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GENERALIZED BURNSIDE ALGEBRA OF TYPE B_n

HASAN ARSLAN AND HIMMET CAN

Abstract. In this paper, we firstly give an alternative method to determine the size of $C(S_n)$ which is the set of elements of type S_n in a finite Coxeter system (W_n, S_n) of type B_n. We also show that all cuspidal classes of W_n are actually the conjugacy classes $\mathcal{K}_\lambda$ for every $\lambda \in \mathcal{DP}^+(n)$. We then define the generalized Burnside algebra $HB(W_n)$ for W_n and construct a surjective algebra morphism between $HB(W_n)$ and Mantaci-Reutenauer algebra $\mathcal{MR}(W_n)$. We obtain a set of orthogonal primitive idempotents e_λ, $\lambda \in \mathcal{DP}(n)$ of $HB(W_n)$, that is, all the characteristic class functions of W_n. Finally, we give an effective formula to compute the number of elements of all the conjugacy classes $\mathcal{K}_\lambda$, $\lambda \in \mathcal{DP}(n)$ of W_n.

1. Introduction

Solomon's descent algebra of a finite Coxeter system (W, S) was introduced by Solomon in 1976 in [11]. In 1992, Bergeron, Bergeron, Howlett and Taylor elegantly reconstructed the Solomon's descent algebra for a finite Coxeter system by using the group structure of Coxeter group and also they introduced a family of orthogonal primitive idempotents of the Solomon's descent algebra by lifting orthogonal primitive idempotents of parabolic Burnside algebra in [1].

Let W_n be the Coxeter group of type B_n. As a convention, throughout this paper, we denote by $HB(W_n)$, $\mathcal{MR}(W_n)$, $\mathcal{SC}(n)$ and $\mathcal{DP}(n)$ the generalized Burnside algebra of type B_n, the Mantaci-Reutenauer algebra, the set of all signed compositions of n and the set of all double partitions of n, respectively.

Mantaci-Reutenauer algebra $\mathcal{MR}(W_n)$, which is a subalgebra of the group algebra $\mathbb{Q}W_n$ and contains the Solomon's descent algebras of type A_n and B_n, was firstly constructed in [10]. In [2], Bonnafé and Hohlweg reconstructed $\mathcal{MR}(W_n)$ by the methods which depend more on the structure of W_n as a Coxeter group. Bonnafé studied the representation theory of Mantaci-Reutenauer algebra in [3].

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In Section 3, we prove that for every positive signed composition A of n, the parabolic closure of the reflection subgroup W_A is W_n. As a result of this, we obtain that the number of all elements of type S_n is equal to $\sum_{\lambda\in\mathcal{DP}^+(n)}|\mathcal{K}_\lambda|$ and realize that all cuspidal classes of W_n are the conjugacy classes $\mathcal{K}_\lambda$ for $\lambda\in\mathcal{DP}^+(n)$.

In Section 4, we introduce the Burnside algebra $HB(W_n)$ generated by isomorphism classes of reflection subgroups of W_n corresponding to signed compositions of n. We call $HB(W_n)$ *generalized Burnside algebra* of type B_n. Generalized Burnside algebra $HB(W_n)$ is isomorphic to the algebra $\mathbb{Q}\mathrm{Irr}W_n$ generated by the irreducible characters of W_n. Then we construct a set of orthogonal primitive idempotents of $HB(W_n)$. These orthogonal primitive idempotents are actually all the characteristic class functions of the Coxeter group W_n. We determine the coefficient of the sign character ε_n of W_n in the expression of the each orthogonal primitive idempotent of $HB(W_n)$ in terms of irreducible characters of W_n. We get a formula to compute the number of elements of all the conjugacy classes $\mathcal{K}_\lambda$, $\lambda\in\mathcal{DP}(n)$ of W_n.

2. Preliminaries

2.1. **Hyperoctahedral group.** Let (W_n, S_n) denote a Coxeter group of type B_n and write its generating set as $S_n=\{t,s_1,\cdots,s_{n-1}\}$. Any element w of W_n acts by the permutation on the set $X_n=\{-n,\cdots,-1,1,\cdots,n\}$ such that for every $i\in I_n$, $w(-i)=-w(i)$. The Dynkin diagram of W_n is as follows:

$$B_n\ :\ \overset{t}{\circ}\Leftarrow\overset{s_1}{\circ}-\overset{s_2}{\circ}-\cdots-\overset{s_{n-1}}{\circ}.$$

If $J\subset S_n$, the subgroup W_J generated by J is called a *standard parabolic subgroup* of W_n. A *parabolic subgroup* of W_n is a subgroup of W_n conjugate to W_J for some $J\subset S_n$. Let $t_1:=t$ and $t_i:=s_{i-1}t_{i-1}s_{i-1}$ for each i, $2\le i\le n$. Put $T_n:=\{t_1,\cdots,t_n\}$. It is well-known that there are the following relations between the elements of S_n and T_n:

(1) $t_i^2=1, s_j^2=1$ for all $i,j,\ 1\le i\le n,\ 1\le j\le n-1$;
(2) $ts_1ts_1=s_1ts_1t$;
(3) $s_is_{i+1}s_i=s_{i+1}s_is_{i+1}$ for all $i,\ 1\le i\le n-2$;
(4) $ts_i=s_it,\ 1<i\le n-1$;
(5) $s_is_j=s_js_i$ for $|i-j|>1$;
(6) $t_it_j=t_jt_i$ for $1\le i,j\le n$.

We denote by $l:W_n\to\mathbb{N}$ the length function attached to S_n. Let $\mathcal{T}_n$ denote the reflection subgroup of W_n generated by T_n. It is also clear that $\mathcal{T}_n$ is a normal subgroup of W_n. Now let $S_{-n}=\{s_1,\cdots,s_{n-1}\}$ and let W_{-n} denote the reflection subgroup of W_n generated by S_{-n}, where W_{-n} is isomorphic to the symmetric group Ξ_n of degree n. Thus $W_n=W_{-n}\ltimes\mathcal{T}_n$.

Let $\{e_1,\cdots,e_n\}$ be the canonical basis of the Euclidian space $\mathbb{R}^n$ over $\mathbb{R}$. Let

$$\Psi_n^+=\{e_i:1\le i\le n\}\cup\{e_j+\lambda e_i:\lambda\in\{-1,1\}\text{ and }1\le i<j\le n\}.$$

Then Ψ_n is a root system of type B_n. For further information about the Coxeter groups of type B_n, see [8], [9].

A *signed composition* of n is an expression of n as a finite sequence $A = (a_1, \cdots, a_k)$ whose each part consists of non-zero integers such that $\sum_{i=1}^{k} |a_i| = n$. Put $|A| = \sum_{i=1}^{k} |a_i|$. We write $\mathcal{SC}(n)$ to denote the set of all signed compositions of n.

