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A Note on Multivariable Humbert Matrix Polynomials

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ABSTRACT

In this paper, we deal with some properties of the matrix extension of the multivariable Humbert polynomials defined by Aktaş et.al [Aktaş, R., Çekim, B. and Şahin, R., The matrix version for the multivariable Humbert polynomials, Miskolc Mathematical Notes, 13(2) (2012), 197-208]. We give differential equations for the products of these matrix polynomials and some other multivariable matrix polynomials, and also we present some new relations for the multivariable Humbert matrix polynomials.

Key words: *Multivariable Humbert matrix polynomials, Chan-Chyan-Srivastava matrix polynomials, Erkus-Srivastava matrix polynomials, multivariable Lagrange-Hermite matrix polynomials, differential equation*

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1. INTRODUCTION

Special matrix functions seen in the study of many area such as statistics, Lie group theory and number theory are well known. Recently, the matrix versions of the classical families orthogonal polynomials such as Jacobi, extended Jacobi, Hermite, Gegenbauer, Laguerre, Bessel and Chebyshev polynomials and some other special functions were introduced by many authors for matrices in $\mathbb{C}^{N\times N}$ and various properties satisfied by them were given from the scalar case, see for example [2,3,6-8,11-16,20,28-32,37,38].

Orthogonal matrix polynomials comprise an emerging field of study with important results in both theory and applications in literature. As theoretical examples, the property of orthogonality, Rodrigues formula, matrix differential equation, a three-term matrix recurrence relation, see [15,16,17,20,21]. As practical points, statistics, group representation theory, scattering theory, interpolation and quadrature and splines, see for example [18,25,27,33,36]. Furthermore, one can see more papers concerning other matrix polynomials. Recently, the matrix version of multivariable extension of generalized Humbert polynomials which are an interesting generalization of Humbert, Gegenbauer, Legendre, Tchebycheff polynomials and several other polynomial systems have been studied in [4]. See also [34] for other matrix extension of Humbert polynomials in one variable. In [4], the authors presented the matrix extension of the multivariable Humbert polynomials generated by

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$$\prod_{i=1}^{r}\left\{\left(C_i-m_ix_it+y_it^{m_i}\right)^{-A_i}\right\}$$
$$=\sum_{n=0}^{\infty}P_n^{(A_1,\dots A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})t^n \tag{1}$$
$$\left(\left|m_ix_it-y_it^{m_i}\right|<\left|C_i\right|,\ C_i\neq 0,\ i=1,2,\dots,r\right)$$

where $A_i\in\mathbb{C}^{N\times N}$, $\mathbf{x}=(x_1,\dots,x_r)$, $\mathbf{y}=(y_1,\dots,y_r)$, $\mathbf{C}=(C_1,\dots,C_r)$ and $\mathbf{m}=(m_1,\dots,m_r),\ m_i\in\mathbb{N}\ \ (i=1,2,\dots,r).$

From (1), the explicit representation of these matrix polynomials is

$$P_n^{(A_1,\dots A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})$$
$$=\sum_{m_1k_1+\dots+m_rk_r+n_1+\dots+n_r=n}\frac{(A_1)_{n_1+k_1}C_1^{-A_1-(n_1+k_1)I}}{n_1!k_1!}$$
$$\times\dots\times\frac{(A_r)_{n_r+k_r}C_r^{-A_r-(n_r+k_r)I}}{n_r!k_r!}$$
$$\times m_1^{n_1}\dots m_r^{n_r}(-1)^{k_1+\dots+k_r}x_1^{n_1}\dots x_r^{n_r}y_1^{k_1}\dots y_r^{k_r}$$
$$=\sum_{m_1k_1+\dots+m_rk_r+n_1+\dots+n_r=n}\prod_{p=1}^{r}\left\{\frac{(A_p)_{n_p+k_p}}{n_p!k_p!}\right.$$
$$\left.\times C_p^{-A_p-(n_p+k_p)I}m_p^{n_p}(-1)^{k_p}x_p^{n_p}y_p^{k_p}\right\}$$

where, as usual, $(A)_n$ denotes the Pochhammer symbol given by [31]

$$(A)_0=I;$$
$$(A)_n=A(A+I)\dots(A+(n-1)I)\quad (n\geq 1.) \tag{2}$$

We notice that since $A_i=\alpha_i\in\mathbb{C}$ in the case $N=1$, these polynomials reduce to the multivariable Humbert polynomials defined by [5].

