Gazi University Journal of Science
GU J Sci
27(1):653-658 (2014)

Fixed Point Theory for Cyclic$^{(\phi)}$ - Contractions in Uniform Spaces

S. FIROUZIAN[1], S. SEDGHI[2,♠], N. SHOBKOLAEI[3]

[1]*Payame Noor University, IRAN*
[2]*Islamic Azad University, Department of Mathematics, Qaemshahr Branch, Qaemshahr, IRAN*
[3]*Islamic Azad University, Department of Mathematics, Babol Branch, Babol, IRAN*

Received: 28.02.2013 Accepted: 08.07.2013

ABSTRACT

In this paper, we apply the concept of cyclic$^{(\phi)}$-contraction for presenting a fixed point theorem on Hausdorff uniform space. Some more general results are also obtained in Hausdorff uniform space.

Key Words: *Fixed point, Uniform Space,Cyclic$^{(\phi)}$-contraction.*

1. INTRODUCTION

Let X be a nonempty set and let ϑ be a nonempty family of subsets of $X \times X$. The pair (X, ϑ) is called a uniform space if it satisfies the following properties:

(i) if G is in ϑ, then G contains the diagonal $\{(x,x) \mid x \in X\}$;

(ii) if G is in ϑ and H is a subset of $X \times X$ which contains G, then H is in ϑ;

(iii) if G and H are in ϑ, then $G \cap H$ is in ϑ;

(iv) if G is in ϑ, then there exists H in ϑ, such that, whenever (x, y) and (y, z) are in H, then (x, z) is in G;

(v) if G is in ϑ, then $\{(y,x) \mid (x,y) \in G\}$ is also in ϑ.

ϑ is called the uniform structure of X and its elements are called entourages or neighbourhoods or surroundings. In Bourbaki [5] and Zeidler [17], (X, ϑ) is called a quasiuniform space if property (v) is omitted. Some authors such as Berinde [3], Jachymski [6], Kada et al [7], Rhoades [12], Rus [13], Wang [16] and Zeidler [17] studied the theory of fixed point or common fixed point for contractive selfmappings in complete metric spaces or Banach spaces in general.

Later, Aamri and El Moutawakil [2] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an A-distance and an E-distance. Diagonal

♠Corresponding author, e-mail: sedghi.gh@qaemshahriau.ac.ir

uniformity introduced by Weil, this approach was largely developed and pursued by Bourbaki [5].

For any set X, the diagonal $\{(x,x) \mid x \in X\}$ will be denoted by Δ where confusion might occur. If $V, W \in X \times X$, then $V \circ W = \{(x,y) \mid \text{ there exists } z \in X : (x,z) \in W \text{ and } (z,y) \in V\}$ and $V^{-1} = \{(x,y) \mid (y,x) \in V\}$.

If $V \in \vartheta$ and $(x,y) \in V, (y,x) \in V$, x and y are said to be V -close, and a sequence $\{x_n\}$ in X is a Cauchy sequence for ϑ, if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V -close for $n,m \geq N$. An uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X \mid (x,y) \in V\}$ when V runs over ϑ.

A sequence $\{x_n\}$ in X is convergent to x for ϑ, if for any $V \in \vartheta$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in V(x)$ for every $n \geq n_0$ and denote by $\lim_{n\to\infty} x_n = x$. A uniform space (X,ϑ) is said to be Hausdorff if and only if the intersection of all the $V \in \vartheta$ reduces to the diagonal Δ of X, i.e., if $(x,y) \in V$ for all $V \in \vartheta$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $V \in \vartheta$ is said to be symmetrical if $V = V^{-1}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x,y) \in W$ then x and y are both W and V -close, then for our purpose, we assume that each $V \in \vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X,ϑ), they always refer to the topological space $(X, \tau(\vartheta))$.

Now, we introduce the concept of A-distance, E-distance and prove fixed point theorems in Uniform spaces which are nice generalization of the known results in metric spaces.

Definition 1. *Let (X,ϑ) be a uniform space. A function $p : X \times X \to \mathbb{R}^+$ is said to be an A-distance if for any $V \in \vartheta$ there exists $\delta > 0$ such that if $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ for some $z \in X$, then $(x,y) \in V$.*

Definition 2. *Let (X,ϑ) be a uniform space. A function $p : X \times X \to \mathbb{R}^+$ is said to be an E-distance if*

(p_1) p is an A-distance,

(p_2)

$$p(x,y) \leq p(x,z) + p(z,y). \quad \forall x,y,z \in X.$$

Let us give some examples of A and E-distance.

