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Comparisons of the Goodness of Fit Tests for the Geometric Distribution

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ABSTRACT

This article gives some goodness of fit tests for the Geometric distribution. The tests used in this study are Anderson Darling test, Cramer-von Mises test, Kolmogorov Smirnov test, test based on partition of Chi-square and some new alternatives based on smooth tests. The performance of these tests is compared by simulation study according to type-I errors and powers of tests. The power comparisons indicate that the modified version of smooth test statistic has the highest power value of test for Negative Binomial, Binomial and Poisson distributions. Also, Kolmogorov Smirnov, Anderson Darling, Cramer-von Mises and V_2^2 test statistics can be preferred for large sample sizes.

Keywords: Anderson Darling test, Cramer-von Mises test, Kolmogorov Smirnov test, Chi-square test.

1. INTRODUCTION

Count data is a type of data in which the observations can take only the non-negative integer values $\{0,1,2,...\}$, here these integers arise from counting rather than ranking. In general, the Poisson distribution is used in studies analyzing count data. The Poisson distribution can be used in quality control statistics to count the number of defects of an item, or biology to count the number of bacteria, in physics to count the number of particles emitted from a radioactive substance [1]. Also, some important application about count data using Poisson distribution can be seen at [2,3]. However, Geometric distribution is one of the best known discrete probability distributions modelling count data. There are some reasons to prefer Geometric distribution to Poisson distribution. Firstly, the time interval of interest, a treatment, is not of fixed duration. Secondly, the Poisson distribution is characterized by equality of the mean and variance of the distribution and finally Poisson model allows for the occurrence of zeros in the data [4].

Geometric distribution has many useful application and these applications are often arise in some areas including agricultural and reliability studies. For example, the pesticide application method you choose depends on the nature and habits of the target pest, the characteristics of target site, the properties of the pesticide, the suitability of the application equipment. Since the number of applications to analyze a given pest is dependent on the probability of achieving successful control with a given application, another important method is the number of pesticide applications made by farmers. So, farmers must decide how many times to apply the chosen pesticides. These frequencies can be

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modelled by Geometric distribution [4]. The Geometric probability function is defined as

$$f(x;q) = (1-q)q^x \text{ , } x\text{=0,1,2,... } q\text{=1-}p \quad (1)$$

where p is the success probability of any trial.

In order to determine whether the random sample of size n comes from a Geometric distribution, null and alternative hypothesis are as seen below:

H_o : The random sample is drawn from population has Geometric distribution.

H_1 : The random sample is not drawn from population has Geometric distribution. (2)

Goodness of fit tests are used to test H_0 against H_1. Pearson's Chi-square (χ^2) is one of the most popular goodness of fit test. However, Pearson's χ^2 approximation which is commonly used is not adequate for small samples. Therefore, further studies have also suggested in the literature.

Choulakian et al. [5] suggested Cramer-Von Mises test statistic in order to test goodness of fit of discrete distributions. Lancaster [6] partitioned into components Pearson's χ^2 statistic by using orthonormal polynomials. Best and Rayner [7] suggested alternative test by taking advantage of these components and defined Kolmogorov-Smirnov statistic for the Geometric distribution take into consideration the approach of Henze [8] for the Poisson distribution. Also Best and Rayner [9,10] proposed the smooth test statistics in order to test goodness of fit of the Geometric distribution.

2. GOODNESS OF FIT TESTS

In this section, goodness of fit tests for Geometric distribution are investigated. Suppose that $X_1, X_2,..., X_n$ be random sample of size n from population with unknown distribution and $x_1, x_2,..., x_n$ be the sample data. By the definition of the Geometric distribution observed values are $j = 0,1,...,k$. In this case, we will have $k+1$ classes. Let N_j be frequencies of these classes ($j = 0,1,...,k$). In order to test the hypothesis in Equation 2, Pearson's χ^2 is given as

$$\chi^2 = \sum_{j=0}^{k} (N_j - np_j)^2 \big/ (np_j) , \quad (3)$$

where $n = N_0 + \cdots + N_k$ and p_j is the probability of an observation lying in the j^{th} class under H_0 . This statistic is approximately distributed as χ_k^2.

The other popular goodness of fit tests for Geometric distribution are Kolmogorov-Smirnov, Cramer-von Mises and Anderson-Darling tests. These tests and alternative tests suggested by Best and Rayner [7,9,10] are given in the rest of this section.

