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A Generalization of the Extended Jacobi Polynomials in Two Variables

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ABSTRACT

The main object of this paper is to construct a two-variable analogue of extended Jacobi polynomials and to give some properties of these polynomials. We obtain various differential formulae for two-variable extended Jacobi polynomials and give recurrence relations involving these polynomials. We derive various families of bilinear and bilateral generating functions. Furthermore, some special cases of the results are presented in this study.

Key Words: Extended Jacobi polynomials, Jacobi polynomials, recurrence relation, generating function, hypergeometric function

1. INTRODUCTION

Some families of polynomials, especially classical orthogonal polynomials including Jacobi, Laguerre, Hermite polynomials, are of considerable interest because of their close connection to applied sciences. In the recent years, multivariable analogues of some well-known polynomials (see [1,5,12]) and their properties, and also various generalizations of classical orthogonal polynomials including Jacobi and Laguerre polynomials (see [3,4]) have been studied. Our research focus on problem to generalize a two-variable analogue of extended Jacobi polynomials (EJPs) $F_n^{(\alpha,\beta)}(x;a,b,c)$ which are an unified presentation of the classical orthogonal polynomials (especially Jacobi, Laguerre and Hermite polynomials) (see [13]). Extended Jacobi polynomials (EJPs) are defined by the Rodrigues formula

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \frac{(-c)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta} \times \frac{d^n}{dx^n}\{(x-a)^{n+\alpha}(b-x)^{n+\beta}\}, \quad (c>0).$$

Szegö ([21]) shown that $F_n^{(\alpha,\beta)}(x;a,b,c)$ polynomials are a constant multiple of the classical Jacobi orthogonal polynomials $P_n^{(\alpha,\beta)}(x)$ in the form:

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \{c(a-b)\}^n P_n^{(\alpha,\beta)}\left(\frac{2(x-a)}{a-b}+1\right) \quad (1)$$

or, equivalently,

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$$P_n^{(\alpha,\beta)}(x) = \{c(a-b)\}^{-n} F_n^{(\alpha,\beta)}\left(\frac{1}{2}\{a+b+(a-b)x\};a,b,c\right).$$

In terms of the hypergeometric function, we get that

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \{c(a-b)\}^n \binom{\alpha+n}{n} \times {}_2F_1\left(-n,\alpha+\beta+n+1;\alpha+1;\frac{x-a}{b-a}\right) \quad (2)$$

or, equivalently, by a finite series

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1+\alpha+\beta)_{n+k}}{k!(n-k)!(1+\alpha)_k (1+\alpha+\beta)_n} \times \{c(a-b)\}^n \left(\frac{x-a}{a-b}\right)^k. \quad (3)$$

We have the following generating functions for the EJPs respectively:

$$\sum_{n=0}^{\infty} (c(a-b))^{-n} F_n^{(\alpha-n,\beta-n)}(x;a,b,c) t^n = \left(1+\frac{(x-b)}{a-b}t\right)^{\alpha} \left(1+\frac{(x-a)}{a-b}t\right)^{\beta}, \quad (4)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} (c(a-b))^{-n} F_n^{(\alpha-n,\beta-n)}(x;a,b,c) t^n = F_1\left[\lambda,-\alpha,-\beta;\mu;\frac{(x-b)}{b-a}t,\frac{(x-a)}{b-a}t\right], \quad (5)$$

$$\left(|t| < \frac{|b-a|}{\max\{|x-b|,|x-a|\}}\right)$$

where $F_1[a,b,b';c;x,y]$ is the first kind of Appell's double hypergeometric function ([8], [9])

$$F_1[a,b,b';c;x,y] = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b)_r(b')_s}{(c)_{r+s}} \frac{x^r}{r!} \frac{y^s}{s!}, \quad \max\{|x|,|y|\} < 1,$$

and

$$\sum_{n=0}^{\infty} \frac{(\sigma)_n (\delta)_n}{(1+\alpha)_n (1+\beta)_n} (c(a-b))^{-n} F_n^{(\alpha,\beta)}(x;a,b,c) t^n = F_4\left[\sigma,\delta;1+\alpha,1+\beta;\frac{(x-a)}{a-b}t,\frac{(x-b)}{a-b}t\right], \quad (6)$$

$$\left(|t|^{1/2} < \frac{|a-b|^{1/2}}{|x-a|^{1/2}+|x-b|^{1/2}}\right)$$

where $F_4[a,b;c,c';x,y]$ is the fourth kind of Appell's double hypergeometric function ([8], [9])

$$F_4[a,b;c,c';x,y] = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b)_{r+s}}{(c)_r(c')_s} \frac{x^r}{r!} \frac{y^s}{s!}, \quad |x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} < 1.$$

It is well-known that the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ are orthogonal over the interval (a,b) with respect to the weight function $\omega(x;a,b) = (x-a)^{\alpha}(b-x)^{\beta}$. Actually, we have

$$\int_a^b (x-a)^{\alpha} (b-x)^{\beta} F_n^{(\alpha,\beta)}(x;a,b,c) \times F_m^{(\alpha,\beta)}(x;a,b,c)\,dx = \frac{c^{2n}(-1)^{\alpha+\beta+1}(a-b)^{2n+\alpha+\beta+1}}{n!(\alpha+\beta+2n+1)} \times \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+n+1)} \delta_{m,n} \quad (7)$$

$$\left(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\} > -1;\ m,n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right),$$

where $\delta_{m,n}$ denotes the Kronecker delta. Furthermore, they hold

$$\int_a^b x^k F_n^{(\alpha,\beta)}(x;a,b,c)(x-a)^{\alpha}(b-x)^{\beta}\,dx = 0 \quad (8)$$

$$(k = 0,1,\ldots,n-1)$$

and

$$\int_a^b x^s F_n^{(\alpha,\beta)}(x;a,b,c)(x-a)^\alpha (b-x)^\beta\, dx \qquad (9)$$
$$= \frac{a^{s-n}c^n s!(b-a)^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)}{n!(s-n)!}$$
$$\times \frac{\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}$$
$$\times\ {}_2F_1\left(-s+n, n+\alpha+1; 2n+\alpha+\beta+2; \tfrac{a-b}{a}\right),$$
$$(s \geq n).$$