Let $A = (a_1, \cdots, a_k) \in \mathcal{SC}(n)$. A is said to be *positive*(resp. *negative*) if $a_i > 0$ (resp. $a_i < 0$) for every $i \geq 1$. If $a_i < 0$ for every $i \geq 2$, then A is called *parabolic*. Let define $A^+ = (|a_1|, \cdots, |a_r|)$. Then A^+ is a positive signed composition of n. The set of positive signed compositions of n is denoted by $\mathcal{SC}^+(n)$.

A *double partition* $\mu = (\mu^+; \mu^-)$ of n consists of a pair of partitions μ^+ and μ^- such that $|\mu| = |\mu^+| + |\mu^-| = n$. If the number of positive parts of n (resp. negative parts of n) is equal to zero, then we write $\emptyset$ instead of μ^+ (resp. μ^-). We denote the set of all double partitions of n by $\mathcal{DP}(n)$. We define $\mathcal{DP}^+(n) = \{\mu = (\mu^+; \mu^-) \in \mathcal{DP}(n) \ : \ \mu^- = \emptyset\}$. For $\mu = (\mu^+; \mu^-) \in \mathcal{DP}(n)$, $\hat{\mu} := \mu^+ * -\mu^-$ is the signed composition obtained by appending the sequence of components of μ^+ to that of $-\mu^-$ [2].

Now let $A \in \mathcal{SC}(n)$. If μ^+ (resp. μ^-) is rearrangement of the positive parts (resp. absolute value of negative parts) of A in decreasing order, then $\boldsymbol{\lambda}(A) := (\mu^+; \mu^-)$ is a double partition of n and also $\boldsymbol{\lambda}(\hat{\mu}) = \mu$ for every $\mu \in \mathcal{DP}(n)$ [2]. In [2], Bonnafé and Hohlweg constructed some reflection subgroups of W_n corresponding to signed compositions of n as an analogue to Ξ_n as follows: For each $A = (a_1, \cdots, a_k) \in \mathcal{SC}(n)$, the reflection subgroup W_A of W_n is generated by S_A, which is

$$\begin{aligned} S_A = &\{s_p \in W_{-n} \ : \ |a_1| + \cdots + |a_{i-1}| + 1 \leq p \leq |a_1| + \cdots + |a_i| - 1\} \\ &\cup \{t_{|a_1|+\cdots+|a_{j-1}|+1} \in T_n\} \mid a_j > 0\} \subset S_n^{'} \end{aligned}$$

where $S_n' = \{s_1 \cdots s_{n-1}, t_1, t_2, \cdots, t_n\}$. By the definition of S_A, there exists an isomorphism $W_A \cong W_{a_1} \times \cdots \times W_{a_k}$ [2]. By taking into account the definition of the generating set S_A and the isomorphism $W_A \cong W_{a_1} \times \cdots \times W_{a_r}$, for $i,\ 1 \leq i \leq r$ if $a_i > 0$ then we have rank $W_{a_i} = a_i$ and if $a_i < 0$ then we have rank $W_{a_i} = |a_i| - 1$. Therefore, we get

$$\text{rank} W_A = |S_A| = n - ng(A),$$

where $ng(A)$ denotes the number of negative parts of A. Because of $\sum_{i=1}^{r} |a_i| = n$, we obtain $\text{rank} W_A = |S_A| \leq n$.

For $A, B \in \mathcal{SC}(n)$, we write $A \subset B$ if $W_A \subset W_B$, where $\subset$ is a partial ordering relation on $\mathcal{SC}(n)$ [2]. For $A \in \mathcal{SC}(n)$ let cox_A be a Coxeter element of W_A in terms of generating set S_A. For $B, B^{'} \subset A$, we write $B \equiv_A B'$ if W_B is conjugate to $W_{B'}$ under W_A and also cox_B and $\text{cox}_{B'}$ are conjugate to each other in W_A if and only if $B \equiv_A B^{'}$ [3]. We write $B \equiv_n B^{'}$ if W_B is conjugate to $W_{B'}$ under W_n. This equivalence is a special case for these kind of reflection subgroups of W_n, because this statement is not true for every reflection subgroup of W_n. Although some two reflection subgroups R and R' of W_n contain W_n-conjugate Coxeter elements cox_R and $\text{cox}_{R'}$ respectively, these subgroups are not able to W_n-conjugate to each other [6]. For every element w of W_n, there exists a unique $\lambda \in \mathcal{DP}(n)$ such that w is W_n-conjugate to $\text{cox}_{\hat{\lambda}}$ [3]. Let $\mathcal{K}_\lambda$ be the conjugacy class of W_n corresponding to $\lambda \in \mathcal{DP}(n)$. Since the number of conjugacy classes of W_n is equal to $|\mathcal{DP}(n)|$, thus we may split up W_n into $|\mathcal{DP}(n)|$ conjugacy classes. In [3], Bonnafé showed that for $A, B \in \mathcal{SC}(n)$, W_A is conjugate to W_B in W_n if and only if $\boldsymbol{\lambda}(A) = \boldsymbol{\lambda}(B)$.

For a subset X of W_n, we denote by $\text{Fix}(X) = \{v \in \mathbb{R}^n : \forall x \in X,\ x(v) = v\}$ the subspace of $\mathbb{R}^n$ fixed by X and let write $W_{\text{Fix}(X)} = \{w \in W_n : \forall v \in \text{Fix}(X),\ w(v) = v\}$ for the stabilizer of $\text{Fix}(X)$ in W_n. By [6], the set $W_{\text{Fix}(X)}$ is called the *parabolic closure* of X and it is denoted by $A(X)$. For any $w \in W_n$, if we take $X = \{w\}$ then we write $\text{Fix}(w)$ and $A(w)$ instead of $\text{Fix}(\{w\})$ and $A(\{w\})$, respectively. By [1], w is said to be an element of *type* J if there exists a $J \subset S_n$ such that $A(w)$ is conjugate to W_J under W_n.