2.1. Kolmogorov-Smirnov Goodness of Fit Test

Best and Rayner [7] modified Kolmogorov-Smirnov test statistic (KS) for the Geometric distribution. Let $m = max(x_1, x_2, \ldots, x_n)$,
$\hat{p}_m = 1 - \hat{p}_0 - \hat{p}_1 - \cdots - \hat{p}_{m-1}$
$R_j = N_0 + N_1 + \cdots + N_j - n(\hat{p}_0 + \hat{p}_1 + \cdots + \hat{p}_j)$ for $j = 0,1,2,\ldots,m.$ Here $\hat{p}_j$ is the estimator of p_j . Then the KS test statistic can be defined as

$$KS = max\left(|R_0|, |R_1|, \ldots, |R_m|\right). \quad (4)$$

2.2. Cramer-von Mises Goodness of Fit Test

Choulakian et al. [5] proposed Cramer-Von Mises test statistic (C^2);

$$C^2 = n^{-1} \sum_{j=0}^{k} Z_j^2 p_j . \quad (5)$$

Here $Z_j = \sum_{i=0}^{j} \left(O_i - E_i\right)$ (O_i and E_i are the observed and expected frequencies in i^{th} class respectively).

2.3. Anderson-Darling Goodness of Fit Test

The Anderson-Darling test statistic (AD) for the Geometric distribution was modified by Best and Rayner [7] by using the work of Spinelli and Stephens [11] for Poisson distribution. Put $j = 0,1,2,\ldots,m$ and $H_j = \hat{p}_0 + \hat{p}_1 + \cdots + \hat{p}_j$, then

$$AD = n \sum_{j=0}^{m} R_j^2 \hat{p}_j \Big/ \left\{H_j (1 - H_j)\right\}. \quad (6)$$

m is determined so that $p_m < 10^{-3}/n$ and $N_m = 0$. Adding other terms to summation formula will not significantly change the value of test statistic [11].

If the KS , AD and C^2 test statistics are greater than their critical values c_α at level α , then hypothesis H_0 is rejected. Since the asymptotic distributions of these statistics under H_0 depend on the unknown parameter, the critical values are estimated from the data. Therefore, the p-values of these tests are obtained by using parametric bootstrap [12].

2.4. Smooth Goodness of Fit Tests

Best and Rayner [9,10] suggested the smooth test statistic S_c which is defined by means of components

$$U_r = \sum_{j=1}^{n} h_r(X_j;\hat{q}) \Big/ \sqrt{n} \quad \text{for } r = 1,2,\ldots,c+1 \tag{7}$$

where $h_r(X_j;\hat{q}) = K \sum_{i=0}^{r} {}^{x}C_{r-i} ({}^{r}C_i)^2 i!(r-i)!(-a)^i$ are Meixner orthonormal polynomials with $a = q/(1-q)$ and $K^{-1} = r!(a^2+a)^{r/2}$. Notation ${}^{n}C_k$ is the number of k-combinations from a given set S of n elements. Then the S_c statistic is given by

$$S_c = U_2^2 + \cdots + U_{c+1}^2. \tag{8}$$

$\hat{q}$ is used as maximum likelihood estimator of q for ungrouped data, the components asymptotically U_r $(r = 2,3,4,\ldots)$ have the standard normal distribution and are asymptotically mutually independent. Thus S_c has asymptotically χ_c^2 distribution. If $S_c > \chi_c^2$, then H_0 is rejected.

Furthermore, a modified version of the smooth test statistic S_1 defined by Best and Rayner [7] is

$$S_1^* = nS_1 \Big/ \sum_{j=1}^{n} h_2^2\left(X_j;\hat{q}\right). \tag{9}$$

Test statistic S_1^* has asymptotically χ_1^2 distribution. If $S_1^* > \chi_1^2$, then H_0 is rejected.

2.5. Goodness of Fit Test Based on Partition of Chi-square

Lancaster [6] considered that χ^2 statistic can be partitioned into components V_r^2 $(r = 1,2,\ldots,k)$. Required notations are given as

$$\mu = \sum_{j=0}^{k} j p_j \,,\ \mu_r = \sum_{j=0}^{k} (j-\mu)^r p_j,\ r = 2,3,\ldots \tag{10}$$

If $b = (\mu_4 - \mu_3^2/\mu_2 - \mu_2^2)^{-0.5}$, then the first three orthonormal polynomials can be written as follows [7]:

$$g_0(j) = 1$$

$$g_1(j) = (j-\mu)/\sqrt{\mu_2}$$

$$g_2(j) = b\left\{(j-\mu)^2 - \mu_3(j-\mu)/\mu_2 - \mu_2\right\}. \tag{11}$$

It is possible to derive orthonormal functions $\{g_r(j)\}$ for $r = 3,4,\ldots$ by using recurrence relations in Emerson [13]. Take

$$V_r = \sum_{j=0}^{k} N_j g_r(j) \Big/ \sqrt{n}\ ,\ r = 1,\ldots,k \tag{12}$$

and then, as in Lancaster [6], the statistic χ^2 is obtained as

$$\chi^2 = V_1^2 + \ldots + V_k^2. \tag{13}$$

V_r^2 $(r = 2,\ldots,k)$ components have asymptotically χ_1^2 distribution. Moreover, these components can be used as test statistic.