On the other hand, multivariable analogues of the Jacobi polynomials seem to be highly nontrivial generalizations of the one-variable case. Koornwinder ([14]) introduced two-variable analogues of the Jacobi polynomials in different ways (see also [11]). One of them is a two-variable analogue of the Jacobi polynomials of class II defined by, for $\gamma > -1$ and $n \geq k \geq 0$,

$${}_2P_{n,k}^{\gamma}(x,y) \qquad (10)$$
$$= P_{n-k}^{\left(\gamma+k+\frac{1}{2},\gamma+k+\frac{1}{2}\right)}(x)\left(1-x^2\right)^{k/2} P_k^{(\gamma,\gamma)}\left(\frac{y}{\sqrt{1-x^2}}\right).$$

Observed that these polynomials are orthogonal with respect to the weight function $\left(1-x^2-y^2\right)^\gamma$ on the unit disk and satisfy the following differential equation [20]

$$\left(x^2-1\right)u_{xx} + 2xy\, u_{xy} + \left(y^2-1\right)u_{yy}$$
$$+(2\gamma+3)\left(xu_x + yu_y\right)$$
$$= n(n+2\gamma+2)u. \qquad (11)$$

These functions are of some interest because they occur in many problems of mathematical physics especially in defraction problems. Main results due to Koornwinder are summarized. Let N be the set of pairs of integers $(n,k),\ n \geq k \geq 0$ with lexicographic ordering defined by

$$(m,l) \leq (n,k) \Leftrightarrow m < n \ \vee\ (m = n \wedge l \leq k)$$

and

$$(m,l) = (n,k) \text{ only if } m = n,\ l = k.$$

A polynomial $p(x,y)$ is said to have a degree

$$(n,k) \in \boldsymbol{N} \text{ if } p(x,y) = \sum_{(i,j)\leq(n,k)} C_{i,j}\, x^{i-j}y^j$$

with $C_{n,k} \neq 0$.

In this paper, we introduce a two-variable analogue of the EJPs with the help of (1) and (10). We give a finite series form and hypergeometric representation for these polynomials. Then, we show the orthogonality and the quadratic norm of the polynomials. We obtain some differential formulas for the polynomials $F_{n,k}^{\gamma}(x,y;a,b,c)$ and then, we give recurrence relations for the polynomials $x.F_{n,k}^{\gamma}(x,y;a,b,c)$ and $y.F_{n,k}^{\gamma}(x,y;a,b,c)$.

Furthermore, we derive various families of bilinear and bilateral generating functions for the polynomials $F_{n,k}^{\gamma}(x,y;a,b,c)$.

2. A TWO-VARIABLE ANALOGUE OF EJPs AND THEIR PROPERTIES

With the help of the equality (1) and two-variable analogue of the Jacobi polynomials given by (10), we define a two-variable analogue of the EJPs as follows:

$$F_{n,k}^{\gamma}(x,y;a,b,c) = \left(c(a-b)\right)^{-k} F_{n-k}^{\left(\gamma+k+\frac{1}{2},\gamma+k+\frac{1}{2}\right)}(x;a,b,c)$$
$$\times\left((x-a)(b-x)\right)^{k/2}$$
$$\times F_k^{(\gamma,\gamma)}\left(\frac{a+b}{2} - \frac{(a-b)^2 y}{4\sqrt{(x-a)(b-x)}}; a,b,c\right) \qquad (12)$$

with degree n for $n \geq k \geq 0.$

By (1) and (11), we can give the next theorem.

Theorem 2. 1. The polynomials $F_{n,k}^{\gamma}(x,y;a,b,c)$ satisfy the following differential equation

$$(x-a)(x-b)v_{xx} + y(2x-a-b)\ v_{xy} + \left(y^2-1\right)v_{yy}$$
$$+ (2\gamma+3)\left(\tfrac{2x-a-b}{2}v_x + yv_y\right) = n(n+2\gamma+2)v.$$

The following results can easily be proved by using (2) and (3).

Theorem 2. 2. For the polynomials $F_{n,k}^{\gamma}(x,y;a,b,c)$, we have

(A)

$$F_{n,k}^{\gamma}\left(x,y;a,b,c\right) = \sum_{m=0}^{k}\sum_{l=0}^{n-k} A_{m,l}^{n,k}\left(a,b,c;\gamma\right) \times\left(x-a\right)^{l+\frac{k-m}{2}}\left(b-x\right)^{\frac{k-m}{2}} \times\left(2\sqrt{\left(x-a\right)\left(b-x\right)}+\left(a-b\right)y\right)^{m},$$

where

$$A_{m,l}^{n,k}\left(a,b,c;\gamma\right) = \frac{\left(-1\right)^{m}c^{n-k}\left(a-b\right)^{n-k-l}\left(1+\gamma\right)_{k}\left(\gamma+k+\frac{3}{2}\right)_{n-k}}{2^{2m}m!\left(k-m\right)!l!\left(n-k-l\right)!\left(1+\gamma\right)_{m}\left(\gamma+k+\frac{3}{2}\right)_{l}} \times\frac{\left(2\gamma+2k+2\right)_{n-k+l}\left(1+2\gamma\right)_{m+k}}{\left(1+2\gamma\right)_{k}\left(2\gamma+2k+2\right)_{n-k}};$$