2.2. **Mantaci-Reutenauer algebra.** For any $A \in \mathcal{SC}(n)$, we set

$$D_A = \{x \in W_n \ :\ \forall\ s \in S_A,\ l(xs) > l(x)\}.$$

By [2] and [7], D_A is the set of distinguished coset representatives of W_A in W_n. Let

$$d_A = \sum_{w \in D_A} w \in \mathbb{Q}W_n,$$

and let

$$\mathcal{MR}(W_n) = \bigoplus_{A \in \mathcal{SC}(n)} \mathbb{Q}d_A.$$

For every $A \in \mathcal{SC}(n)$, from [2] $\Phi_n : \mathcal{MR}(W_n) \to \mathbb{Q}\text{Irr}W_n$ is a surjective algebra morphism such that $\Phi_n(d_A) = \text{Ind}_{W_A}^{W_n} 1_A$, where 1_A stands for the trivial character of W_A. It is well-known from [2] that the radical of $\mathcal{MR}(W_n)$ is $\text{Ker}\Phi_n = \bigoplus_{A,B \in \mathcal{SC}(n),\ A \equiv_n B} \mathbb{Q}(d_A - d_B)$

By [2], for $A, B \in \mathcal{SC}(n)$, the set of distinguished double coset representatives is defined as $D_{AB} = D_A^{-1} \cap D_B$ and for any $x \in D_{AB}$,

$$W_A \cap {}^xW_B = W_{A \cap {}^xB}.$$

For $A, B \in \mathcal{SC}(n)$, let define [3] the sets $D_{AB}^{\subseteq} = \{x \in D_{AB} :\ {}^{x^{-1}}W_A \subset W_B\}$ and $D_{AB}^{\equiv} = \{x \in D_{AB} : W_A =^x W_B\}$.

The following proposition proved by Bonnafé in [3] gives the ring multiplication structure in $\mathcal{MR}(W_n)$.

Proposition 1 ([3])**.** *Let A and B be any two signed composition of n. Then,*

i. *There is a map $f_{AB} : D_{AB} \to \mathcal{SC}(n)$ satisfying the following conditions:*
   - *For every $x \in D_{AB}$, $f_{AB}(x) \subset B$ and $f_{AB}(x) \equiv_B {}^{x^{-1}}A \cap B$.*
   - $d_Ad_B - \sum_{x \in D_{AB}} d_{f_{AB}(x)} \in \mathcal{MR}_{\subsetneq_{\boldsymbol{\lambda}} A}(W_n) \cap \mathcal{MR}_{\prec B}(W_n) \cap Ker\Phi_n$.

ii. *If A parabolic or B is semi-positive, then $f_{AB}(x) = {}^{x^{-1}}A \cap B$ for $x \in D_{AB}$ and $d_A d_B = \sum_{x \in D_{AB}} d_{x^{-1}A \cap B}$.*
iii. $\tau_{\boldsymbol{\lambda}(A)}(d_B) = |D^{\subseteq}_{AB}|$.
iv. $D^{\equiv}_{AB} = \{x \in W_n \ : \ S_A =^x S_B\}$.
v. $\mathcal{W}(B) = \{w \in W_n :^w S_B = S_B\}$.
vi. *$\mathcal{W}(B)$ is a subgroup of $N_{W_n}(W_B)$.*
vii. $N_{W_n}(W_B) = \mathcal{W}(B) \ltimes W_B$.

In the Proposition 1, the symbols $\subset_{\boldsymbol{\lambda}}$ and $\prec$ denote a pre-order and an ordering defined on $\mathcal{SC}(n)$, respectively. If $A \equiv_n B$, then it is clear $D^{\equiv}_{AB} = D^{\subseteq}_{AB}$ and $\mathcal{W}(A) = D^{\subseteq}_{AA}$. Thus $\mathcal{MR}(W_n)$ is called *Mantaci-Reutenauer algebra* of W_n.

For $\lambda \in \mathcal{DP}(n)$, the map $\tau_\lambda \ : \ \mathcal{MR}(W_n) \to \mathbb{Q}, \ x \mapsto \Phi_n(x)(\mathrm{cox}_{\hat{\lambda}})$ is independent of the choice of $\mathrm{cox}_{\hat{\lambda}} \in \mathcal{K}_\lambda$ and it is also an algebra morphism [2].

3. Some Properties of Coxeter group of type B_n

Let $A \in \mathcal{SC}(n)$ and let $l_A : W_A \to \mathbb{N}$ be the length function of W_A in terms of its generating set S_A. When A is not a parabolic signed composition of n, the value $l_A(w)$ is not equal to $l(w)$ for some $w \in W_A$. The following lemma gives a relation between these two length functions. The proof of the following lemma is clear from the fact that $l(t_i) = 2i - 1$ for i, $1 \leq i \leq n$.

Lemma 2. *Let $A \in \mathcal{SC}(n)$. Then for every $w \in W_A$*

$$l(w) \equiv l_A(w) \ (mod\ 2).$$

Let ε_n and ε_A be the sign character of W_n and W_A, respectively. As a result of the previous lemma, we get

$$\varepsilon_n(w) = (-1)^{l(w)} = (-1)^{l_A(w)} = \varepsilon_A(w).$$

Since the restriction of ε_n to W_A, that is $\mathrm{res}^{W_n}_{W_A}\varepsilon_n$, is an irreducible character of W_A for every $A \in \mathcal{SC}(n)$ and Lemma 2, then we have $\mathrm{res}^{W_n}_{W_A}\varepsilon_n = \varepsilon_A$.

Example 3. For a concrete example, let $A = (-2, 3, -1, -3, 1) \in \mathcal{SC}(10)$. Then $S_A = \{s_1\} \cup \{t_3, s_3, s_4\} \cup \{s_7, s_8\} \cup \{t_{10}\} \subset S'_{10}$ and $S'_A = W_A \cap S'_{10} = \{s_1\} \cup \{t_3, s_3, s_4, t_4, t_5\} \cup \{s_7, s_8\} \cup \{t_{10}\}$. Thus $W_A \cong W_{-2} \times W_3 \times W_{-1} \times W_{-3} \times W_1$. For $w = s_7 t_3 s_3 s_1 t_{10} \in W_A$, $l_A(w) = 5$ and also

$$w = s_7 t_3 s_3 s_1 t_{10} = s_7 s_2 s_1 t_1 s_1 s_2 s_3 s_1 s_9 s_8 s_7 s_6 s_5 s_4 s_3 s_2 s_1 t_1 s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 \in W_{10},$$

so $l(w) = 27$. It follows that $l(w) \equiv l_A(w) \equiv 1(mod\ 2)$.

Proposition 4. *If $B \in \mathcal{SC}^+(n)$, then the parabolic closure of W_B is $A(W_B) = W_n$.*

Proof. Since $B \in \mathcal{SC}^+(n)$, we have $\mathcal{T}_n \leq W_B$ and so $w_n \in W_B$. By considering w_n as a linear map $-id_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$, we obtain $\mathrm{Fix}(w_n) = \{\vec{0}\}$. Thus, the parabolic closure of w_n is $A(w_n) = W_{\mathrm{Fix}(w_n)} = W_n$. Because of the relation $w_n \in W_B \subset A(\mathrm{cox}_B) = A(W_B)$, we get $w_n \in A(\mathrm{cox}_B)$. By [11], the inclusion $A(w_n) \subset$

$A(\mathrm{cox}_B) = A(W_B)$ holds. If we take into account the fact that $A(w_n) = W_n$, then we have $A(W_B) = W_n$. This completes the proof. □

As a consequence of Proposition 4, if $B \in \mathcal{SC}^+(n)$, then the parabolic closure of W_B is W_n and each element of $\mathcal{K}_{\boldsymbol{\lambda}(B)}$ is of type S_n.