Example 1. *Let (X,ϑ) be a uniform space and let d be a distance on X : Clearly (X,ϑ_d) is a uniform space where ϑ_d is the set of all subsets of $X \times X$ containing a "band" $B_\varepsilon = \{(x,y) \in X^2 \mid d(x,y) < \varepsilon\}$ for some $\varepsilon > 0$. Moreover, if $\vartheta \subseteq \vartheta_d$, then d is an E-distance on (X,ϑ).*

The following Lemma contain some useful properties of A-distances. It is stated in [7] for metric spaces and in [1, 2] for uniform spaces. The proof is straightforward.

Lemma 1. *Let (X,ϑ) be a Hausdorff uniform space and p be an A-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x,y,z \in X$, the following holds:*

(a) If $p(x_n,y) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$,

(b) if $p(x_n,y_n) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z,

(c) if $p(x_n,x_m) \leq \alpha_n$ for all $n,m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence in (X,ϑ).

Let (X,ϑ) be a uniform space with an A-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. That is, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $p(x_n,x_m) < \varepsilon$ for all $n,m \geq n_0$. There are several concepts of completeness in this setting

Definition 3. *Let (X,ϑ) be a uniform space and p be an A-distance on X.*

- X is S-complete if every p-Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n\to\infty} p(x_n,x) = 0$.

- X is p-Cauchy complete if every p-Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n\to\infty} x_n = x$ respect to $\tau(\vartheta)$.

Remark 1. *Let (X,ϑ) be a Hausdorff uniform space and let $\{x_n\}$ be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x\in X$ such that $\lim_{n\to\infty} p(x_n,x)=0$. Lemma 1(b) then gives $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$. Therefore S-completeness implies p-Cauchy completeness.*

Definition 4. *Let (X,ϑ) be a Hausdorff uniform space and p be an A-distance on X. Two selfmappings f and g of X are said to be weak compatible if they commute at their coincidence points, that is, $fx = gx$ implies that $fgx = gfx$.*

One of the most important results used in nonlinear analysis is the well-known Banach's contraction principle. Generalization of the above principle has been a heavily investigated branch research. Particularly, in [10] the authors introduced the following definition.

Definition 5. *Let X be a nonempty set, m a positive integer and $T: X\to X$ a mapping. $X=\cup_{i=1}^m A_i$ is said to be a cyclic representation of X with respect to T if*

- $A_i, i=1,2,\cdots,m$ are nonempty sets.
- $T(A_1)\subset A_2,\cdots,T(A_{m-1})\subset A_m, T(A_m)\subset A_1$.

Remark 2. *For convenience, we denote by F the class of functions $\phi:[0,\infty)\to[0,\infty)$ nondecreasing and continuous satisfying $\phi(t)>0$ for $t\in(0,\infty)$ and $\phi(0)=0$.*

Recently, fixed point theorems for operators T defined on a complete metric space X with a cyclic representation of X with respect to T have appeared in the literature (see e.g. [8, 9, 11, 14, 15]). Now, we present a modification the main result of [11]. Previously, we need the following definition.

Definition 6. *Let (X,d) be a metric space, m a positive integer, $A_1, A_2,\cdots, A_m$ nonempty subsets of X and $X=\cup_{i=1}^m A_i$. An operator $T: X\to X$ is a cyclic (ϕ)-contraction if*

- $X=\cup_{i=1}^m A_i$ is a cyclic representation of X with respect to T,
- $d(Tx,Ty)\le \phi(d(x,y))$, for any $x\in A_i$, $y\in A_{i+1}$, $i=1,2,\cdots,m$, where $A_{m+1}=A_1$ and $\phi\in \mathsf{F}$.

Previously, we need the following definitions.

Definition 7. *[13]. A function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is called a comparison function if it satisfies:*

- ϕ is increasing, i.e., $t_1\le t_2$ implies $\phi(t_1)\le\phi(t_2)$, for $t_1,t_2\in\mathbb{R}^+$;
- $\{\phi^n(t)\}_{n\in\mathbb{N}}$ converges to 0 as $n\to\infty$, for all $t\in\mathbb{R}^+$.

Definition 8. *[4]. A function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is called a (c)-comparison function if:*

- ϕ is increasing,
- there exist $k_0\in\mathbb{N}$, $a\in(0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\phi^{k+1}(t)\le a\phi^k(t)+v_k, \quad (1.2)$$

for $k\ge k_0$ and any $t\in\mathbb{R}^+$.