(B)

$$F_{n,k}^{\gamma}\left(x,y;a,b,c\right) = \left\{c\left(a-b\right)\right\}^{n-k}\binom{\gamma+n+\frac{1}{2}}{n-k}\binom{\gamma+k}{k}\left(\left(x-a\right)\left(b-x\right)\right)^{k/2} \times {}_{2}F_{1}\left(k-n,2\gamma+k+n+2;\gamma+k+\frac{3}{2};\frac{x-a}{b-a}\right) \times {}_{2}F_{1}\left(-k,2\gamma+k+1;\gamma+1;\frac{1}{2}-\frac{\left(b-a\right)y}{4\sqrt{\left(x-a\right)\left(b-x\right)}}\right).$$

Now we have the following

Theorem 2.3. A two-variable analogue of the EJPs given by (12) is orthogonal with respect to the weight function

$$\omega(x,y;a,b,\gamma) = \left(1-\left(\tfrac{2x-a-b}{b-a}\right)^{2}-y^{2}\right)^{\gamma}$$

over the domain

$$\Omega:\ \left\{(x,y):\left(\tfrac{2x-a-b}{b-a}\right)^{2}+y^{2}\le 1\right\}.$$

Proof . By (12) and (7), we have

$$\begin{aligned}&\iint_{\Omega} F_{n,k}^{\gamma}(x,y;a,b,c)\,F_{m,l}^{\gamma}(x,y;a,b,c)\,\omega(x,y;a,b,\gamma)\,dxdy\\
&=\{c(a-b)\}^{-k-l}\iint_{\Omega}\Big\{F_{n-k}^{\left(\gamma+k+\frac{1}{2},\gamma+k+\frac{1}{2}\right)}(x;a,b,c)\\
&\times F_{k}^{(\gamma,\gamma)}\left(\frac{a+b}{2}-\frac{(a-b)^{2}y}{4\sqrt{(x-a)(b-x)}};a,b,c\right)((x-a)(b-x))^{\frac{k+l}{2}}\\
&\times F_{m-l}^{\left(\gamma+l+\frac{1}{2},\gamma+l+\frac{1}{2}\right)}(x;a,b,c)F_{l}^{(\gamma,\gamma)}\left(\frac{a+b}{2}-\frac{(a-b)^{2}y}{4\sqrt{(x-a)(b-x)}};a,b,c\right)\\
&\times\left(1-\left(\tfrac{2x-a-b}{b-a}\right)^{2}-y^{2}\right)^{\gamma}\Big\}dxdy\\
&=\frac{4^{2\gamma+1}}{(b-a)^{4\gamma+2}}\{c(a-b)\}^{-k-l}\int_{a}^{b}F_{n-k}^{\left(\gamma+k+\frac{1}{2},\gamma+k+\frac{1}{2}\right)}(x;a,b,c)\\
&\times F_{m-l}^{\left(\gamma+l+\frac{1}{2},\gamma+l+\frac{1}{2}\right)}(x;a,b,c)((x-a)(b-x))^{\frac{k+l+1}{2}+\gamma}dx\\
&\times\int_{a}^{b}F_{k}^{(\gamma,\gamma)}(u;a,b,c)F_{l}^{(\gamma,\gamma)}(u;a,b,c)((u-a)(b-u))^{\gamma}du\\
&=0\end{aligned}$$

for $\left(n,k\right)\neq\left(m,l\right)$. The proof is completed.

Theorem 2.4. The quadratic norm of the polynomials $F_{n,k}^{\gamma}\left(x,y;a,b,c\right)$ is determined as

$$\left\|F_{n,k}^{\gamma}\left(x,y;a,b,c\right)\right\|^{2} = \iint_{\Omega}\left[F_{n,k}^{\gamma}\left(x,y;a,b,c\right)\right]^{2}w\left(x,y;a,b,\gamma\right)dxdy = \frac{2^{4\gamma+1}\left(b-a\right)^{2n+1}c^{2(n-k)}\Gamma^{2}\left(\gamma+n+\frac{3}{2}\right)}{\left(n-k\right)!k!\left(\gamma+n+1\right)\left(2\gamma+2k+1\right)} \times\frac{\Gamma^{2}\left(\gamma+k+1\right)}{\Gamma\left(2\gamma+n+k+2\right)\Gamma\left(2\gamma+k+1\right)}.$$

Proof. It can easily be proved by (12) and (7).

3. RECURRENCE RELATIONS FOR TWO-VARIABLE ANALOGUE OF EJPs

Using the similar technique to ([18]) and considering (1), we have:

$$2n(\alpha+\beta+n)(\alpha+\beta+2n-2)F_n^{(\alpha,\beta)}(x;a,b,c)$$
$$=-2(\alpha+n-1)(\beta+n-1)(\alpha+\beta+2n)$$
$$\times\{c(a-b)\}^2 F_{n-2}^{(\alpha,\beta)}(x;a,b,c) \qquad (13)$$
$$+\left[\alpha^2-\beta^2+\tfrac{(2x-a-b)}{a-b}(\alpha+\beta+2n)(\alpha+\beta+2n-2)\right]$$
$$\times(\alpha+\beta+2n-1)c(a-b)F_{n-1}^{(\alpha,\beta)}(x;a,b,c),$$

$$(\alpha+\beta+2n)(x-a)(b-x)\frac{\partial}{\partial x}F_n^{(\alpha,\beta)}(x;a,b,c)$$
$$=(\alpha+n)(\beta+n)c(a-b)^2F_{n-1}^{(\alpha,\beta)}(x;a,b,c)$$
$$+n(a\alpha+b\beta+n(a+b))F_n^{(\alpha,\beta)}(x;a,b,c)$$
$$-n(\alpha+\beta+2n)xF_n^{(\alpha,\beta)}(x;a,b,c) \qquad (14)$$

and

$$\frac{\partial}{\partial x}F_n^{(\alpha,\beta)}(x;a,b,c) \qquad (15)$$
$$=c(\alpha+\beta+n+1)F_{n-1}^{(\alpha+1,\beta+1)}(x;a,b,c).$$

