Completion of Vector Metric Spaces

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ABSTRACT

In this study a completion theorem for vector metric spaces is proved. The completion spaces are defined by means of an equivalence relation obtained by order convergence via the module of the Riesz space *E*.

Keywords: Vector metric space, order-Cauchy sequence, order-Cauchy completion, Riesz space, Banach lattice.

1. INTRODUCTION

Let E be a Riesz space. If (a_n) is a decreasing sequence in E such that $\inf a_n = a$, we will write $a_n \downarrow a$. A sequence (b_n) is said to *order convergence* (or *o-convergence*) to b if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $|b_n - b| \le a_n$ for all n, and written $b_n \xrightarrow{o} b$ or $o\text{-}\lim b_n = b$. Furthermore (b_n) is said to be *order-Cauchy* (or *o-Cauchy*) if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $|b_n - b_{n+p}| \le a_n$ holds for all n and p. E is said to be *o-Cauchy complete* if every *o*-Cauchy sequence is *o*-convergence. If every nonempty bounded above countable subset of E has a supremum, then E is called *Dedekind σ-complete*. An operator $T : E \to F$ between two Riesz spaces is a *lattice homomorphism* if $T(x \vee y) = T(x) \vee T(y)$ for all $x, y \in E$. For notations and other facts regarding Riesz spaces we refer to [1] and [9].

In [3], a vector metric space is defined with a distance map having values in a Riesz space, and some results in metric space theory are generalized to vector metric space theory. Some fixed point theorems in vector metric spaces are given in [2,3,10,12,13]. In [4], new notions of vectorial and topological continuities are defined and some new basic results are presented. In Section 2, we recall basic concepts some results of the metric spaces theory in vector metric spaces.

Section 3 is an exposition of the fundamentals of completeness and completion by *E*-Cauchy sequences in vector metric spaces. In [5], order-Cauchy completeness and completions of Archimedean Riesz spaces and rings of real-valued continuous functions are discussed. Most of the material presented in [5] is due to Everett [6] and Papangelou [11] in the case of *l*-groups. Every Archimedean Riesz space or *f*-algebra has an *o*-Cauchy completion in a sense defined precisely in [5]. Among other things, conditions are given under which the order-Cauchy completion and the Dedekind completion coincide. Completions akin to the order-Cauchy completion are described also in [14]. We give an abstract characterization of *E*-completion for vector metric which valued *o*-Cauchy complete Riesz space.

2. VECTOR METRIC SPACES AND CONTINUITY

In this section we recall vector metric spaces and prove some properties.

Definition 1. [3] Let X be a nonempty set and E be a Riesz space. The function $d : X \times X \to E$ is said to be a *vector metric* (or *E-metric*) if it is satisfying the following properties:

(vm1) $d(x, y) = 0$ if and only if $x = y$,

(vm2) $d(x, y) \le d(x, z) + d(y, z)$

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for all $x, y, z \in X$. Also the triple (X, d, E) is said to be *vector metric space*.

Proposition 1. [3] For arbitrary elements x, y, z, w of a vector metric space, the following properties hold:

(i) $0 \leq d(x, y)$,

(ii) $d(x, y) = d(y, x)$,

(iii) $\left| d(x, z) - d(y, z) \right| \leq d(x, y)$,

(iv) $\left| d(x, z) - d(y, w) \right| \leq d(x, y) + d(z, w)$.

Now we give some examples of vector metric spaces.

Example. (a) A Riesz space E is a vector metric space with $d : E \times E \to E$ defined by $d(x, y) = |x - y|$. This vector metric is called to be absolute valued metric on E.

(b) The complexification E of a real Banach lattice E is a vector metric space with $d : E \times E \to E$ defined by $d(x, y) = |x - y|$.

(c) Suppose we have a finite number of vector metric spaces (X_i, d_i, E), $i = 1, \ldots, k$. On the cartesian product $X = X_1 \times \cdots \times X_k$ various vector metrics can be defined. Let $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ be two elements of the product space X. We define

$$d_{(1)}(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i) \text{ and}$$

$$d_{\infty}(x, y) = \sup\{d_i(x_i, y_i) : i = 1, \ldots, k\}.$$

It can be verified easily that $d_{(1)}$ and d_{∞} are vectorial distance functions on the product space X. If E is Banach lattice and $1 < p < \infty$ where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then by the functional calculus of J.L.Krivine, $\left(\sum_{i=1}^{k} d_i(x_i, y_i)^p \right)^{\frac{1}{p}} \in E$ with

$$\left(\sum_{i=1}^{k} d_i(x_i, y_i)^p \right)^{\frac{1}{p}}$$

$$= \sup \left\{ \sum_{i=1}^{k} |a_i| d_i(x_i, y_i) : (a_1, \ldots, a_k) \in \quad^k, \sum_{i=1}^{k} |a_i|^q \leq 1 \right\}$$

(see [7], [8] pp. 42-44). We define

$$d_{(p)}(x, y) = \left(\sum_{i=1}^{k} \left(d_i(x_i, y_i) \right)^p \right)^{\frac{1}{p}}.$$

Thus, $d_{(p)}$ is a vector metric on the product space X.