In [4] the following are also proved:

Lemma 2. *[4]. If $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is a (c)-comparison function, then the following hold:*

- ϕ is comparison function,
- $\phi(t)<t$, for any $t\in\mathbb{R}^+$,
- ϕ is continuous at 0,
- the series $\sum_{k=0}^{\infty}\phi^k(t)$ converges for any $t\in\mathbb{R}^+$.

Our main result is the following.

The main aim of this paper is to present a generalization of Theorem 2.1[11].

2. MAIN RESULT

First, we present the following definition.

Definition 9. *Let X be a nonempty set, m a positive integer and $T_i: X\to X$ be m mappings. $X=\cup_{i=1}^m A_i$ is said to be a cyclic representation of X with respect to T_i if*

- $A_i, i = 1,2,\cdots,m$ are nonempty sets.
- $T_1(A_1) \subset A_2, \cdots, T_{m-1}(A_{m-1}) \subset A_m, T_m(A_m) \subset A_1$.

Definition 10. *Let (X,ϑ) be a uniform space, m a positive integer, $A_1, A_2, \cdots, A_m$ nonempty subsets of X and $X = \cup_{i=1}^m A_i$. m operators $T_i : X \to X$ are cyclic ϕ-contraction if*

- $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to T_i,
- $\max\{p(T_i x, T_{i+1} y), p(T_{i+1} y, T_i x)\} \le \min\{\phi(p(x,y)), \phi(p(y,x))\}$,

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\cdots,m$, where $A_n = A_r$, $T_n = T_r$ for $r \in \{1,2,\cdots,m\}$ such that $n \equiv^m r$ and $\phi \in \mathsf{F}$.

Our main result is the following.

Theorem 1. *Let (X,ϑ) be a S-complete Hausdorff uniform space such that p be a E-distance on X and m a positive integer, $A_1, A_2, \cdots, A_m$ nonempty subsets of X and $X = \cup_{i=1}^m A_i$. Let $\phi : \mathsf{R}^+ \to \mathsf{R}^+$ is a (c)-comparison function and $T_i : X \to X$ be m cyclic ϕ-contraction such that A_i closed subsets of X respect to $\tau(\vartheta)$. Then there exists $z \in \cap_{i=1}^m A_i$ such that $T_i z = z$ and z is unique.*

Proof. Let x_1 be an arbitrary point in A_1. By cyclic representation of X with respect to T_i, we choose a point x_2 in A_2 such that $T_1(x_1) = x_2$. For this point x_2 there exists a point x_3 in A_3 such that $T_2(x_2) = x_3$, and so on there exist a point $x_m \in A_m$ such that $T_m(x_m) = x_{m+1} \in A_1$ and $T_1(x_{m+1}) = x_{m+2} \in A_2$. Continuing in this manner we can define a sequence $\{x_n\}$ as follows

$$T_r(x_{mk+r}) = x_{mk+r+1} \quad or \quad T_r(x_n) = x_{n+1},$$

where $n \equiv^m r$ for $r \in \{1,2,\cdots,m\}$.We prove that $\{x_n\}$ is a Cauchy sequence. Then, since $X = \cup_{i=1}^m A_i$, for any $n > 0$ there exists $i_n \in \{1,2,\cdots,m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Since T_i is a cyclic (ϕ)-contraction, we have

$$p(x_n, x_{n+1}) = p(T_r x_{n-1}, T_{r+1} x_n)$$

$$\le \max\{p(T_r x_{n-1}, T_{r+1} x_n), p(T_{r+1} x_n, T_r x_{n-1})\}$$

$$\le \min\{\phi(p(x_{n-1}, x_n)), \phi(p(x_n, x_{n-1}))\}$$

$$\le \phi(p(x_{n-1}, x_n)),$$

where $n \equiv^m r$ for $r \in \{1,2,\cdots,m\}$. From (1) and taking into account that ϕ is (c)-comparison, we get by induction that

$$p(x_n, x_{n+1}) \le \phi^{n-1}(p(x_1, x_2)) \ for\ any\ n = 1,2,\cdots.$$

Then by above inequality we obtain that

$$p(x_n, x_{n+1}) \to 0 \quad as \quad n \to \infty.$$

Therefore,since p is a E-distance we obtain that

$$p(x_n, x_m) \le p(x_n, x_{n+1}) + \cdots + p(x_{m-1}, x_m), \quad (2.2)$$

hence

$$p(x_n, x_m) \le \phi^{n-1}(p(x_1, x_2)) + \cdots + \phi^{n+m-2}(p(x_1, x_2)).$$

In the sequel, we will prove that $\{x_n\}$ is a p-Cauchy sequence.