Theorem3.1. For the polynomials $F_{n,k}^{\gamma}(x,y;a,b,c)$, we get

$$4(n-k)(2\gamma+k+n+1)F_{n,k}^{\gamma}(x,y;a,b,c)$$
$$=4c(\gamma+n)(2x-a-b)(2\gamma+2n+1)F_{n-1,k}^{\gamma}(x,y;a,b,c)$$
$$-(2\gamma+2n-1)(2\gamma+2n+1)\{c(a-b)\}^2F_{n-2,k}^{\gamma}(x,y;a,b,c).$$

Proof. Use (13) and (12).

Theorem 3.2. The polynomials $F_{n,k}^{\gamma}(x,y;a,b,c)$ satisfy the following differential formulas

$$2(x-a)(b-x)\frac{\partial}{\partial x}F_{n,k}^{\gamma}(x,y;a,b,c)$$
$$=\frac{1}{4}(2\gamma+k+1)(-2x+a+b)(a-b)yF_{n-1,k-1}^{\gamma+1}(x,y;a,b,c)$$
$$+c(a-b)^2\left(\gamma+n+\tfrac{1}{2}\right)F_{n-1,k}^{\gamma}(x,y;a,b,c)$$
$$+n(-2x+a+b)F_{n,k}^{\gamma}(x,y;a,b,c),$$

$$\frac{\partial}{\partial y}F_{n,k}^{\gamma}(x,y;a,b,c)=\frac{b-a}{4}(2\gamma+k+1)F_{n-1,k-1}^{\gamma+1}(x,y;a,b,c),$$

$$\frac{\partial^k}{\partial y^k}F_{n,k}^{\gamma}(x,y;a,b,c)$$
$$=\left(\frac{b-a}{4}\right)^k(2\gamma+k+1)_k F_{n-k,0}^{\gamma+k}(x,y;a,b,c)$$

and

$$\frac{\partial^2}{\partial x\partial y}F_{n,k}^{\gamma}(x,y;a,b,c)$$
$$=\frac{(2\gamma+k+1)}{2(x-a)(b-x)}$$
$$\times\left\{\frac{(b-a)(n-1)}{4}(-2x+a+b)F_{n-1,k-1}^{\gamma+1}(x,y;a,b,c)\right.$$
$$-\frac{c(a-b)^3}{4}\left(\gamma+n+\frac{1}{2}\right)F_{n-2,k-1}^{\gamma+1}(x,y;a,b,c)$$
$$-\left(\frac{a-b}{4}\right)^2y(-2x+a+b)$$
$$\left.\times(2\gamma+k+2)F_{n-2,k-2}^{\gamma+2}(x,y;a,b,c)\right\}.$$

Proof. Differentiating (12) with respect to x and y and using the results (14) and (15), we obtain the desired relations.

The next result is a straightforward consequence of Theorem 3.2.

Corollary 3.1. For the polynomials $F_{n,k}^{\gamma}(x,y;a,b,c)$, we have

$$S_1^{\gamma}\left(F_{n,k}^{\gamma}(x,y;a,b,c)\right)=-\frac{c(a-b)^3}{16}(2\gamma+k+1)(2\gamma+2n+1)$$
$$\times F_{n-2,k-1}^{\gamma+1}(x,y;a,b,c)$$

and

$$S_2^{\gamma}\left(F_{n,k}^{\gamma}(x,y;a,b,c)\right)=c(a-b)^2\left(\gamma+n+\frac{1}{2}\right)F_{n-1,k}^{\gamma}(x,y;a,b,c)$$

where

$$S_1^\gamma = (x-a)(b-x)\frac{\partial^2}{\partial x \partial y} - \frac{(n-1)(-2x+a+b)}{2}\frac{\partial}{\partial y} + \frac{(-2x+a+b)y}{2}\frac{\partial^2}{\partial y^2}$$

and

$$S_2^\gamma = 2(x-a)(b-x)\frac{\partial}{\partial x} + (-2x+a+b)y\frac{\partial}{\partial y} - n(-2x+a+b).$$

4. RECURRENCE RELATION FOR THE POLYNOMIAL $x.F_{n,k}^\gamma(x,y;a,b,c)$

The polynomials $x.F_{m,s}^\gamma(x,y;a,b,c)$ have the series expansion ([20])

$$x\,F_{m,s}^\gamma(x,y;a,b,c) = \sum_{(i,j)\le(m+1,s)} c_{i,j}(m,s;\gamma;a,b,c)F_{i,j}^\gamma(x,y;a,b,c), \tag{16}$$