Definition 2. [3] (a) A sequence (x_n) in a vector metric space (X, d, E) *vectorially converges* (or is *E-conver-*

ges) to some $x \in X$, written $x_n \overset{d,E}{\to} x$, if there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x) \leq a_n$ for all n.

(b) A sequence (x_n) is called an *E-Cauchy sequence* whenever there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$ holds for all n and p.

(c) A vector metric space (X, d, E) is called *E-complete* if each *E*-Cauchy sequence in X *E*-converges to a limit in X.

(d) A subset Y of a vector metric space (X, d, E) is said to be *E-closed* whenever $(x_n) \subseteq Y$ and $x_n \overset{d,E}{\to} x$ imply $x \in Y$.

Remark. More explicit (and overly cumbersome) terminology would perhaps be sequentially *E*-Cauchy complete, to distinguish from the corresponding notion for nets. However, this paper is concerned exclusively with sequences, so dropping "sequentially" introduces no ambiguity here.

Theorem 1. [3] If $x_n \overset{d,E}{\to} x$, then the followings hold:

(i) The limit x is unique.

(ii) Every subsequence of (x_n) *E*-converges to x.

(iii) If also $y_n \overset{d,E}{\to} y$, then $d(x_n, y_n) \overset{o}{\to} d(x, y)$.

Also, we have the following theorem.

Theorem 2. [3] For the vector metric space (X, d, E), the followings hold:

(i) Every *E*-convergent sequence is an *E*-Cauchy sequence;

(ii) Every *E*-Cauchy sequence is *E*-bounded;

(iii) If an *E*-Cauchy sequence (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \overset{d,E}{\to} x$, then $x_n \overset{d,E}{\to} x$;

(iv) If (x_n) and (y_n) are *E*-Cauchy sequences, then $(d(x_n, y_n))$ is an order Cauchy sequence.

When $E = \quad$, the concepts of vectorial convergence and convergence in metric are the same. When also $X = E$ and d is the concepts of absolute valued vector metric, vectorial convergence and convergence in order are the same. When $E = \quad$, the concepts of *E*-Cauchy sequence and Cauchy sequence are the same.

Now, let us fix a vector metric space (X, d, E). For two elements a and b in E, we shall write $a < b$ to indicate that $a \leq b$ but $a \neq b$, while $b > a$ will stand for $a < b$.

Definition 3. [3] (a) A subset Y of X is called $\tau_{d,E}$-*dense* whenever $B(x,r) \cap Y \neq \varnothing$ for each $x \in X$ and $0 < r \in E$.

(b) A subset Y of X is called *E-dense* whenever for every $x \in X$ there exists a sequence (x_n) in Y satisfying $x_n \xrightarrow{d,E} x$.

We have already following results.

Corollary 1. [3] Let Y be a subset of a vector metric space (X,d,E) with E Archimedean. If Y is $\tau_{d,E}$-dense, then Y is E-dense.

The relationships between the concepts of boundedness and diameter of a subset of a vector metric space are a different from the usual. For a nonempty subset A of a vector metric space (X,d,E) its *E-diameter* defined by $d(A) = \sup\{d(x,y) : x,y \in A\}$ if $\sup\{d(x,y) : x,y \in A\}$ in E. Furthermore, if there exists an $a > 0$ in E such that $d(x,y) \leq a$ for $x,y \in A$, then A is called *E-bounded* set. If E is Dedekind complete, then every E-bounded set of (X,d,E) has a diameter.

Definition 4. [4] Let (X,d,E) and (Y,ρ,F) be vector metric spaces, and let $x \in X$.

(a) A function $f : X \to Y$ is said to be *topologically continuous at x* if for every $b > 0$ in F there exists some a in E such that $\rho(f(x),f(y)) < b$ whenever $y \in X$ and $d(x,y) < a$. The function f is said to be *topologically continuous* if it is topologically continuous at each point of X.

(b) A function $f : X \to Y$ is said to be *vectorially continuous at x* if $x_n \xrightarrow{d,E} x$ in X implies $f(x_n) \xrightarrow{\rho,F} f(x)$ in Y. The function f is said to be *vectorially continuous* if it is vectorially continuous at each point of X.