Lemma 5. *Let A be a signed composition of n. Then w_n belongs to W_A if and only if $A \in \mathcal{SC}^+(n)$.*

Proof. When A is a positive signed composition of n, we can easily see from the proof of Proposition 4 that w_n is an element of W_A. Conversely, let w_n be in W_A. We suppose that $A = (a_1, \cdots, a_i, \cdots, a_r)$ is not a positive signed composition of n. Then there exists $a_i < 0$ for some i, $1 \leq i \leq r$. Thus from the definition of W_A, we obtain $t_{|a_1|+\cdots+|a_{i-1}|+1}, \cdots, t_{|a_1|+\cdots+|a_i|} \notin S'_A = W_A \cap S'_n$. Hence for any $x \in W_A$ and $e_{|a_1|+\cdots+|a_{i-1}|+1} + \cdots + e_{|a_1|+\cdots+|a_i|} \in \mathbb{R}^n$, we have $x(e_{|a_1|+\cdots+|a_{i-1}|+1} + \cdots + e_{|a_1|+\cdots+|a_i|}) = e_{|a_1|+\cdots+|a_{i-1}|+1} + \cdots + e_{|a_1|+\cdots+|a_i|}$ and so $e_{|a_1|+\cdots+|a_{i-1}|+1} + \cdots + e_{|a_1|+\cdots+|a_i|} \in \mathrm{Fix}(W_A)$. This is a contradiction, because the subspace $\mathrm{Fix}(W_A)$ consists of only $\vec{0}$. Therefore, we get $A \in \mathcal{SC}^+(n)$. □

Theorem 6. *If the set $\mathcal{C}(S_n)$ denotes the set of all elements of W_n of type S_n, then we have*

$$\mathcal{C}(S_n) = \coprod_{\lambda \in \mathcal{DP}^+(n)} \mathcal{K}_\lambda. \tag{1}$$

Proof. For each $\lambda \in \mathcal{DP}^+(n)$, we have $\hat{\lambda} \in \mathcal{SC}^+(n)$. From Proposition 4, for every element of $\mathcal{K}_\lambda$ is of type S_n and so the reverse inclusion holds. Now let $w \in \mathcal{C}(S_n)$. Then w is W_n-conjugate to cox_A for some $A \in \mathcal{SC}(n)$. Thus we get $A(w) = A(\mathrm{cox}_A) = A(W_A) = W_n$. From here, for every $x \in W_n$ and every $v \in \mathrm{Fix}(W_A)$ we obtain $x(v) = v$. In particular, if we take $w_n = -id_{\mathcal{R}^n} \in W_n$, then it is seen that $\mathrm{Fix}(W_A)$ includes just $\{\vec{0}\}$. Thus w_n is an element of W_A. Otherwise, if $A \notin \mathcal{SC}^+(n)$, then from the proof of Lemma 5 we get $\mathrm{Fix}(W_A) \neq \{\vec{0}\}$, which is a contradiction. Hence $A \in \mathcal{SC}^+(n)$. By taking the definition of $\boldsymbol{\lambda}$ into account, we get a $\lambda \in \mathcal{DP}^+(n)$ such that $\boldsymbol{\lambda}(A) = \lambda$. Thus w belongs to $\mathcal{K}_\lambda$ and so it is seen that the inclusion $\mathcal{C}(S_n) \subset \coprod_{\lambda \in \mathcal{DP}^+(n)} \mathcal{K}_\lambda$ satisfies. It is required. □

Since the exponents of W_n are in turn $1, 3, \cdots, 2n-1$, then from [1] the number of elements of type S_n is equal to the product of exponents of W_n and so $|\mathcal{C}(S_n)| = 1 \cdot 3 \cdots 2n-1$. By the equation (1), we obtain the formula

$$|\mathcal{C}(S_n)| = \sum_{\mu \in \mathcal{DP}^+(n)} |\mathcal{K}_\mu|.$$

Thus Theorem 6 gives us an alternative method to compute the number of elements of type S_n. We will give a formula in Corollary 19 to find the number of elements of every conjugacy class $\mathcal{K}_\lambda$, $\lambda \in \mathcal{DP}(n)$ of W_n. Moreover, we will give an example for Theorem 6 in Section 5.

A conjugacy class of a finite Coxeter group W which does not contain an element of a proper standard parabolic subgroup of W is called a *cuspidal class* of W [8].

Corollary 7. *Let A be a positive signed composition of n. Then the conjugacy class $\mathcal{K}_{\boldsymbol{\lambda}(A)}$ is a cuspidal class of W_n.*

If we consider the proof of Proposition 4 and Corollary 7, then all cuspidal classes of W_n are the conjugacy classes $\mathcal{K}_{\boldsymbol{\lambda}(A)}$ for every $A \in \mathcal{SC}^+(n)$. From Theorem 6, the set $\mathcal{C}(S_n)$ is disjoint union of cuspidal classes of W_n. Therefore, each element of W_n of type S_n belongs to a unique cuspidal class of W_n.

4. Generalized Burnside Algebra of W_n

Let A, B be any two signed compositions of n. Then, we have that

$$A \equiv_n B \;\Leftrightarrow\; W_A \sim_{W_n} W_B \;\Leftrightarrow\; [W/W_A] = [W/W_B]$$

where $[W/W_A]$ represents the isomorphism class of W_n-set W/W_A. The orbits of W_n on $W/W_A \times W/W_B$ are of the form $(W_A x, W_B)$ where $x \in D_{AB}$. The stabilizer of $(W_A x, W_B)$ in W_n is ${}^{x^{-1}}W_A \cap W_B = W_{{}^{x^{-1}}A \cap B}$. Therefore

$$[W/W_A].[W/W_B] = [W/W_A \times W/W_B] = \sum_{x \in D_{AB}} [W/W_{{}^{x^{-1}}A \cap B}].$$

Thus, we are now in a position to give the following definition.

Definition 8. *The generalized Burnside algebra* of W_n is $\mathbb{Q}$-spanned by the set $\{[W/W_A] : A \in \mathcal{SC}(n)\}$ and it is denoted by $\mathrm{HB}(W_n)$.