The case $r=1$ of the polynomials gives Humbert matrix polynomials, which is the matrix version of generalized Humbert polynomials given by Gould [26] and they are generated by

$$\left(C-mxt+yt^m\right)^{-A}$$
$$=\sum_{n=0}^{\infty}P_n^{(A)}(m,x,y,C)t^n \tag{3}$$

where $\left|mxt-yt^m\right|<|C|,\ C\neq 0, A\in\mathbb{C}^{N\times N}$, m is a positive integer.

For the special cases of (3), including Gegenbauer matrix polynomials, see [30].

It is clear that in the special case

$$C_i=1,\ x_i=0,\ y_i=-x_i,\ i=1,2,\dots,r,$$

the matrix polynomials given by (1) reduce to the matrix version of the Erkus-Srivastava multivariable polynomials generated by [22]:

$$\prod_{i=1}^{r}\left\{\left(1-x_it^{m_i}\right)^{-A_i}\right\}$$
$$=\sum_{n=0}^{\infty}u_n^{(A_1,\dots A_r)}(x_1,\dots,x_r)t^n \tag{4}$$
$$\left\{|t|<\min\left\{|x_1|^{-1/m_1},|x_2|^{-1/m_2},\dots,|x_r|^{-1/m_r}\right\}\right\}$$

where $A_i\in\mathbb{C}^{N\times N}\ (i=1,2,\dots,r)$.

For $N=1$ we get $A_i=\alpha_i\in\mathbb{C}$ in (4) so, it gives the generating function for the Erkus-Srivastava multivariable polynomials [24]. Taking $m_i=1$ and $m_i=i$ $(i=1,2,\dots,r)$ respectively in (4) gives the matrix versions of the Chan-Chyan-Srivastava polynomials and multivariable Lagrange-Hermite polynomials, which are generated by

$$\prod_{i=1}^{r}\left\{\left(1-x_it\right)^{-A_i}\right\}$$
$$=\sum_{n=0}^{\infty}g_n^{(A_1,\dots A_r)}(x_1,\dots,x_r)t^n \tag{5}$$
$$\left(|t|<\min\left\{|x_1|^{-1},|x_2|^{-1},\dots,|x_r|^{-1}\right\}\right)$$

and

$$\prod_{i=1}^{r}\left\{\left(1-x_i t^i\right)^{-A_i}\right\}$$
$$=\sum_{n=0}^{\infty} h_n^{(A_1,\dots,A_r)}\left(x_1,\dots,x_r\right)t^n \qquad (6)$$
$$\left(|t|<\min\left\{|x_1|^{-1},|x_2|^{-1/2},\dots,|x_r|^{-1/r}\right\}\right)$$

where $A_i \in \mathbb{C}^{N\times N}$ $(i=1,2,\dots,r)$, respectively [22]. For $N=1$ we get $A_i=\alpha_i\in\mathbb{C}$ that these polynomials turn out Chan-Chyan-Srivastava and multivariable Lagrange-Hermite polynomials, respectively [9,10].

The purpose of this paper is to give various differential equations for the products of the multivariable Humbert matrix polynomials and some well-known multivariable matrix polynomials such as the matrix extensions of Chan-Chyan-Srivastava, Erkus-Srivastava and multivariable Lagrange-Hermite polynomials. Furthermore, we present some new equalities for the multivariable Humbert matrix polynomials.