Since $\sum_{n=1}^{\infty} \phi^n(t)$ is convergent for each $t > 0$, then $\{x_n\}$ is a p-Cauchy sequence in the uniform space (X,ϑ). Since (X,ϑ) is S-complete then from Remark 1, the sequence $\{x_n\}$ is p-complete, therefore there exists $x \in X$ such that

$$p(x_n, x) \to 0. \quad (2)$$

In fact, since $\lim_{n\to\infty} x_n = x$ and, as $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to T_i, the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1,2,\cdots,m\}$. Since A_i is closed for every i, it follows that $x \in \bigcap_{i=1}^m A_i$, thus we take a subsequence x_{n_k} of $\{x_n\}$ with $x_{n_k} \in A_{i+1}$ (the existence of this subsequence is guaranteed by the above mentioned comment). Using the contractive condition, we can obtain

$$p(T_r x, x) \leq p(T_r x, x_{n_k+1}) + p(x_{n_k+1}, x)$$

$$\leq p(T_r x, T_{r+1} x_{n_k}) + p(x_{n_{n_k+1}}, x)$$

$$\leq \min\{\phi(p(x, x_{n_k})), \phi(p(x_{n_k}, x))\} + p(x_{n_k}, x)$$

$$\leq p(x_{n_k}, x) + p(x_{n_k}, x),$$

where $n_k \equiv^m r+1$ for $r \in \{1,2,\cdots,m\}$. Since $x_{n_k} \to x$ and ϕ is (c)-comparison, letting $n_k \to \infty$ in the last inequality, we have $p(T_r x, x) = 0$. Similarly we can show that $p(T_r x, T_r x) = 0$ therefore, x is a fixed point of T_r.

Finally, in order to prove the uniqueness of the fixed point, we have $y, z \in X$ with y and z fixed points of T_r. The cyclic character of T_r and the fact that $y, z \in X$ are fixed points of T_r, imply that $y, z \in \cap_{i=1}^m A_i$. Using the contractive condition we obtain

$$p(y,z) = p(T_r y, T_{r+1} z) \leq \max\{p(T_r y, T_{r+1} z), p(T_{r+1} z, T_r y)\}$$

$$\leq \min\{\phi(p(y,z)), \phi(p(z,y))\} < p(y,z)$$

and from the last inequality

$$p(y,z) = 0$$

Similarly we can show that $p(y,y) = 0$ and, consequently, $y = z$. This finishes the proof.

Corollary 1. *Let (X, ϑ) be a S-complete Hausdorff uniform space such that p be a E-distance on X and m a positive integer, $A_1, A_2, \cdots, A_m$ nonempty subsets of X and $X = \cup_{i=1}^m A_i$. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a (c)-comparison function and $T : X \to X$ is a cyclic ϕ-contraction such that A_i closed subsets of X respect to $\tau(\vartheta)$. Then there exists $z \in \cap_{i=1}^m A_i$ such that $Tz = z$ and z is unique.*

Proof. Take $T_i = T$ in Theorem 1.

Corollary 2. *Let (X, d) be a complete metric space and m a positive integer, $A_1, A_2, \cdots, A_m$ nonempty closed subsets of X and $X = \cup_{i=1}^m A_i$. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a (c)-comparison function and $T_i : X \to X$ be m cyclic ϕ-contraction. Then there exists $z \in \cap_{i=1}^m A_i$ such that $T_i z = z$ and z is unique.*

Proof. Take $\vartheta = \vartheta_d$ in Theorem 1.

Corollary 3. *Let (X, d) be a complete metric space and m a positive integer, $A_1, A_2, \cdots, A_m$ nonempty closed subsets of X and $X = \cup_{i=1}^m A_i$. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a (c)-comparison function and $T : X \to X$ is a cyclic ϕ-contraction. Then there exists $z \in \cap_{i=1}^m A_i$ such that $Tz = z$ and z is unique.*

Proof. Take $T_i = T$ in Corollary 2.

Corollary 3 is a generalization of the main results of [11](see [[11], Theorem 2.1).