From part (i) of Proposition 1 and the structure of $\mathrm{Ker}(\Phi_n)$, the ring multiplication rule in $\mathcal{MR}(W_n)$ may be expressed by

$$d_A d_B = \sum_{x \in D_{AB}} d_{f_{AB}(x)} + \sum\nolimits_{N \equiv_n N'} a_{NN'}(d_N - d_{N'}),$$

where $a_{NN'} \in \mathbb{Z}$; $N, N' \subsetneq_{\boldsymbol{\lambda}} A$; $N, N' \prec B$; $f_{AB}(x) \subset B$ and $f_{AB}(x) \equiv_B {}^{x^{-1}}A \cap B$.
Now we define

$$\psi \;:\; \mathcal{MR}(W_n) \to HB(W_n),\; d_A \mapsto [W/W_A].$$

Thus ψ is well-defined and surjective linear map. By considering the structure of $\mathrm{Ker}\Phi_n$ and $f_{AB}(x) \equiv_B {}^{x^{-1}}A \cap B \Rightarrow W_{f_{AB}(x)} \sim_{W_B} W_{{}^{x^{-1}}A \cap B}$, we get

$$\begin{aligned}\psi(d_A d_B) &= \psi\Big(\sum_{x \in D_{AB}} d_{f_{AB}(x)} + \sum\nolimits_{N \equiv_n N'} a_{NN'}(d_N - d_{N'})\Big)\\ &= \sum_{x \in D_{AB}} [W/W_{f_{AB}(x)}]\\ &= \psi(d_A)\psi(d_B).\end{aligned}$$

Then the map ψ is an algebra morphism. Since $\dim_{\mathbb{Q}}\mathrm{HB}(W_n) = \dim_{\mathbb{Q}}\mathbb{Q}\mathrm{Irr}W_n = |\mathcal{DP}(n)|$, then there is an algebra isomorphism between $\mathrm{HB}(W_n)$ and $\mathbb{Q}\mathrm{Irr}W_n$ such that

$$\mathrm{HB}(W_n) \to \mathbb{Q}\mathrm{Irr}W_n,\ [W/W_A] \mapsto \mathrm{Ind}_{W_A}^{W_n} 1_A.$$

Now let $\lambda, \mu \in \mathcal{DP}(n)$ and let $\varphi_\lambda = \mathrm{Ind}_{W_{\hat{\lambda}}}^{W_n} 1_{\hat{\lambda}}$. From part (iii) of Proposition 1, $\varphi_\lambda(\mathrm{cox}_{\hat{\lambda}}) = \tau_\lambda(d_{\hat{\lambda}}) = |D^{\subseteq}_{\hat{\lambda}\hat{\lambda}}| \neq 0$ and $\tau_\lambda(d_{\hat{\mu}}) = 0$ if $\lambda \nsubseteq \mu$. Thus the matrix $(\tau_\lambda(d_{\hat{\lambda}}))_{\lambda,\mu\in\mathcal{DP}(n)}$ is lower diagonal. Then $(\varphi_\lambda(\mathrm{cox}_{\hat{\mu}}))_{\lambda,\mu}$ is upper diagonal and also has positive diagonal entries. Therefore $(\varphi_\lambda(\mathrm{cox}_{\hat{\mu}}))_{\lambda,\mu}$ is invertible and we write $(u_{\lambda\mu})_{\lambda,\mu\in\mathcal{DP}(n)}$ for the inverse of $(\varphi_\lambda(\mathrm{cox}_{\hat{\mu}}))_{\lambda,\mu}$. We define

$$e_\lambda = \sum_{\mu\in\mathcal{DP}(n)} u_{\lambda\mu}\varphi_\mu.$$

By definition of e_λ and $(\varphi_\lambda(\mathrm{cox}_{\hat{\mu}}))^{-1} = (u_{\lambda\mu})$, we obtain that

$$e_\lambda(\mathrm{cox}_{\hat{\mu}}) = \sum_{\gamma\in\mathcal{DP}(n)} u_{\lambda\gamma}\varphi_\gamma(\mathrm{cox}_{\hat{\mu}}) = \delta_{\lambda,\mu}.$$

Hence the set $\{e_\lambda : \lambda \in \mathcal{DP}(n)\}$ is a collection of orthogonal primitive idempotents of $\mathrm{HB}(W_n)$. Consequently, we have $HB(W_n) = \oplus_{\lambda\in\mathcal{DP}(n)}\mathbb{Q}e_\lambda$.

For each $A \in \mathcal{SC}(n)$,

$$s_A : HB(W_n) \to \mathbb{Q},\ s_A([X]) = |{}^{W_A}X|$$

is an algebra map, where ${}^{W_A}X = \{x \in X : W_A x = x\}$. Since $HB(W_n)$ is semisimple and commutative algebra, then all algebra maps $HB(W_n) \to \mathbb{Q}$ are of the form s_A for every $A \in \mathcal{SC}(n)$. The proof of the following lemma is immediately seen from [5].

Lemma 9. *For $A, B \in \mathcal{SC}(n)$, we have that*

$$s_A = s_B \ \Leftrightarrow\ \boldsymbol{\lambda}(A) = \boldsymbol{\lambda}(B).$$

Thus the dual basis of $HB(W_n)$ is $\{s_{\hat{\lambda}} : \lambda \in \mathcal{DP}(n)\}$. For any $\lambda,\ \mu \in \mathcal{DP}(n)$, we have the following equality

$$s_{\hat{\lambda}}(e_\mu) = \delta_{\lambda,\mu}, \tag{2}$$

and so any element x in $HB(W_n)$ can be expressed as $x = \sum_{\lambda\in\mathcal{DP}(n)} s_{\hat{\lambda}}(x)e_\lambda$.

Let A be a signed composition of n. Induction and restriction of characters give rise to two maps between $HB(W_A)$ and $HB(W_n)$. For any $A, B \in \mathcal{SC}(n)$ such that $B \subset A$, we have $\mathrm{Ind}_{W_A}^{W_n}([W_A/W_B]) = [W_n/W_B]$.

Definition 10. Let $A, B \in \mathcal{SC}(n)$ be such that $B \subset A$. The *restriction* is a linear map

$$\mathrm{res}_{W_B}^{W_A} : HB(W_A) \to HB(W_B),\ \mathrm{res}_{W_B}^{W_A}([W_A/W_C]) = \sum_{d\in W_A\cap D_{CB}} [W_B/W_{B\cap {}^{d^{-1}}C}].$$

Before going into a further discussion of the restriction and induced character theories of generalized Burnside algebra, we shall first mention the number of elements of the conjugacy class of W_A in W_n.