where the coefficients $c_{i,j}(m,s;\gamma;a,b,c)$ are given by

$$\begin{aligned} c_{i,j}(m,s;\gamma;a,b,c) &:= \frac{1}{\left\|F_{i,j}^\gamma(x,y;a,b,c)\right\|^2} \\ &\times \int\int_\Omega x\,F_{m,s}^\gamma(x,y;a,b,c)\; F_{i,j}^\gamma(x,y;a,b,c) \\ &\times w(x,y;a,b,\gamma)\,dxdy \\ &= \frac{2^{4\gamma+2}\left(c(a-b)\right)^{-s-j}}{(b-a)^{4\gamma+2}}\frac{\lambda_1(i,j;a,b,c)\lambda_2(i,j;a,b,c)}{\left\|F_{i,j}^\gamma(x,y;a,b,c)\right\|^2}, \end{aligned} \tag{17}$$

where

$$\lambda_1(i,j;a,b,c) := \int_a^b F_s^{(\gamma,\gamma)}(u;a,b,c)F_j^{(\gamma,\gamma)}(u;a,b,c)(b-u)^\gamma(u-a)^\gamma du$$

and

$$\begin{aligned} \lambda_2(i,j;a,b,c) &:= \int_a^b xF_{m-s}^{\left(\gamma+s+\frac12,\gamma+s+\frac12\right)}(x;a,b,c) \\ &\times F_{i-j}^{\left(\gamma+j+\frac12,\gamma+j+\frac12\right)}(x;a,b,c) \\ &\times \left((b-x)(x-a)\right)^{\frac{s+j+1}{2}+\gamma}dx. \end{aligned}$$

For the non-zero values of $c_{i,j}(m,s;\gamma;a,b,c)$, we have to use $j=s \;\wedge\; m-1\le i\le m+1.$ From (7), we have

$$\lambda_1(i,s;a,b,c) = \frac{c^{2s}(b-a)^{2s+2\gamma+1}\Gamma^2(\gamma+s+1)}{s!(2\gamma+2s+1)\Gamma(2\gamma+s+1)} \tag{18}$$

for every $i\in\{m-1,m,m+1\}$. Using the results (3), (8) and (9), we get

$$\begin{aligned} &\lambda_2(m-1,s;a,b,c) \\ &= \frac{c^{2m-2s-1}(2\gamma+2s+2)_{2(m-s-1)}(b-a)^{2m+2\gamma+2}}{(m-s-1)!(2\gamma+2s+2)_{m-s-1}} \\ &\times \frac{\Gamma^2\left(\gamma+m+\frac32\right)}{\Gamma(2m+2\gamma+3)}, \end{aligned} \tag{19}$$

$$\begin{aligned} &\lambda_2(m,s;a,b,c) \\ &= \frac{c^{2m-2s}(a+b)(b-a)^{2m+2\gamma+2}\Gamma^2\left(\gamma+m+\frac32\right)}{2(m-s)!\Gamma(2m+2\gamma+3)} \\ &\times(2\gamma+m+s+2)_{m-s} \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\lambda_2(m+1,s;a,b,c) \\ &= \frac{c^{2m-2s+1}(2\gamma+2s+2)_{2m-2s}(b-a)^{2m+2\gamma+4}}{(m-s)!(2\gamma+2s+2)_{m-s}} \\ &\times \frac{\Gamma^2\left(\gamma+m+\frac52\right)}{\Gamma(2m+2\gamma+5)}. \end{aligned} \tag{21}$$

The values of the norms are given by

$$\begin{cases} \left\|F^{\gamma}_{m-1,s}(x,y;a,b,c)\right\|^2 \qquad (22) \\ =\dfrac{(b-a)^{2m-1}c^{2(m-s-1)}2^{4\gamma+1}}{(m-s-1)!s!(m+\gamma)(2s+2\gamma+1)} \\ \times\dfrac{\Gamma^2(\gamma+s+1)\Gamma^2(\gamma+m+\frac{1}{2})}{\Gamma(2\gamma+m+s+1)\Gamma(2\gamma+s+1)}, \\ \left\|F^{\gamma}_{m,s}(x,y;a,b,c)\right\|^2 \\ =\dfrac{(b-a)^{2m+1}c^{2(m-s)}2^{4\gamma+1}}{(m-s)!s!(m+\gamma+1)(2s+2\gamma+1)} \\ \times\dfrac{\Gamma^2(\gamma+s+1)\Gamma^2(\gamma+m+\frac{3}{2})}{\Gamma(2\gamma+m+s+2)\Gamma(2\gamma+s+1)}, \\ \left\|F^{\gamma}_{m+1,s}(x,y;a,b,c)\right\|^2 \\ =\dfrac{(b-a)^{2m+3}c^{2m-2s+2}2^{4\gamma+1}}{(m-s+1)!s!(m+\gamma+2)(2s+2\gamma+1)} \\ \times\dfrac{\Gamma^2(\gamma+s+1)\Gamma^2(\gamma+m+\frac{5}{2})}{\Gamma(2\gamma+m+s+3)\Gamma(2\gamma+s+1)}. \end{cases}$$

It follows from (17) - (22) that

$$c_{m-1,s}=\frac{2^{4\gamma+2}(c(a-b))^{-2s}}{(b-a)^{4\gamma+2}}\frac{\lambda_1(m-1,s;a,b,c)\lambda_2(m-1,s;a,b,c)}{\left\|F^{\gamma}_{m-1,s}(x,y;a,b,c)\right\|^2},$$

$$c_{m,s}=\frac{2^{4\gamma+2}(c(a-b))^{-2s}}{(b-a)^{4\gamma+2}}\frac{\lambda_1(m,s;a,b,c)\lambda_2(m,s;a,b,c)}{\left\|F^{\gamma}_{m,s}(x,y;a,b,c)\right\|^2},$$

$$c_{m+1,s}=\frac{2^{4\gamma+2}(c(a-b))^{-2s}}{(b-a)^{4\gamma+2}}\frac{\lambda_1(m+1,s;a,b,c)\lambda_2(m+1,s;a,b,c)}{\left\|F^{\gamma}_{m+1,s}(x,y;a,b,c)\right\|^2}.$$

Therefore, by the series expansion (16), we can give the next theorem for $xF^{\gamma}_{m,s}(x,y;a,b,c)$.