Throughout this paper, for a matrix $A\in\mathbb{C}^{N\times N}$ its spectrum is denoted by $\sigma(A)$. The two-norm of A, which will be denoted by $\|A\|$, is defined by

$$\|A\|=\sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2}$$

where, for a vector $y\in\mathbb{C}^N$, $\|y\|_2=\left(y^T y\right)^{1/2}$ is the Euclidean norm of y. I and θ will denote the identity matrix and the null matrix in $\mathbb{C}^{N\times N}$, respectively. A matrix A in $\mathbb{C}^{N\times N}$ is a positive stable if $\mathrm{Re}(\lambda)>0$ for all $\lambda\in\sigma(A)$ where $\sigma(A)$ is the set of all eigenvalues of A. If $A_0,A_1,\dots,A_n$ are elements of $\mathbb{C}^{N\times N}$ and $A_n\neq\theta$, then we call

$$P(x)=A_n x^n + A_{n-1}x^{n-1}+\dots+A_1 x + A_0$$

a matrix polynomial of degree n in x. For any matrix A in $\mathbb{C}^{N\times N}$ we will exploit the following relation due to [31]

$$(1-x)^{-A}=\sum_{n=0}^{\infty}\frac{(A)_n}{n!}x^n\ ,\quad |x|<1 \qquad (7)$$

where $(A)_n$ denotes the Pochhammer symbol given by (2).

2. SOME RESULTS FOR THE MATRIX POLYNOMIALS $P_n^{(A_1,\dots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})$

Firstly, we discuss differential equations satisfied by the products of the multivariable Humbert matrix polynomials and some other multivariable matrix polynomials by considering the method given in the multivariate case in [23], which is motivated by the method given in [35] in one variable. Before giving main theorem, recall that the multivariable Humbert matrix polynomials verify the following equation [4]:

$$\sum_{j=1}^{r}\left(x_j\frac{\partial}{\partial x_j}+m_j y_j\frac{\partial}{\partial y_j}\right)P_n^{(A_1,\dots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})$$
$$=nP_n^{(A_1,\dots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}). \qquad (8)$$

From [22], for Erkus-Srivastava matrix polynomials, differential equation

$$\sum_{j=1}^{r} m_j x_j\frac{\partial}{\partial x_j}u_n^{(A_1,\dots,A_r)}\left(x_1,\dots,x_r\right)$$
$$=nu_n^{(A_1,\dots,A_r)}\left(x_1,\dots,x_r\right) \qquad (9)$$

holds. For $m_j=j$ and $m_j=1$ $(j=1,2,\dots,r)$, the equality (9) gives the differential equation for the multivariable Lagrange-Hermite matrix polynomials and Chan-Chyan-Srivastava matrix polynomials, respectively [22]:

$$\sum_{j=1}^{r} jx_j\frac{\partial}{\partial x_j}h_n^{(A_1,\dots,A_r)}\left(x_1,\dots,x_r\right)$$
$$=nh_n^{(A_1,\dots,A_r)}\left(x_1,\dots,x_r\right) \qquad (10)$$

and

$$\sum_{j=1}^{r} x_j\frac{\partial}{\partial x_j}g_n^{(A_1,\dots,A_r)}\left(x_1,\dots,x_r\right)$$
$$=ng_n^{(A_1,\dots,A_r)}\left(x_1,\dots,x_r\right). \qquad (11)$$

Let's consider the following differential operators:

$$L=\sum_{i=1}^{r}\left(a_i(x_i)\frac{\partial}{\partial x_i}+b_i(y_i)\frac{\partial}{\partial y_i}\right)$$

and

$$N=\sum_{j=1}^{s}\left(c_j(u_j)\frac{\partial}{\partial u_j}+d_j(v_j)\frac{\partial}{\partial v_j}\right). \quad (12)$$

Theorem 2.1 Let $\left\{S_n^{(A_1,\dots,A_r)}(\mathbf{x};\mathbf{y})\right\}\Big|_{n=0}^{\infty}$ and $\left\{Q_n^{(B_1,\dots,B_s)}(\mathbf{u};\mathbf{v})\right\}\Big|_{n=0}^{\infty}$ be matrix polynomials satisfying

$$L\left[S_n^{(A_1,\dots,A_r)}(\mathbf{x};\mathbf{y})\right]=\lambda_n S_n^{(A_1,\dots,A_r)}(\mathbf{x};\mathbf{y})$$
$$=n\,S_n^{(A_1,\dots,A_r)}(\mathbf{x};\mathbf{y})$$

and

$$N\left[Q_n^{(B_1,\dots,B_s)}(\mathbf{u};\mathbf{v})\right]=\eta_n Q_n^{(B_1,\dots,B_s)}(\mathbf{u};\mathbf{v})$$
$$=n\,Q_n^{(B_1,\dots,B_s)}(\mathbf{u};\mathbf{v}),$$

respectively, where $\mathbf{u}=(u_1,\dots,u_s)$ and $\mathbf{v}=(v_1,\dots,v_s)$.