Proposition 11. *Let $A \in \mathcal{SC}(n)$ and $\boldsymbol{\lambda}(A) = \lambda$. The number of all reflection subgroups of W_n which are conjugate to W_A is*

$$|[W_A]| = |D_A| \cdot u_{\lambda,\lambda}.$$

Proof. Put $[W_A] = \{{}^xW_A : x \in W_n\}$. Now we note that $xW_Ax^{-1} = yW_Ay^{-1}$ if and only if $x \in yN_{W_n}(W_A)$. Thus, the number of distinct conjugates of W_A in W_n is $[W_n : N_{W_n}(W_A)]$. Since also $N_{W_n}(W_A) = \mathcal{W}(A) \ltimes W_A$, we have

$$|[W_A]| = \frac{|W_n|}{|\mathcal{W}(A)| \cdot |W_A|} = \frac{|D_A|}{|\mathcal{W}(A)|}.$$

Furthermore, from the fact that $\tau_{\boldsymbol{\lambda}(A)}(d_A) = |D^{\subseteq}_{AA}| = |\mathcal{W}(A)|$ and $\varphi_\lambda(\mathrm{cox}_{\hat{\lambda}}) = \tau_{\boldsymbol{\lambda}(A)}(d_A) = \frac{1}{u_{\lambda,\lambda}}$, as desired. □

Example 12. We consider the set $D_{(2,1)} = \{1, s_2, s_1s_2\}$ consisting of the distinguished coset representatives of reflection subgroup $W_{(2,1)}$ in W_3. The number of all reflection subgroups conjugate to $W_{(2,1)}$ in W_3 is

$$|[W_{(2,1)}]| = |D_{(2,1)}| \cdot u_{(2,1;\emptyset),(2,1;\emptyset)} = 3 \cdot 1 = 3.$$

These are explicitly $W_{(2,1)}$, $W_{(1,2)}$ and ${}^{s_2}W_{(2,1)} = \langle s_2s_1s_2, t_1, t_2\rangle$. We note that the reflection subgroup ${}^{s_2}W_{(2,1)}$ does not coincide with any subgroup of W_3 corresponding to any signed composition of 3.

Remark 13. *For $A, B \in \mathcal{SC}(n)$ such that $B \subset A$ and for any $x \in HB(W_n)$, by using the definition of s_A one can see that there exists the relation $s^A_B(res^{W_n}_{W_A}(x)) = s_B(x)$.*

We can now give the following proposition.

Proposition 14. *Let be $A, B \in \mathcal{SC}(n)$ and let $A_1, A_2, \cdots, A_r$ be representatives of the W_A-equivalent classes of subsets of A, which are W_n-equivalent to B. Then,*

$$res^{W_n}_{W_A} e_B = \sum_{i=1}^{r} e^A_{A_i}.$$

If B is not W_n-equivalent to any subset of A then $res^{W_n}_{W_A} e_B = 0$.

Proof. Since $\mathrm{res}^{W_n}_{W_A} e_B$ is an element of $HB(W_A)$, then we have

$$\mathrm{res}^{W_n}_{W_A} e_B = \sum_{A_i \subset A} s^A_{A_i}(\mathrm{res}^{W_n}_{W_A}(e_B))e^A_{A_i}.$$

Then by using Remark 13 and the relation (2), we get

$$\mathrm{res}^{W_n}_{W_A} e_B = \sum_{A_i \subset A} s_{A_i}(e_B)e^A_{A_i}$$

$$= \sum_{\substack{A_i \subset A \\ A_i \equiv_A B}} e^A_{A_i}$$
$$= \sum_{i=1}^{r} e^A_{A_i}.$$

□

Proposition 15. *Let $A, B \in \mathcal{SC}(n)$ and let $B \subset A$. Then we have*

$$Ind^{W_n}_{W_A} e^A_B = \frac{|\mathcal{W}(B)|}{|W_A \cap \mathcal{W}(B)|} \cdot e_B.$$

Proof. Firstly, we assume that $A = B$ and cox_A is a Coxeter element of W_A. Since the image of cox_A under permutation character of W_n on the cosets of W_A is $|\mathcal{W}(A)|$ then it follows from the fact that

$$x^{-1}\text{cox}_A x \in W_A \Leftrightarrow x \in N_{W_n}(W_A).$$

Thus we obtain

$$\begin{aligned}\text{Ind}^{W_n}_{W_A} e^A_A(\text{cox}_A) &= |D_A \cap N_{W_n}(W_A)| \\ &= |\mathcal{W}(A)|.\end{aligned}$$

As $\text{Ind}^{W_n}_{W_A} e^A_A$ takes value zero except for the elements conjugate to cox_A and so we get

$$\text{Ind}^{W_n}_{W_A} e^A_A = |\mathcal{W}(A)| e_A.$$

By transitivity of induced characters, we generally get

$$\begin{aligned}\text{Ind}^{W_n}_{W_A} e^A_B &= \text{Ind}^{W_n}_{W_A}\left(\frac{1}{|W_A \cap \mathcal{W}(B)|}|W_A \cap \mathcal{W}(B)| e^A_B\right) \\ &= \text{Ind}^{W_n}_{W_A}\left(\frac{1}{|W_A \cap \mathcal{W}(B)|}\text{ind}^{W_A}_{W_B} e^B_B\right) \\ &= \frac{|\mathcal{W}(B)|}{|W_A \cap \mathcal{W}(B)|} e_B.\end{aligned}$$

□

Furthermore, there is also the equality $\text{Ind}^{W_n}_{W_A} e^A_B = |N_{W_n}(W_B) : N_{W_A}(W_B)| e_B$.

Theorem 16. *Let $A, B \in \mathcal{SC}(n)$ be such that $\boldsymbol{\lambda}(B) \subset \boldsymbol{\lambda}(A)$. If $B_1, B_2, \cdots, B_r$ are the representatives of the W_A-equivalence classes of subsets of A which are W_n-equivalent to B, then for $cox_B \in W_n$,*

$$Ind^{W_n}_{W_A} 1_A(cox_B) = \sum_{i=1}^{r} \frac{|\mathcal{W}(B)|}{|W_A \cap \mathcal{W}(B_i)|}.$$

Proof. Let $A, B \in \mathcal{SC}(n)$. If $A \equiv_n B$ then it is easy to prove that $|\mathcal{W}(A)| = |\mathcal{W}(B)|$. We write $1_A = \sum_E e_E^A$, where $E \in \mathcal{SC}(n)$ runs over W_A-conjugacy classes of subsets of A. From Proposition 15, we have

$$\mathrm{Ind}_{W_A}^{W_n} 1_A = \sum_E \mathrm{Ind}_{W_A}^{W_n} e_E^A \Rightarrow \mathrm{Ind}_{W_A}^{W_n} 1_A = \sum_E \frac{|\mathcal{W}(E)|}{|W_A \cap \mathcal{W}(E)|} \cdot e_E.$$

Since each B_i is W_n-equivalent to B, then $e_E(\mathrm{cox}_B) = 1$ if and only if $E \equiv_{W_A} B_i$. Thus we obtain that

$$\mathrm{Ind}_{W_A}^{W_n} 1_A(\mathrm{cox}_B) = \sum_{i=1}^{r} \frac{|\mathcal{W}(B)|}{|W_A \cap \mathcal{W}(B_i)|}.$$

Hence the theorem is proved. □

Theorem 17 and Proposition 18 give us a useful computation to determine the coefficient of the sign character ε_n in the expression of the orthogonal primitive idempotent $e_\lambda,\ \lambda \in \mathcal{DP}(n)$ in terms of irreducible characters of W_n.

