^x(g_x)\beta_{n,c,\alpha}(x,y) + \int_0^y \beta_{n,c,\alpha}(x,t)d_t\left(-V_t^x(g_x)\right).$$

Lemma 4.2 implies that

$$|E_1| \le V_y^x(g_x)\frac{\alpha\lambda x(1+cx)}{n(x-y)^2} + \frac{\alpha\lambda x(1+cx)}{n}\int_0^y \frac{1}{(x-t)^2}d_t\left(-V_t^x(g_x)\right).$$

Integrating the last term by parts, we get

$$|E_1| \le \frac{\alpha\lambda x(1+cx)}{n}\left[\frac{V_0^x(g_x)}{x^2} + 2\int_0^y \frac{V_t^x(g_x)}{(x-t)^3}dt\right].$$

Now replacing the variable t in the last integral by $x - x/\sqrt{u}$ we obtain

$$|E_1| \le \frac{\alpha\lambda x(1+cx)}{n}\left[x^{-2}V_0^x(g_x) + x^{-2}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^x(g_x)\right].$$

Hence,

$$|E_1| \le \frac{2\alpha\lambda(1+cx)}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^x(g_x). \tag{23}$$

Finally we estimate E_3. We put

$$\nu_{n,c,\alpha}(x,t) = \begin{cases} 1-\beta_{n,c,\alpha}(x,t), & 0 \le t < 2x \\ 0, & t = 2x \end{cases}.$$

and $z = x + x/\sqrt{n}$, then

$$\begin{aligned} E_3 &= -\int_z^{2x} g_x(t)d_t\left(\nu_{n,c,\alpha}(x,t)\right) - g_x(2x)\int_{2x}^{\infty} K_{n,c,\alpha}(x,t)dt + \int_{2x}^{\infty} g_x(t)d_t\left(\beta_{n,c,\alpha}(x,t)\right) \\ &= E_{31} + E_{32} + E_{33}. \end{aligned} \tag{24}$$

We write

$$E_{31} = g_x(z)\nu_{n,c,\alpha}(x,t) + \int_z^{2x} \nu_{n,c,\alpha}^{\sim}(x,t)d_t\left(g_x(t)\right)$$

where $\nu_{n,c,\alpha}^{\sim}(x,t)$ is normalized form of $\nu_{n,c,\alpha}(x,t)$. Since $\nu_{n,c,\alpha}(x,z-) = \nu_{n,c,\alpha}^{\sim}(x,z)$ and $g_x(z-) \le V_x^{z-}(g_x)$, we have

$$E_{31} \le V_x^{z-}(g_x)\nu_{n,c,\alpha}(x,z) + \int_z^{2x} \nu_{n,c,\alpha}^{\sim}(x,t)d_t\left(-V_x^t(g_x)\right).$$

Applying Lemma 4.2

$$\begin{aligned} |E_{31}| &\le V_x^{z-}(g_x)\frac{\alpha\lambda x(1+cx)}{n(z-x)^2} + \frac{\alpha\lambda x(1+cx)}{n}\int_z^{2x-}\frac{1}{(x-t)^2}d_t\left(-V_x^t(g_x)\right) \\ &\quad + \frac{1}{2}V_x^{2x-}(g_x) + \int_{2x}^{\infty} K_{n,c,\alpha}(x,u)du \\ &\le V_x^{z-}(g_x)\frac{\alpha\lambda x(1+cx)}{n(z-x)^2} + \frac{\alpha\lambda x(1+cx)}{n}\int_z^{2x-}\frac{1}{(x-t)^2}d_t\left(-V_x^t(g_x)\right) \\ &\quad + \frac{1}{2}V_x^{2x-}(g_x) + \frac{\lambda x(1+cx)}{n}. \end{aligned}$$

Thus arguing similarly as in estimate of E_1, we get

$$\left|E_{31}\right| \le \frac{2\alpha\lambda(1+x)}{nx}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \tag{25}$$

Again Lemma 4.2, we get

$$\left|E_{32}\right| \le \frac{\alpha\lambda(1+x)}{nx}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \tag{26}$$

Finally for $n > m,$ we can write

$$\left|E_{33}\right| \le M\int_{2x}^{\infty} K_{n,c,\alpha}(x,t)\left[(1+t)^m + (1+x)^m\right]dt.$$

For $t \ge 2x,$ using the inequalities $(1+t)^m \le 2^m \frac{(1+x)^m}{x^{2m}}(t-x)^{2m}$ and $(1+x)^m \le 2^m \frac{(1+x)^m}{x^{2m}}(t-x)^{2m},$ considering Lemma 2.1 and Lemma 2.2, we obtain

$$\left|E_{33}\right| \le C(f,\alpha,m,x)\alpha 2^m \frac{(1+x)^m}{x^{2m}} O(n^{-m}). \tag{27}$$

Combining the estimates of (19)-(27) we reach the required result. This completes proof of the theorem.

ACKNOWLEDGEMENTS

The author is thankful to the reviewers for making valuable suggestions, leading to a better presentation of the paper

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