Theorem 4.1. *We have*

$$xF^{\gamma}_{m,s}(x,y;a,b,c)=c_{m-1,s}F^{\gamma}_{m-1,s}(x,y;a,b,c)+c_{m,s}F^{\gamma}_{m,s}(x,y;a,b,c)+c_{m+1,s}F^{\gamma}_{m+1,s}(x,y;a,b,c).$$

This formula may be regarded as corresponding to recurrence relation connecting successively members of the set of orthogonal polynomials in two variables. As a consequence of Theorem 4.1 , we obtain the following result.

Corollary 4.1. For the polynomials $xF^{\gamma}_{m,s}(x,y;a,b,c)$, we have

$$\iint_{\Omega}\left(xF^{\gamma}_{m,s}(x,y;a,b,c)\right)^2 w(x,y;a,b,\gamma)\,dxdy$$
$$=\left(c_{m-1,s}\left\|F^{\gamma}_{m-1,s}(x,y;a,b,c)\right\|\right)^2+\left(c_{m,s}\left\|F^{\gamma}_{m,s}(x,y;a,b,c)\right\|\right)^2+\left(c_{m+1,s}\left\|F^{\gamma}_{m+1,s}(x,y;a,b,c)\right\|\right)^2.$$

5. RECURRENCE RELATION FOR THE POLYNOMIAL $y.F^{\gamma}_{n,k}(x,y;a,b,c)$

$yF^{\gamma}_{m,s}(x,y;a,b,c)$ has the following series expansion ([20])

$$yF^{\gamma}_{m,s}(x,y;a,b,c)=\sum_{(i,j)\le(m+1,s+1)}b_{i,j}(m,s;\gamma;a,b,c)F^{\gamma}_{i,j}(x,y;a,b,c) \qquad (23)$$

where the coefficients $b_{i,j}(m,s;\gamma;a,b,c)$ are given by

$$b_{i,j}(m,s;\gamma;a,b,c):=\frac{1}{\left\|F^{\gamma}_{i,j}(x,y;a,b,c)\right\|^2}\times\iint_{\Omega}yF^{\gamma}_{m,s}(x,y;a,b,c)F^{\gamma}_{i,j}(x,y;a,b,c)\times w(x,y;a,b,\gamma)\,dxdy \qquad (24)$$
$$=\frac{2^{4\gamma+3}(c(a-b))^{-s-j}}{(b-a)^{4\gamma+4}}\times\frac{\xi_1(i,j;a,b,c)\xi_2(i,j;a,b,c)}{\left\|F^{\gamma}_{i,j}(x,y;a,b,c)\right\|^2},$$

where

$$\xi_1(i,j;a,b,c) := \int_a^b (a+b-2u) F_s^{(\gamma,\gamma)}(u;a,b,c) \times F_j^{(\gamma,\gamma)}(u;a,b,c)(b-u)^\gamma (u-a)^\gamma \, du$$

and

$$\xi_2(i,j;a,b,c) := \int_a^b F_{m-s}^{\left(\gamma+s+\frac{1}{2},\gamma+s+\frac{1}{2}\right)}(x;a,b,c) F_{i-j}^{\left(\gamma+j+\frac{1}{2},\gamma+j+\frac{1}{2}\right)}(x;a,b,c) \times ((b-x)(x-a))^{\frac{s+j}{2}+\gamma+1} dx.$$

For the non-zero values of $b_{i,j}(m,s;\gamma;a,b,c),$ we have to use

$$s-1 \le j \le s+1 \ \wedge \ m-1 \le i \le m+1.$$

Using the results (3), (8) and (9), we obtain

$$\xi_1(i,s-1;a,b,c) = -\frac{2c^{2s-1}(b-a)^{2\gamma+2s+1}}{(s-1)!(1+2\gamma)_{s-1}} \times \frac{(1+2\gamma)_{2s-2}\Gamma^2(\gamma+s+1)}{\Gamma(2\gamma+2s+2)}, \qquad (25)$$

$$\xi_1(i,s;a,b,c) = 0 \qquad (26)$$

and

$$\xi_1(i,s+1;a,b,c) = -\frac{2c^{2s+1}(b-a)^{2\gamma+2s+3}(1+2\gamma)_{2s}\Gamma^2(\gamma+s+2)}{s!(1+2\gamma)_s \Gamma(2\gamma+2s+4)}. \qquad (27)$$