Theorem 17. $u_{(n;\emptyset),(\emptyset;1,\cdots,1)} = \frac{(-1)^n}{2n}$.

Proof. Let $\chi_{reg} : W_n \to \mathbb{Z}$ be the regular character of W_n. For $A = (-1, \cdots, -1)$ it is satisfied $\mathrm{Ind}_{W_A}^{W_n} 1_A = \chi_{reg}$. The character ε_n is contained in χ_{reg} with the property that its coefficient is just 1, thus we have

$$\langle \mathrm{Ind}_{W_A}^{W_n} 1_A, \varepsilon_n \rangle = 1.$$

Now let $A \neq (-1, \cdots, -1)$. By using Frobenius Reciprocity and the formula $\mathrm{res}_{W_A}^{W_n} \varepsilon_n = \varepsilon_A$, it is obtained that $\langle \mathrm{Ind}_{W_A}^{W_n} 1_A, \varepsilon_n \rangle = 0$. If w is conjugate to cox_{W_n} under W_n, then we have $e_{(n;\emptyset)}(w) = 1$ and $\varepsilon_n(w) = \varepsilon_n(\mathrm{cox}_{W_n}) = (-1)^{l(w)} = (-1)^n$. Let $\mathrm{ccl}_{W_n}(\mathrm{cox}_{W_n})$ denote the conjugacy class of cox_{W_n} in W_n. By considering the formula $|\mathrm{ccl}_{W_n}(\mathrm{cox}_{W_n})| = \frac{|W_n|.n}{2N}$ in [4], we obtain

$$\langle e_{(n;\emptyset)}, \varepsilon_n \rangle = \frac{(-1)^n}{2n}.$$

On the other hand, $\langle e_{(n;\emptyset)}, \varepsilon_n \rangle = \sum_{\mu \in \mathcal{DP}(n)} u_{(n;\emptyset)\mu} \langle \varphi_\mu, \varepsilon_n \rangle = u_{(n;\emptyset),(\emptyset;1,\cdots,1)}$ and so the proof is completed. □

Proposition 18. *For $\lambda \in \mathcal{DP}(n)$ and $\lambda \neq (n;\emptyset)$, then we have*

$$u_{\lambda,(\emptyset;1,\cdots,1)} = (-1)^{|S_{\hat{\lambda}}|} \cdot \frac{|\mathcal{K}_\lambda|}{|W_n|}.$$

Proof. Since the sign character is constant on the conjugacy classes, then we have

$$\begin{aligned} \langle e_\lambda, \varepsilon_n \rangle &= \frac{1}{|W_n|} \sum_{w \in \mathcal{K}_\lambda} (-1)^{l(w)} \ (\mathrm{rank} W_{\hat{\lambda}} = |S_{\hat{\lambda}}|) \\ &= (-1)^{|S_{\hat{\lambda}}|} \cdot \frac{|\mathcal{K}_\lambda|}{|W_n|}. \end{aligned}$$

Note that $\langle \varphi_\mu, \varepsilon_n \rangle$ has value 1 for $\mu = (\emptyset; 1, \cdots, 1)$ and zero for the others. Henceforth, we obtain $\langle e_\lambda, \varepsilon_n \rangle = \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} \langle \varphi_\mu, \varepsilon_n \rangle = u_{\lambda,(\emptyset;1,\cdots,1)}$. Eventually, we have $u_{\lambda,(\emptyset;1,\cdots,1)} = (-1)^{|S_{\hat{\lambda}}|} \cdot \frac{|\mathcal{K}_\lambda|}{|W_n|}$. □

Notice that calculation of the inner product $\langle e_\lambda, 1_{W_n} \rangle$ leads to the following corollary.

Corollary 19. *Let $\lambda \in \mathcal{DP}(n)$. Then*

$$|W_n| \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda,\mu} = |\mathcal{K}_\lambda|.$$

By means of Corollary 19 and the matrix $(u_{\lambda\mu})_{\lambda,\mu \in \mathcal{DP}(n)}$, one can readily determine the sizes of all the conjugacy classes of W_n.

Theorem 20. *Let $A \in \mathcal{SC}(n)$ and $\lambda \in \mathcal{DP}(n)$. Then*

$$\sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} a_{\hat{\mu}A(-1,\cdots,-1)} = (-1)^{|S_{\hat{\lambda}}|} \frac{|\mathcal{K}_\lambda \cap W_A|}{|W_A|},$$

where $a_{\hat{\mu}A(-1,\cdots,-1)} = |\{x \in D_{\hat{\mu}A} : \ {}^{x^{-1}}\hat{\mu} \cap A = (-1, \cdots, -1)\}|$.

Proof. The term $d_{(-1,\cdots,-1)}$ in the multiplication $d_{\hat{\mu}} d_A$ lies in the summand $\sum_{x \in D_{\hat{\mu}A}} d_{f_{\hat{\mu}A}(x)}$ from the structure of $\mathrm{Ker}\Phi_n$ and part (i) of Proposition 1. If we write the coefficient of $d_{(-1,\cdots,-1)}$ in this summand as $a_{\hat{\mu}A(-1,\cdots,-1)}$, and so we get

$$a_{\hat{\mu}A(-1,\cdots,-1)} = |\{x \in D_{\hat{\mu}A} : f_{\hat{\mu}A}(x) = (-1, \cdots, -1)\}|.$$

By using part (i) of Proposition 1 along with the fact $f_{\hat{\mu}A}(x) \equiv_A {}^{x^{-1}}\hat{\mu} \cap A$, it is seen that there is the equivalence ${}^{x^{-1}}\hat{\mu} \cap A \equiv_A (-1, \cdots, -1)$. Since no element in $\mathcal{SC}(n)$ is congruent to $(-1, \cdots, -1)$ except for $(-1, \cdots, -1)$, it then follows that ${}^{x^{-1}}\hat{\mu} \cap A = (-1, \cdots, -1)$. Hence we have deduced the equality $a_{\hat{\mu}A(-1,\cdots,-1)} = |\{x \in D_{\hat{\mu}A} : \ {}^{x^{-1}}\hat{\mu} \cap A = (-1, \cdots, -1)\}|$ holds. Therefore, by Frobenius Reciprocity and Mackey Theorem, we have

$$\begin{aligned}
\langle e_\lambda, \mathrm{ind}_{W_A}^{W_n} \varepsilon_A \rangle &= \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} \sum_{x \in D_{\hat{\mu}A}} \langle \mathrm{ind}_{W_{x^{-1}\hat{\mu} \cap A}}^{W_A} 1_{x^{-1}\hat{\mu} \cap A}, \varepsilon_A \rangle \\
&= \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} \sum_{\substack{x \in D_{\hat{\mu}A} \\ {}^{x^{-1}}\hat{\mu} \cap A = (-1,\cdots,-1)}} 1_{x^{-1}\hat{\mu} \cap A} \\
&= \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda\mu} a_{\hat{\mu}A(-1,\cdots,-1)}.
\end{aligned}$$