Then for the product matrix polynomials

$$\left\{\Phi_n^{(A_1,\dots,A_r;B_1,\dots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u},\mathbf{v})\right\}\Big|_{n=0}^{\infty}$$
$$=\left\{S_{n-k}^{(A_1,\dots,A_r)}(\mathbf{x};\mathbf{y})\,Q_k^{(B_1,\dots,B_s)}(\mathbf{u};\mathbf{v})\right\}\Big|_{k=0,n=0}^{n,\infty},$$

the following differential equation holds:

$$L[w]+N[w]=nw. \quad (13)$$

Proof. Considering

$$w(\mathbf{x},\mathbf{y};\mathbf{u},\mathbf{v})=S_{n-k}^{(A_1,\dots,A_r)}(\mathbf{x};\mathbf{y})\,Q_k^{(B_1,\dots,B_s)}(\mathbf{u};\mathbf{v}),$$

we can write that

$$L[w]=\lambda_{n-k}w=(n-k)w$$

and

$$N[w]=\eta_k w=kw.$$

Then, it follows

$$L[w]+N[w]=nw$$

which completes the proof.

Now, we apply this theorem for the products of the multivariable Humbert matrix polynomials and Erkus-Srivastava, Chan-Chyan-Srivastava, multivariable Lagrange-Hermite matrix polynomials.

Remark 2.1 Let $\left\{S_n^{(A_1,\dots,A_r)}(\mathbf{x};\mathbf{y})\right\}\Big|_{n=0}^{\infty}$ and $\left\{Q_k^{(B_1,\dots,B_s)}(\mathbf{u};\mathbf{v})\right\}\Big|_{k=0}^{\infty}$ be the multivariable Humbert matrix polynomials so that

$$\left\{\Phi_n^{(A_1,\dots,A_r;B_1,\dots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u},\mathbf{v})\right\}\Big|_{n=0}^{\infty} \quad (14)$$
$$=\left\{P_{n-k}^{(A_1,\dots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\,P_k^{(B_1,\dots,B_s)}(\mathbf{n},\mathbf{u},\mathbf{v},\mathbf{C})\right\}\Big|_{k=0,n=0}^{n,\infty}.$$

Then, it follows that from (8)

$$L=\sum_{j=1}^{r}\left(x_j\frac{\partial}{\partial x_j}+m_j y_j\frac{\partial}{\partial y_j}\right) \text{ and}$$
$$M=\sum_{j=1}^{s}\left(u_j\frac{\partial}{\partial u_j}+n_j v_j\frac{\partial}{\partial v_j}\right).$$

Hence, differential equation for the matrix polynomials given by the equality (14) is in the following form:

$$\sum_{j=1}^{r}\left(x_j\frac{\partial}{\partial x_j}\Phi_n^{(A_1,\dots,A_r;B_1,\dots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u},\mathbf{v})\right.$$
$$\left.+m_j y_j\frac{\partial}{\partial y_j}\Phi_n^{(A_1,\dots,A_r;B_1,\dots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u},\mathbf{v})\right)$$
$$+\sum_{j=1}^{s}\left(u_j\frac{\partial}{\partial u_j}\Phi_n^{(A_1,\dots,A_r;B_1,\dots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u},\mathbf{v})\right.$$
$$\left.+n_j v_j\frac{\partial}{\partial v_j}\Phi_n^{(A_1,\dots,A_r;B_1,\dots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u},\mathbf{v})\right)$$
$$=n\,\Phi_n^{(A_1,\dots,A_r;B_1,\dots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u},\mathbf{v}).$$