Note: If not much pooling is done then the V_r will be numerically close to the corresponding smooth components U_r. In addition, a large value of U_r indicates deviations of data from the hypothesized distribution in the r^{th} moment [14].

In this study, we used MATLAB R2009a software to compare KS, AD, C^2, χ^2, V_2^2, S_2, S_3, S_1^*, U_2^2 and U_3^2 tests in terms of type I errors and powers of tests. Obtained results and their discussions are given the following section.

3. SIMULATION STUDY

In this section, we compared goodness of fit tests described in the previous section in terms of type-I errors and powers of tests. For this purpose, 2000 random numbers of size n from the Geometric distribution with $q = 0.5$ were generated for different sample sizes $n = 25(25)100$ and different nominal values $\alpha = 0.05, 0.1$. Also, similar results were obtained for $q = 0.25, 0.4, 0.6, 0.75$. Thus these results were not given in this study. On the other hand, since the theoretical distribution of the test statistics KS, AD, C^2, V_2^2, S_2, S_3, S_1^*, U_2^2 and U_3^2 could not be obtained, the parametric bootstrap method was used by performing also 2000 repetition to obtain p values for these tests.

Same processes for power values of these tests were performed by generating random numbers of size n from Negative Binomial, Binomial and Poisson distributions with different parameters. Probability functions of these distributions are given as

Distributions	**Probability Functions**
Negative Binomial	$f(x;k,p)=\binom{k+x-1}{x}p^k(1-p)^x,\ x=0,1,...$
Binomial	$f(x;m,p)=\binom{m}{x}p^x(1-p)^{m-x},\ x=0,1,...,m$
Poisson	$f(x;\lambda)=\dfrac{e^{-\lambda}\lambda^x}{x!},\ x=0,1,...$

Table 1. Empirical type-I errors of the tests for $\alpha=0.05, 0.1$

	n	KS	AD	C^2	χ^2	V_2^2	S_2	S_3	S_1^*	$S_1=U_2^2$	U_3^2
$\alpha=0.05$	25	0.0420	0.0385	0.0385	0.0280	0.0505	0.0450	0.0445	0.0440	0.0420	0.0440
	50	0.0600	0.0580	0.0595	0.0335	0.0545	0.0550	0.0520	0.0565	0.0485	0.0520
	75	0.0485	0.0425	0.0475	0.0310	0.0460	0.0480	0.0485	0.0590	0.0530	0.0450
	100	0.0490	0.0455	0.0460	0.0315	0.0485	0.0505	0.0450	0.0455	0.0475	0.0485
$\alpha=0.1$	25	0.0905	0.0835	0.0900	0.0645	0.1015	0.0895	0.0935	0.0915	0.0940	0.0890
	50	0.0965	0.0940	0.1010	0.0605	0.1010	0.0945	0.0925	0.0995	0.0950	0.1005
	75	0.1165	0.1205	0.1195	0.0765	0.1130	0.1140	0.1060	0.1140	0.1110	0.1105
	100	0.0955	0.0940	0.0985	0.0595	0.1075	0.0965	0.0975	0.0980	0.0995	0.0985

Type-I errors of the tests are given in Table 1. It is observed that for both $\alpha=0.05$ and $\alpha=0.1$, type-I error rates of the tests except Pearson's χ^2 test statistic are close to the value of nominal α.

Table 2. Power values of tests under the alternative Negative Binomial distribution for $\alpha = 0.05$