(25), (26) and (27) are valid for every i satisfying $m-1 \le i \le m+1.$

Similarly, by (3), (8) and (9), we get

$$\begin{cases} \xi_2(m-1,s-1;a,b,c) = \dfrac{c^{2m-2s}(2\gamma+2s)_{2(m-s)}(b-a)^{2\gamma+2m+2}\Gamma^2\left(\gamma+m+\frac{3}{2}\right)}{(m-s)!(2\gamma+2s)_{m-s}\Gamma(2\gamma+2m+3)}, \\ \xi_2(m,s-1;a,b,c) = 0, \\ \xi_2(m+1,s-1;a,b,c) = -\dfrac{c^{2m-2s+2}(2\gamma+2s)_{2(m-s+1)}(b-a)^{2\gamma+2m+4}}{4(\gamma+m+2)(m-s)!(2\gamma+2s)_{m-s+2}} \times \dfrac{\Gamma\left(\gamma+m+\frac{3}{2}\right)\Gamma\left(\gamma+m+\frac{5}{2}\right)}{\Gamma(2\gamma+2m+3)}, \\ \xi_2(m-1,s+1;a,b,c) = -\dfrac{c^{2m-2s-2}(2\gamma+2s+2)_{2(m-s-1)}(b-a)^{2\gamma+2m+2}}{4(\gamma+m+1)(m-s-2)!(2\gamma+2s+2)_{m-s}} \times \dfrac{\Gamma\left(\gamma+m+\frac{3}{2}\right)\Gamma\left(\gamma+m+\frac{1}{2}\right)}{\Gamma(2\gamma+2m+1)}, \\ \xi_2(m,s+1;a,b,c) = 0, \\ \xi_2(m+1,s+1;a,b,c) = \dfrac{c^{2m-2s}(2\gamma+2s+2)_{2(m-s)}(b-a)^{2\gamma+2m+4}}{(m-s)!(2\gamma+2s+2)_{m-s}} \times \dfrac{\Gamma^2\left(\gamma+m+\frac{5}{2}\right)}{\Gamma(2\gamma+2m+5)}. \end{cases} \qquad (28)$$

Similarly, we can evaluate $\xi_2(m,s;a,b,c)$, $\xi_2(m-1,s;a,b,c)$ and $\xi_2(m+1,s;a,b,c).$

Using the equalities from (24) to (28) , we obtain the following coefficients:

$$b_{m-1,s-1}\left(m,s;\gamma;a,b,c\right) = \frac{2^{4\gamma+3}\left(c\left(a-b\right)\right)^{-2s+1}}{\left(b-a\right)^{4\gamma+4}} \times \frac{\xi_1\left(m-1,s-1;a,b,c\right)\xi_2\left(m-1,s-1;a,b,c\right)}{\left\|F^{\gamma}_{m-1,s-1}\left(x,y;a,b,c\right)\right\|^2},$$

$$b_{m,s-1}\left(m,s;\gamma;a,b,c\right)=0,$$

$$b_{m+1,s-1}\left(m,s;\gamma;a,b,c\right) = \frac{2^{4\gamma+3}\left(c\left(a-b\right)\right)^{-2s+1}}{\left(b-a\right)^{4\gamma+4}} \times \frac{\xi_1\left(m+1,s-1;a,b,c\right)\xi_2\left(m+1,s-1;a,b,c\right)}{\left\|F^{\gamma}_{m+1,s-1}\left(x,y;a,b,c\right)\right\|^2},$$

$$b_{m-1,s+1}\left(m,s;\gamma;a,b,c\right) = \frac{2^{4\gamma+3}\left(c\left(a-b\right)\right)^{-2s-1}}{\left(b-a\right)^{4\gamma+4}} \times \frac{\xi_1\left(m-1,s+1;a,b,c\right)\xi_2\left(m-1,s+1;a,b,c\right)}{\left\|F^{\gamma}_{m-1,s+1}\left(x,y;a,b,c\right)\right\|^2},$$

$$b_{m,s+1}\left(m,s;\gamma;a,b,c\right)=0,$$

$$b_{m+1,s+1}\left(m,s;\gamma;a,b,c\right) = \frac{2^{4\gamma+3}\left(c\left(a-b\right)\right)^{-2s-1}}{\left(b-a\right)^{4\gamma+4}} \times \frac{\xi_1\left(m+1,s+1;a,b,c\right)\xi_2\left(m+1,s+1;a,b,c\right)}{\left\|F^{\gamma}_{m+1,s+1}\left(x,y;a,b,c\right)\right\|^2}.$$

Since $\xi_1\left(i,s;a,b,c\right)=0$ for $i=m-1,m,m+1,$ we have

$$b_{m-1,s}\left(m,s;\gamma;a,b,c\right)=b_{m,s}\left(m,s;\gamma;a,b,c\right) = b_{m+1,s}\left(m,s;\gamma;a,b,c\right)=0.$$

Thus, by (23), we obtain the next theorem.

Theorem 5.1. For the polynomials $y\,F^{\gamma}_{m,s}\left(x,y;a,b,c\right),$ we have the following recurrence relation:

$$\begin{aligned} &y\,F^{\gamma}_{m,s}(x,y;a,b,c) \\ &= b_{m-1,s-1}\,F^{\gamma}_{m-1,s-1}(x,y;a,b,c)+b_{m+1,s-1}\,F^{\gamma}_{m+1,s-1}(x,y;a,b,c) \\ &\quad + b_{m-1,s+1}\,F^{\gamma}_{m-1,s+1}(x,y;a,b,c)+b_{m+1,s+1}\,F^{\gamma}_{m+1,s+1}(x,y;a,b,c). \end{aligned}$$

If $a=-1,\ b=1$ and $c=1/2$ in (12), then we have the polynomials ${}_2P^{\gamma}_{n,k}\left(x,y\right)$ given by (10). For $a=-1,\ b=1$ and $c=1/2,$ the relations given by Theorem 3.1, Theorem 3.2 and Corollary 3.1 are respectively reduced to the following results given by Malave and Bhonsle ([15], [16]).