Remark 2.2 Let $\left\{S_n^{(A_1,\dots,A_r)}(\mathbf{x};\mathbf{y})\right\}\Big|_{n=0}^{\infty}$ be the multivariable Humbert matrix polynomials and $\left\{Q_k^{(B_1,\dots,B_s)}(\mathbf{u};\mathbf{v})\right\}\Big|_{k=0}^{\infty}$ be Erkus-Srivastava matrix polynomials. Then we have the matrix polynomials

$$\left\{ \Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}) \right\} \Big|_{n=0}^{\infty} = \left\{ P_{n-k}^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\, u_k^{(B_1,\ldots,B_s)}(\mathbf{u}) \right\} \Big|_{k=0,n=0}^{n,\infty}$$

so that we get the differential operators from (8) and (9)

$$L = \sum_{j=1}^{r} \left(x_j \frac{\partial}{\partial x_j} + m_j y_j \frac{\partial}{\partial y_j} \right) \text{ and}$$

$$N = \sum_{j=1}^{s} n_j u_j \frac{\partial}{\partial u_j}.$$

Thus, the differential equation (13) becomes

$$\sum_{j=1}^{r} \left(x_j \frac{\partial}{\partial x_j} \Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}) + m_j y_j \frac{\partial}{\partial y_j} \Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}) \right) + \sum_{j=1}^{s} n_j u_j \frac{\partial}{\partial u_j} \Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}) = n\,\Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}). \tag{15}$$

Remark 2.3 By taking $n_j = j \ (j=1,2,\ldots,s)$ in the equality (15), for the product of the multivariable Humbert matrix polynomials and multivariable Lagrange-Hermite matrix polynomials given by

$$\left\{ \Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}) \right\} \Big|_{n=0}^{\infty} = \left\{ P_{n-k}^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\, h_k^{(B_1,\ldots,B_s)}(\mathbf{u}) \right\} \Big|_{k=0,n=0}^{n,\infty},$$

the differential equation is

$$\sum_{j=1}^{r} \left(x_j \frac{\partial}{\partial x_j} \Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}) + m_j y_j \frac{\partial}{\partial y_j} \Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}) \right) + \sum_{j=1}^{s} j u_j \frac{\partial}{\partial u_j} \Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}) = n\,\Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}).$$

Remark 2.4 The case $n_j = 1 \ (j=1,2,\ldots,s)$ in the relation (15) gives the differential equation

$$\sum_{j=1}^{r} \left(x_j \frac{\partial}{\partial x_j} \Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}) + m_j y_j \frac{\partial}{\partial y_j} \Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}) \right) + \sum_{j=1}^{s} u_j \frac{\partial}{\partial u_j} \Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}) = n\,\Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u})$$

for the products of the multivariable Humbert matrix polynomials and Chan-Chyan-Srivastava matrix polynomials defined by

$$\left\{ \Phi_n^{(A_1,\ldots,A_r;B_1,\ldots,B_s)}(\mathbf{x},\mathbf{y};\mathbf{u}) \right\} \Big|_{n=0}^{\infty} = \left\{ P_{n-k}^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\, g_k^{(B_1,\ldots,B_s)}(\mathbf{u}) \right\} \Big|_{k=0,n=0}^{n,\infty}.$$

Since we have $A_i = \alpha_i \in \mathbb{C} \ (i=1,2,\ldots,r)$ and $B_j = \beta_j \in \mathbb{C} \ (j=1,2,\ldots,s)$ in the case $N=1$, Remark 2.1-2.4 reduces to the results given by [1] for the multivariable Humbert polynomials, Erkus-Srivastava, Chan-Chyan-Srivastava and multivariable Lagrange-Hermite polynomials.

Now, we get some relations for the matrix polynomials $P_n^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})$.

Theorem 2.2 For the multivariable Humbert matrix polynomials, we have the following equalities

(i)

$$P_n^{\left(\sum_{i=1}^{k} B_i^1,\ldots,\sum_{i=1}^{k} B_i^r\right)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) = \sum_{n_1+n_2+\ldots+n_k=n} P_{n_1}^{(B_1^1,\ldots,B_1^r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) \times \ldots \times P_{n_k}^{(B_k^1,\ldots,B_k^r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})$$

where $B_i^j \in \mathbb{C}^{N\times N}, \ (i=1,2,\ldots,k \ ; \ j=1,2,\ldots,r)$ and these matrices are commutative.