$[k,p]$	n	KS	AD	C^2	χ^2	V_2^2	S_2	S_3	S_1^*	$S_1 = U_2^2$	U_3^2
$[2,0.9]$	25	0.0215	0.0210	0.0085	0.0165	0.0195	0.0175	0.0185	0.0205	0.0155	0.0215
	50	0.0260	0.0215	0.0225	0.0080	0.0360	0.0230	0.0230	0.0665	0.0255	0.0340
	75	0.0400	0.0170	0.0320	0.0095	0.0370	0.0260	0.0240	0.0930	0.0200	0.0310
	100	0.0480	0.0180	0.0435	0.0125	0.0525	0.0350	0.0280	0.1095	0.0275	0.0370
$[2,0.7]$	25	0.1175	0.0970	0.1210	0.0530	0.1140	0.0710	0.0680	0.1730	0.0465	0.0930
	50	0.1760	0.1475	0.1770	0.0505	0.1705	0.1020	0.0855	0.2570	0.0745	0.1060
	75	0.2415	0.2105	0.2435	0.0600	0.2050	0.1390	0.1200	0.3185	0.1195	0.1300
	100	0.2865	0.2800	0.2965	0.0825	0.2730	0.1870	0.1610	0.3885	0.1755	0.1525
$[2,0.5]$	25	0.1900	0.2040	0.2145	0.1190	0.1650	0.1345	0.1190	0.3055	0.0910	0.1665
	50	0.3375	0.3810	0.3845	0.1360	0.3240	0.2560	0.2265	0.4685	0.2160	0.2650
	75	0.4875	0.5195	0.5275	0.1780	0.4525	0.3695	0.3345	0.6105	0.3280	0.3420
	100	0.6035	0.6505	0.6570	0.2290	0.5770	0.5040	0.4645	0.7035	0.4740	0.4360
$[2,0.3]$	25	0.2935	0.2555	0.2630	0.2035	0.1755	0.2485	0.2255	0.4615	0.1700	0.2970
	50	0.5230	0.5055	0.5265	0.2620	0.4385	0.4660	0.4390	0.7015	0.3905	0.4745
	75	0.7265	0.7205	0.7420	0.3470	0.6270	0.6620	0.6405	0.8275	0.5855	0.6430
	100	0.8400	0.8320	0.8495	0.4050	0.7655	0.7925	0.7890	0.8945	0.7455	0.7625
$[3,0.9]$	25	0.0455	0.0275	0.0320	0.0110	0.0475	0.0375	0.0375	0.0585	0.0210	0.0480
	50	0.0760	0.0300	0.0715	0.0210	0.0660	0.0505	0.0485	0.1350	0.0325	0.0655
	75	0.1155	0.0255	0.1100	0.0205	0.0940	0.0630	0.0550	0.1960	0.0490	0.0705
	100	0.1390	0.0330	0.1315	0.0205	0.1105	0.0675	0.0580	0.2220	0.0605	0.0655
$[3,0.7]$	25	0.2730	0.2725	0.2925	0.1385	0.2425	0.1990	0.1830	0.3625	0.1350	0.2290
	50	0.4335	0.4525	0.4565	0.1615	0.4095	0.3200	0.2875	0.5550	0.2625	0.3310
	75	0.6235	0.6470	0.6490	0.2465	0.6030	0.5125	0.4665	0.7455	0.4670	0.4790
	100	0.7395	0.7680	0.7655	0.3280	0.7210	0.6420	0.6010	0.8310	0.6200	0.5770
$[3,0.5]$	25	0.4530	0.4840	0.5005	0.2800	0.3085	0.4265	0.4045	0.6660	0.3150	0.4845
	50	0.7620	0.7865	0.7965	0.4275	0.6760	0.7300	0.7035	0.8745	0.6520	0.7410
	75	0.9305	0.9435	0.9485	0.6085	0.8950	0.9130	0.8965	0.9640	0.8715	0.8970
	100	0.9845	0.9835	0.9875	0.7380	0.9665	0.9725	0.9760	0.9835	0.9580	0.9665
$[3,0.3]$	25	0.6725	0.5070	0.5190	0.3795	0.1020	0.6605	0.6530	0.8645	0.5420	0.7200
	50	0.9450	0.8340	0.8595	0.6605	0.7690	0.9310	0.9310	0.9820	0.8770	0.9480
	75	0.9950	0.9715	0.9760	0.8410	0.9570	0.9970	0.9945	0.9975	0.9835	0.9945
	100	1.0000	0.9940	0.9980	0.9130	0.9925	1.0000	0.9995	0.9995	0.9990	0.9995

From Table 2, one can see that, when the sample sizes are large, S_1^* has the highest power value among the others for $k=2$ and all p parameters. On the other hand, KS, AD and C^2 perform better results compared to the other tests when p is small. When p is small and sample sizes are large, it is observed that S_1^* has the highest power value for $k=3$. In addition, the powers of KS, AD, C^2 and S_1^* are quite close to each other when p is small and sample sizes are large.