Also, $\varepsilon_n(w)$ is the same value for every $w \in \mathcal{K}_\lambda$ and so $\varepsilon_n(w) = \varepsilon_n(\mathrm{cox}_{\hat{\lambda}}) = (-1)^{|S_{\hat{\lambda}}|}$. Therefore, by Lemma 2, we have

$$\langle e_\lambda, \mathrm{ind}_{W_A}^{W_n} \varepsilon_A \rangle = \frac{1}{|W_A|} \sum_{w \in \mathcal{K}_\lambda \cap W_A} (-1)^{l_A(w^{-1})}$$

$$= \frac{1}{|W_A|} \sum_{w \in \mathcal{K}_\lambda \cap W_A} (-1)^{l(w)} = \frac{1}{|W_A|} (-1)^{|S_{\hat{\lambda}}|} |\mathcal{K}_\lambda \cap W_A|$$

Putting these two results together, we see that theorem is proved. □

5. Example

We consider the Coxeter group W_3. For all $\lambda, \mu \in \mathcal{DP}(3)$, by means of the character table of $\mathcal{MR}(W_3)$ in [3], we can write the values $\varphi_\lambda(\text{cox}_{\hat{\mu}})$ as in the following table:

	$c_{(3;\emptyset)}$	$c_{(\emptyset;3)}$	$c_{(2,1;\emptyset)}$	$c_{(2;1)}$	$c_{(1;2)}$	$c_{(\emptyset;2,1)}$	$c_{(1,1,1;\emptyset)}$	$c_{(1,1;1)}$	$c_{(1;1,1)}$	$c_{(\emptyset;1,1,1)}$
$\varphi_{(3;\emptyset)}$	1	1	1	1	1	1	1	1	1	1
$\varphi_{(\emptyset;3)}$	0	2	0	0	0	4	0	0	0	8
$\varphi_{(2,1;\emptyset)}$	0	0	1	1	1	1	3	3	3	3
$\varphi_{(2;1)}$	0	0	0	2	0	2	0	2	4	6
$\varphi_{(1;2)}$	0	0	0	0	2	2	0	0	4	12
$\varphi_{(\emptyset;2,1)}$	0	0	0	0	0	4	0	0	0	24
$\varphi_{(1,1,1;\emptyset)}$	0	0	0	0	0	0	6	6	6	6
$\varphi_{(1,1;1)}$	0	0	0	0	0	0	0	4	8	12
$\varphi_{(1;1,1)}$	0	0	0	0	0	0	0	0	8	24
$\varphi_{(\emptyset;1,1,1)}$	0	0	0	0	0	0	0	0	0	48

The matrices $(u_{\lambda,\mu})_{\lambda,\mu \in \mathcal{DP}(n)}$ is

$$\begin{pmatrix} 1 & -1/2 & -1 & 0 & 0 & 1/2 & 1/3 & 0 & 0 & -1/6 \\ 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 1/6 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1/4 & -1/2 & 1/4 & 1/4 & -1/8 \\ 0 & 0 & 0 & 1/2 & 0 & -1/4 & 0 & -1/4 & 0 & 1/8 \\ 0 & 0 & 0 & 0 & 1/2 & -1/4 & 0 & 0 & -1/4 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & -1/4 & 1/8 & -1/48 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & -1/4 & 1/16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & -1/16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/48 \end{pmatrix}.$$

For $\lambda = (3;\emptyset),\ (2,1;\emptyset),\ (1,1,1;\emptyset) \in \mathcal{DP}(3)$, the size of $\mathcal{K}_\lambda$ is calculated by means of Corollary 19 and matrix $(u_{\lambda,\mu})_{\lambda,\mu \in \mathcal{DP}(n)}$ the above. Since $|\mathcal{K}_{(3;\emptyset)}| = 8$, $|\mathcal{K}_{(2,1;\emptyset)}| = 6$ and $|\mathcal{K}_{(1,1,1;\emptyset)}| = 1$, then we have found that the number of elements of type S_3 is $|\mathcal{C}(S_3)| = 15$.

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References

[1] Bergeron, F., Bergeron, N., Howlett, R.B., Taylor, D.E., A Decomposition of the Descent Algebra of a Finite Coxeter Group, *Journal of Algebraic Combinatorics*, 1(1) 1992, 23–44.

[2] Bonnafé, C., Hohlweg, C., Generalized descent algebra and construction of irreducible characters of hyperoctahedral groups, *Ann. Inst. Fourier (Grenoble)* 56(1) 2006, 131–181.

[3] Bonnafé, C., Representation theory of Mantaci-Reutenauer algebras, *Algebras and Representation Theory* 11(4) (2008), 307–346.
[4] Carter, R.W., Conjugacy Classes in the Weyl Groups, *Compositio Math.*, 25 1972, 1–59.
[5] Curtis, C.W., Reiner, I., Methods of Representation Theory with Applications to Finite Groups and Orders, Vol. II, John Wiley and Sons, 1987.
[6] Douglass, J.M., Pfeiffer, G., Rohrle, G., On reflection subgroups of finite Coxeter groups, *Comm. Algebra* 41(7) 2013, 2574–2592.
[7] Fleischmann, P., On pointwise conjugacy of distinguished coset representatives in Coxeter groups, *J. Group Theory*, 5 (2002), 269–283.
[8] Geck, M., Pfeiffer, G., Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras, London Mathematical Society Monographs, New Series, vol. 21, The Clarendon Press, Oxford University Press, New York, 2000.
[9] Humphreys, J.E., Reflection Groups and Coxeter Groups, Cambridge Studies in Advanced Math., vol. 29, Cambridge University Press, 1990.
[10] Mantaci, R., Reutenauer, C., A generalization of Solomon's algebra for hyperoctahedral groups and other wreath products, *Comm. Algebra*, 23(1) (1995), 27–56.
[11] Solomon, L., A Mackey formula in the group ring of a Coxeter group, *J. Algebra*, 41(2) (1976), 255–264.

Current address: Department of Mathematics, Faculty of Science, Erciyes University, 38039, Kayseri, Turkey.
E-mail address: hasanarslan@erciyes.edu.tr
ORCID Address: http://orcid.org/0000-0002-0430-8737
Current address: Department of Mathematics, Faculty of Science, Erciyes University, 38039, Kayseri, Turkey.
E-mail address: can@erciyes.edu.tr
ORCID Address: http://orcid.org/0000-0001-8485-6815