(ii)

$$\sum_{n_1+n_2+...+n_p=n} P_{n_1}^{(A_1,...,A_{i-1},A_i+I,A_{i+1},...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})$$
$$\times...\times P_{n_p}^{(A_1,...,A_{i-1},A_i+I,A_{i+1},...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})$$
$$=C_i^{-p}\sum_{k=0}^{n}\sum_{l=0}^{[(n-k)/m_i]}\frac{(p)_{k+l}(m_i x_i)^k y_i^l(-1)^l}{C_i^{k+l}k!l!}$$
$$\times P_{n-k-m_i l}^{(pA_1,...,pA_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})$$

where $A_j \in \mathbb{C}^{N\times N}$ $(j=1,2,...,r)$ and $A_jA_i=A_iA_j$ $(i,j=1,2,...,r)$.

Proof. (i) From the generating function given by (1), one can easily prove this.

(ii) Taking pA_j $(p\in\mathbb{N})$ instead of A_j in the generating function (1), we get

$$F(\mathbf{x},\mathbf{y};t):=\prod_{i=1}^{r}\left\{\left(C_i-m_ix_it+y_it^{m_i}\right)^{-pA_i}\right\}$$
$$=\sum_{n=0}^{\infty}P_n^{(pA_1,...,pA_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})t^n.$$

If we differentiate $F(\mathbf{x},\mathbf{y};t)$ with respect to x_i $(i=1,2,...,r)$ p times, we obtain

$$\frac{\partial^p}{\partial x_i^{\,p}}F(\mathbf{x},\mathbf{y};t) \tag{16}$$
$$=(pA_i)_p(m_it)^p\left(C_i-m_ix_it+y_it^{m_i}\right)^{-p(A_i+I)}$$
$$\times\prod_{\substack{j=1\\ j\neq i}}^{r}\left\{\left(C_j-m_jx_jt+y_jt^{m_j}\right)^{-pA_j}\right\}$$
$$=(pA_i)_p(m_it)^p\left(C_i-m_ix_it+y_it^{m_i}\right)^{-pI}$$
$$\times\sum_{n=0}^{\infty}P_n^{(pA_1,...,pA_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})t^n$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{[(n-k)/m_i]}\frac{(p)_{k+l}(m_ix_i)^k y_i^l(-1)^l}{C_i^{k+l}k!l!}$$
$$\times(pA_i)_p(m_it)^pC_i^{-p}P_{n-k-m_il}^{(pA_1,...,pA_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})t^n.$$

On the other hand, we can write the relation (16) again in the form

$$\frac{\partial^p}{\partial x_i^{\,p}}F(\mathbf{x},\mathbf{y};t)$$
$$=(pA_i)_p(m_it)^p$$
$$\times\left(\sum_{n=0}^{\infty}P_n^{(A_1,...,A_{i-1},A_i+I,A_{i+1},...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})t^n\right)^p$$
$$=(pA_i)_p(m_it)^p$$
$$\times\sum_{n=0}^{\infty}\left(\sum_{n_1+n_2+...+n_p=n}P_{n_1}^{(A_1,...,A_{i-1},A_i+I,A_{i+1},...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\right.$$
$$\left.\times...\times P_{n_p}^{(A_1,...,A_{i-1},A_i+I,A_{i+1},...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\right)t^n.$$

From these two last equalities, we have the desired addition formula.

Remark 2.5 The special cases of Theorem 2.2 give similar results for the matrix versions of Chan-Chyan-Srivastava, Erkus-Srivastava and Lagrange-Hermite multivariable polynomials defined by [22].

Remark 2.6 Since

$A_j=\alpha_j\in\mathbb{C}$ $(j=1,2,...,r)$ and

$B_i^j=b_i^j\in\mathbb{C}$ $(i=1,2,...,k\ ;\ j=1,2,...,r)$

for $N=1$, Theorem 2.2 reduces to the results given for the multivariable Humbert polynomials in [1].

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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