Theorem 3. [4] Let (X,d,E) and (Y,ρ,F) be vector metric spaces where F is Archimedean. If a function $f : X \to Y$ is topologically continuous, then f is vectorially continuous.

Corollary 2. [4] For a function $f : X \to Y$ between two vector metric spaces (X,d,E) and (Y,ρ,F) the following statements hold:

(a) If F is Dedekind σ-complete and f is vectorially continuous, then $\rho(f(x_n),f(x)) \downarrow 0$ whenever $d(x_n,x) \downarrow 0$.

(b) If E is Dedekind σ-complete and $\rho(f(x_n),f(x)) \downarrow 0$ whenever $d(x_n,x) \downarrow 0$, then the function f is vectorially continuous.

(c) Suppose that E and F are Dedekind σ-complete. Then, the function f is vectorially continuous if and only if $\rho(f(x_n),f(x)) \downarrow 0$ whenever $d(x_n,x) \downarrow 0$.

Definition 5. [4] Let (X,d,E) and (Y,ρ,F) be vector metric spaces. A function $f : X \to Y$ is said to be a *vector isometry* if there exists a linear operator $T_f : E \to F$ satisfying the following two conditions,

(i) $T_f(d(x,y)) = \rho(f(x),f(y))$ for all $x,y \in X$,

(ii) $T_f(a) = 0$ implies $a = 0$ for all $a \in E$.

If the function f is onto, and the operator T_f is a lattice homomorphism, then the vector metric spaces (X,d,E) and $(Y,\rho,T_f(E))$ are called *vector isometric*.

If $E = F$ in Definition 5, then the map $f : X \to Y$ between two vector metric spaces (X,d,E) and (Y,ρ,E) is called *E-isometry* if $d(x, y) = \rho(i(x), i(y))$ holds for all $x,y \in X$. If in addition f is onto, then (X,d,E) and (Y,ρ,E) are called *E-isometric*.

3. ORDER-CAUCHY COMPLETIONS OF VECTOR METRIC SPACES

Given a vector metric space (X,d,E) which is not E-complete, we are going to construct a E-complete vector metric space $(\hat{X},\hat{d},E)$ such that there is an E-isometry $i : X \to \hat{X}$ with the property that $i(X)$ is E-dense in $\hat{X}$. We call $(\hat{X},\hat{d},E)$ the *order-Cauchy completion* of (X,d,E).

Let E be an order-Cauchy complete Riesz space and let (x_n) and (y_n) be two E-Cauchy sequences in X. Then we define a equivalence relation denoted by

$$d(x_n, y_n) \xrightarrow{o} 0.$$

Let $\hat{X}$ be the set of all equivalence classes of E-Cauchy sequences of X. Since $d(x_n, y_n)$ is an order Cauchy sequence whenever (x_n) and (y_n) are E-Cauchy sequences, by E is order-Cauchy we define

$$\hat{d}(\hat{x},\hat{y}) = o\text{-}\lim d(x_n, y_n)$$

in E for all $\hat{x},\hat{y} \in \hat{X}$ with $(x_n) \in \hat{x}$ and $(y_n) \in \hat{y}$. It is not difficult to see that $(\hat{X},\hat{d},E)$ is a vector metric space.

For an element x of X consider the equivalence class $i(x) \in \hat{X}$ which is generated by the constant sequence whose each term is x. This way we define a map $i : X \to \hat{X}$. Then (X,d,E) and $(i(X),\hat{d},E)$ are E-isometric.

Lemma 1. $(i(X), \hat{d}, E)$ is E-dense in $(\hat{X}, \hat{d}, E)$.

Proof Let $(x_n) \in \hat{x}$ and for fixed n_0 consider $i(x_{n_0})$ as an element $i(X)$. $i(x_{n_0})$ is of course the equivalence class generated by the constant $(x_{n_0}, x_{n_0}, \ldots)$. For every $n \geq n_0$ there exists a number p such that $n = n_0 + p$. Since (x_n) is E-Cauchy, there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and

$$\hat{d}(i(x_{n_0}), \hat{x}) = o\text{-}\lim d(x_{n_0}, x_n)$$
$$= o\text{-}\lim d(x_{n_0}, x_{n_0+p}) \leq a_{n_0}$$

holds for all $n \geq n_0$ with $n = n_0 + p$. Hence $\hat{d}(i(x_k), \hat{x}) \leq a_k$ for all k; i.e., $i(x_k) \xrightarrow{\hat{d}, E} \hat{x}$.

We have the following theorem.

Theorem 4. $(\hat{X}, \hat{d}, E)$ is unique E-complete vector metric space generated by (X, d, E).