Example 2. *Let (X, ϑ) be a S-complete Hausdorff uniform space where $p(x,y) = y$, $X = \{\frac{1}{n}\} \cup \{0\}$ and $\vartheta = \tau_d$ such that $d = |,|$. Set $A_1 = \{\frac{1}{2n}\} \cup \{0,1\}$ and $A_2 = \{\frac{1}{2n+1}\} \cup \{0,1\}$. If define $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(t) = kt$ for $0 < k < 1$ and $T_i : X \to X$ by $T_1(0) = T_1(1) = 0$, $T_1(\frac{1}{n}) = \frac{1}{6n+1}$ and $T_2(0) = T_2(1) = 0$, $T_2(\frac{1}{2n+1}) = \frac{1}{6n}$. Then for every $x, y \neq 0,1$ we have*

$$\frac{1}{6n} = p(T_1(\frac{1}{2n}), T_2(\frac{1}{2n+1}))$$

$$= \max\{\frac{1}{6n+1}, \frac{1}{6n})\}$$

$$\leq kp(\frac{1}{2n}, \frac{1}{2n+1})$$

$$= k\min\{p(\frac{1}{2n}, \frac{1}{2n+1}), p(\frac{1}{2n+1}, \frac{1}{2n})\}.$$

Also, for $x, y = 0,1$ the above inequality obviously is hold for $\frac{1}{2} \leq k < 1$. This shows that the contractive condition of Corollary 1 is satisfied for $\phi(t) = kt$ and 0 is a fixed point for T_1, T_2.

REFERENCES

[1] Aamri, M., Bennani, S., El Moutawakil, D., "Fixed points and variational principle in uniform spaces", ***Siberian Electronic Mathematical Reports***, 3: 137–142 (2006).

[2] Aamri, M., El Moutawakil, D., "Common Fixed Point Theorems for E-contractive or E-expansive Maps in Uniform Spaces", ***Acta Math. Acad. Paedagog. Nyh´azi (N.S.),*** 20(1): 83-91(2004).

[3] Berinde, V., "Iterative Approximation of Fixed Points", ***Editura Efemeride***, Baia Mare, (2002).

[4] Berinde, V., "Contractii Generalizate si Aplicatii", vol. 22, ***Editura Cub Press***, Baia Mare, (1997).

[5] Bourbaki, N., "´E l´ements de math´ematique. Fasc. II. Livre III: Topologie g´en´erale. Chapitre 1: Structures topologiques. Chapitre 2: Structures uni-formes", Quatri´eme ´edition., ***Actualit´es Scientifiques et Industrielles***, No. 1142, Hermann, Paris, (1965).

[6] Jachymski, J., "Fixed Point Theorems for Expansive Mappings", ***Math. Japon.***, 42(1):131-136(1995).

[7] Kada, O., Suzuki, T., Takahashi, W., "Nonconvex Minimization Theorems and Fixed Point Theorems in Complete Metric Spaces", ***Math. Japon.***, 44(2): 381-391(1996).

[8] Karapnar, E., "Fixed point theory for cyclic weak ϕ-contraction", ***Appl.Math. Lett.***, 24(6): 822-825 (2011).

[9] Karap nar, E., Sadarangani, K., "Fixed point theory for cyclic $(\phi - \psi)$-contractions", ***Fixed Point Theory and Applications,*** 69, (2011) doi: 10.1186/1687-1812-2011-69.

[10] Kirk, W.A., Srinivasan, P.S., Veeramani, P., "Fixed points for mappings satisfying cyclical weak contractive conditions", ***Fixed Point Theory***, 4(1): 79–89(2003).

[11] Pacurar, M., Rus, I.A., "Fixed point theory for cyclic φ-contractions", ***Nonlinear Amal.,*** 72: 1181-1187(2010).

[12] Rhoades, B.E., "A Comparison of Various Definitions of Contractive Mappings", ***Trans. Amer. Math. Soc.***, 226: 257-290 (1977).

[13] Rus, I.A., "Generalized Contractions and Applications", ***Cluj University Press***, Cluj-Napoca, (2001).

[14] Rus, I.A., "Cyclic representations and fixed points", *Ann. T. Popoviciu,* ***Seminar Funct. Eq. Approx. Convexity,*** 3: 171-178(2005).

[15] De La Sen, M., "Linking contractive self-mappings and cyclic Meir-Keeler contractions with Kannan self-mappings", ***Fixed Point Theory and Applications***, Article ID 572057(2010).

[16] Wang, S.Z., Li, B. Y., Gao, Z. M., Is´eki, K., "Some Fixed Point Theorems on Expansion Mappings", ***Math. Japon.***, 29(4): 631-636(1984).

[17] Zeidler, E., "Nonlinear Functional Analysis and its Applications", Vol. 1, Springer- Verlag, New York, (1986).