Table 3. Power values of tests under the alternative Binomial distribution for $\alpha = 0.05$

$[m,p]$	n	KS	AD	C^2	χ^2	V_2^2	S_2	S_3	S_1^*	$S_1 = U_2^2$	U_3^2
$[10, 0.1]$	25	0,6025	0,6035	0,6335	0,3405	0,6240	0,5510	0,5180	0,7500	0,4430	0,6020
	50	0,8905	0,8915	0,9015	0,6080	0,9035	0,8655	0,8295	0,9615	0,8195	0,8635
	75	0,9685	0,9760	0,9750	0,8125	0,9745	0,9670	0,9525	0,9925	0,9595	0,9555
	100	0.9945	0.9970	0.9960	0.9190	0.9965	0.9935	0.9880	0.999	0.9935	0.9860
$[20, 0.1]$	25	0.9060	0.9310	0.9345	0.7355	0.8905	0.9295	0.9170	0.9785	0.8810	0.9395
	50	0.9995	0.9995	0.9995	0.9740	0.9980	1.0000	1.0000	1.0000	0.9990	1.0000
	75	1.0000	1.0000	1.0000	0.9975	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

From Table 3, it can be see that particularly S_1^* has higher power values of test than the others for $m=10$ and all sample sizes. Also, the powers of AD, KS, C^2 and V_2^2 give better results than the other tests when the sample sizes are large. In addition, S_1^* has the highest power values of test for $m=20$ and $n=25$. However, when sample sizes are large all tests have similar results with respect to power values of tests.

Table 4. Power values of tests under the alternative Poisson distribution for $\alpha = 0.05$

λ	n	KS	AD	C^2	χ^2	V_2^2	S_2	S_3	S_1^*	$S_1 = U_2^2$	U_3^2
0.25	25	0.0365	0.0140	0.0215	0.0080	0.0335	0.0255	0.0260	0.0445	0.0125	0.0375
	50	0.0670	0.0105	0.0625	0.0085	0.0590	0.0415	0.0365	0.1555	0.0205	0.0670
	75	0.1225	0.0150	0.1100	0.0215	0.1140	0.0775	0.0615	0.2465	0.0590	0.0855
	100	0.1700	0.0215	0.1630	0.0310	0.1490	0.0995	0.0735	0.2865	0.0785	0.0860
0.5	25	0.1770	0.1035	0.1600	0.0520	0.1730	0.1120	0.1135	0.2170	0.0630	0.1580
	50	0.3610	0.2105	0.3260	0.0985	0.3105	0.2455	0.2120	0.4630	0.1815	0.2680
	75	0.4955	0.3090	0.4730	0.1600	0.4505	0.3595	0.3165	0.6080	0.3215	0.3515
	100	0.6045	0.4045	0.5910	0.2175	0.5785	0.4695	0.4025	0.7175	0.4515	0.4080
0.75	25	0.3220	0.2685	0.3250	0.1270	0.3230	0.2440	0.2290	0.4345	0.1665	0.2930
	50	0.5910	0.5310	0.5870	0.2475	0.5730	0.4835	0.4330	0.7200	0.4100	0.4900
	75	0.7855	0.7460	0.7850	0.3905	0.7790	0.6855	0.6320	0.8755	0.6485	0.6595
	100	0.8845	0.8675	0.8915	0.5450	0.8950	0.8345	0.7855	0.9525	0.8235	0.7720
1	25	0.4745	0.4655	0.4995	0.2295	0.4710	0.3980	0.3710	0.6195	0.2825	0.4665
	50	0.7695	0.7815	0.7860	0.4210	0.7820	0.7150	0.6635	0.8810	0.6535	0.7225

	75	0.9260	0.9320	0.9330	0.6285	0.9320	0.8985	0.8655	0.9705	0.8775	0.8745
	100	0.9780	0.9810	0.9825	0.8010	0.9755	0.9650	0.9575	0.9900	0.9615	0.9485
2	25	0.8430	0.8665	0.8730	0.6390	0.8060	0.8570	0.8390	0.9480	0.7830	0.8880
	50	0.9950	0.9955	0.9955	0.9205	0.9905	0.9945	0.9925	0.9995	0.9880	0.9955
	75	1.0000	1.0000	1.0000	0.9920	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

From Table 4, especially S_1^* has higher power value of test than the others for different λ and sample sizes. In addition, the test statistics KS , C^2, AD and V_2^2 perform better results according to other tests for higher values of λ .

4. CONCLUSION

In this study, we investigated classical tests KS , C^2, AD goodness of fit test based on partition of Chi-square and alternative goodness of fit tests suggested by Best and Rayner [1,2,11] for Geometric distribution. Furthermore, we compared these test statistics with respect to type-I error and power values. It is observed that generally S_1^* test statistic has the highest power value among the others. Also, KS, AD, C^2, V_2^2 test statistics can be used for large sample sizes.

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