Proof Let $(\hat{x}_n)$ be a E-Cauchy sequence in $\hat{X}$. Using the fact that $i(X)$ is E-dense in $\hat{X}$ for each n we find y_n in X and a_n in E with

$$\hat{d}(i(y_n), \hat{x}_n) \leq a_n.$$

We will show that (y_n) is a E-Cauchy sequence in X and if $\hat{y}$ is the equivalence class generated by (y_n), then $\hat{x}_n \to \hat{y}$ in $(\hat{X}, \hat{d}, E)$. We have

$$d(y_n, y_{n+p}) = \hat{d}(i(y_n), i(y_{n+p}))$$
$$\leq \hat{d}(i(y_n), \hat{x}_n)) + \hat{d}(\hat{x}_n, \hat{x}_{n+p}) + \hat{d}(\hat{x}_{n+p}, i(y_{n+p}))$$
$$\leq 3a_n$$

for all n and p. By (1) we have $i(x_n) \xrightarrow{\hat{d}, E} \hat{y}$. Hence,

$$\hat{d}(\hat{y}, \hat{x}_n) \leq \hat{d}(\hat{y}, i(y_n)) + d(i(y_n), \hat{x}_n) \leq 2a_n$$

which implies $x_n \xrightarrow{\hat{d}, E} \hat{y}$. Therefore, $(\hat{X}, \hat{d}, E)$ E-complete.

Let $(\tilde{X}, \tilde{d}, E)$ be another E-complete vector metric space such that there is an E-isometry $j : X \to \tilde{X}$ with $j(X)$ is E-dense in $\tilde{X}$. We will show that $(\tilde{X}, \tilde{d}, E)$ is then isometrically equivalent to the completion $(\hat{X}, \hat{d}, E)$. Hence in this sense the completion is unique.

We define first $h_0 : i(X) \to j(X)$ by $h_0(i(x)) = j(x)$ for all $x \in X$. It is clear that $i(X)$ and $j(X)$ are E-isometric. Let $\hat{x} \in \hat{X}$. Then there exists a E-Cauchy

sequence $i(x_n)$ in $i(X)$ such that $i(x_n) \xrightarrow{\hat{d}, E} \hat{x}$. Since h_0 is an E-isometry, then $(j(x_n)) = (h_0 \circ i(x_n))$ is also an E-Cauchy sequence in $\tilde{X}$. Because $\tilde{X}$ is E-complete, there exists $\tilde{x}$ in $\tilde{X}$ such that $j(x_n) \xrightarrow{\tilde{d}, E} \tilde{x}$. Hence we define

$$h : \hat{X} \to \tilde{X} \text{ with } h(\hat{x}) = \tilde{x}.$$

For two elements $\hat{x}$ and $\hat{y}$ in $\hat{X}$ there exist two sequences $(i(x_n))$ and $(i(y_n))$ in $i(X)$ such that $i(x_n) \xrightarrow{\hat{d}, E} \hat{x}$ and $i(y_n) \xrightarrow{\hat{d}, E} \hat{y}$. Then $j(x_n) \xrightarrow{\tilde{d}, E} h(\hat{x})$ and $j(y_n) \xrightarrow{\tilde{d}, E} h(\hat{y})$ hold. Since h_0 is an E-isometry, by Theorem 2 (4) we have

$$\tilde{d}(h(\hat{x}), h(\hat{y})) = o\text{-}\lim \tilde{d}(j(x_n), j(y_n))$$
$$= o\text{-}\lim \hat{d}(i(x_n), i(y_n)) = \hat{d}(\hat{x}, \hat{y}).$$

Finally, we will show that h is onto. If $\tilde{x}$ in $\tilde{X}$, then there exists a E-Cauchy sequence $j(x_n)$ in $j(X)$ such that $j(x_n) \xrightarrow{\tilde{d}, E} \tilde{x}$. Since $i(X)$ E-isometric to $j(X)$, $i(x_n)$ is also an E-Cauchy sequence in $\hat{X}$. If $\hat{x}$ is $\hat{d}$-limit of $i(x_n)$, then $h(i(x_n)) \xrightarrow{\tilde{d}, E} h(\hat{x})$ since h is an E-isometry. By using $j(x_n) = h(i(x_n))$ we have

$$\tilde{d}(\tilde{x}, h(\hat{x})) \leq \tilde{d}(\tilde{x}, j(x_n)) + \tilde{d}(h(\hat{x}), h(i(x_n)))$$

for each n. Therefore, $h(\hat{x}) = \tilde{x}$ holds in $\tilde{X}$.

CONFLICT OF INTEREST

No conflict of interest was declared by the author.

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