Corollary 5.1..For the polynomials ${}_2P^{\gamma}_{n,k}\left(x,y\right),$ we have

$$\begin{aligned} &4(n-k)(2\gamma+k+n+1)\ {}_2P^{\gamma}_{n,k}(x,y) \\ &= 4x(\gamma+n)(2\gamma+2n+1)\ {}_2P^{\gamma}_{n-1,k}(x,y) \\ &-(2\gamma+2n-1)(2\gamma+2n+1)\ {}_2P^{\gamma}_{n-2,k}(x,y), \end{aligned}$$

$$\begin{aligned} 2(1-x^2)\frac{\partial}{\partial x}\ {}_2P^{\gamma}_{n,k}(x,y) &= (2\gamma+k+1)\,xy\ {}_2P^{\gamma+1}_{n-1,k-1}(x,y) \\ &+(2\gamma+2n+1)\ {}_2P^{\gamma}_{n-1,k}(x,y)-2nx\ {}_2P^{\gamma}_{n,k}(x,y), \end{aligned}$$

$$\frac{\partial}{\partial y}\ {}_2P^{\gamma}_{n,k}(x,y) = \frac{(2\gamma+k+1)}{2}\ {}_2P^{\gamma+1}_{n-1,k-1}(x,y),$$

$$\frac{\partial^k}{\partial y^k}\ {}_2P^{\gamma}_{n,k}(x,y) = \frac{(2\gamma+k+1)_k}{2^k}\ {}_2P^{\gamma+k}_{n-k,0}(x,y)$$

and

$$\frac{\partial^2}{\partial x \partial y}\ {}_2P_{n,k}^{\gamma}(x,y) = \frac{(2\gamma+k+1)}{4(1-x^2)}\Big\{-2(n-1)x\ {}_2P_{n-1,k-1}^{\gamma+1}(x,y) + (2\gamma+2n+1)\ {}_2P_{n-2,k-1}^{\gamma+1}(x,y) + (2\gamma+k+2)xy\ {}_2P_{n-2,k-2}^{\gamma+2}(x,y)\Big\}.$$

Corollary 5.2. The polynomials ${}_2P_{n,k}^{\gamma}(x,y)$ satisfy

$$\left\{(1-x^2)\frac{\partial^2}{\partial x\partial y} + (n-1)x\frac{\partial}{\partial y} - xy\frac{\partial^2}{\partial y^2}\right\}{}_2P_{n,k}^{\gamma}(x,y) = \frac{(2\gamma+k+1)(2\gamma+2n+1)}{4}\ {}_2P_{n-2,k-1}^{\gamma+1}(x,y)$$

and

$$\left\{2(1-x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y} + 2nx\right\}{}_2P_{n,k}^{\gamma}(x,y) = (2\gamma+2n+1)\ {}_2P_{n-1,k}^{\gamma}(x,y).$$

On the other hand, for $a=-1,\ b=1$ and $c=1/2,$ the relations given by Theorem 4.1 and Theorem 5.1 are reduced to recurrence relations for $x.\ {}_2P_{m,s}^{\gamma}(x,y)$ and $y.\ {}_2P_{m,s}^{\gamma}(x,y)$ given by Malave and Bhonsle ([15], [16]).

6. INTEGRAL REPRESENTATIONS FOR TWO-VARIABLE EJPs $F_{n,k}^{\gamma}(x,y;a,b,c)$

The EJPs $F_n^{(\alpha,\beta)}(x;a,b,c)$ have the following integral representation given by Altın et. al. [6]:

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \frac{\{c(a-b)\}^n}{\Gamma(\alpha+\beta+n+1)}\int_0^\infty t^{\alpha+\beta+n}e^{-t}L_n^{(\alpha)}\left(\frac{x-a}{b-a}t\right)dt,$$
$$\left(\mathrm{Re}(\alpha+\beta)>-1\ ,\ n\in\mathbb{N}_0\right)$$

or, equivalently,

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \frac{\{c(a-b)\}^n}{\Gamma(\alpha+\beta+n+1)} \times\int_0^1\left(\log\tfrac{1}{t}\right)^{\alpha+\beta+n}L_n^{(\alpha)}\left(\tfrac{x-a}{b-a}\log\tfrac{1}{t}\right)dt, \quad (29)$$
$$\left(\mathrm{Re}(\alpha+\beta)>-1\ ,\ n\in\mathbb{N}_0\right)$$

where $L_n^{(\alpha)}$ is the Laguerre polynomial of degree n. As a result of this formula, we obtain the next result.

Theorem 6.1. For the polynomials $F_{n,k}^{\gamma}(x,y;a,b,c),$ we have

$$F_{n,k}^{\gamma}(x,y;a,b,c) = \int_0^\infty\int_0^\infty t^{2\gamma+n+k+1}s^{2\gamma+k}e^{-t-s}L_{n-k}^{\left(\gamma+k+\frac{1}{2}\right)}\left(\frac{x-a}{b-a}t\right) \times L_k^{(\gamma)}\left(\frac{s}{2}-\frac{(b-a)ys}{4\sqrt{(x-a)(b-x)}}\right)dtds \times\frac{\{c(a-b)\}^{n-k}\left((x-a)(b-x)\right)^{k/2}}{\Gamma(2\gamma+n+k+2)\Gamma(2\gamma+k+1)}$$

for $\mathrm{Re}(\gamma)>-\frac{1}{2}$ and integers $n\geq k\geq 0.$

Theorem 6.2. For the polynomials $F_{n,k}^{\gamma}(x,y;a,b,c),$ we have

$$F_{n,k}^{\gamma}(x,y;a,b,c) = \frac{2^{2\gamma+k+n+\frac{1}{2}}\Gamma\left(\gamma+n+\frac{3}{2}\right)\Gamma(\gamma+k+1)}{\pi^2\Gamma(2\gamma+n+k+2)\Gamma(2\gamma+k+1)} \times\frac{\left(c(a-b)\right)^{n-k}}{\Gamma\left(2\gamma+n+k+\frac{3}{2}\right)}\left((x-a)(b-x)\right)^{k/2} \times\int_0^1\int_0^1\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\left(\log\frac{1}{u}\right)^{2\gamma+n+k+1}\left(\log\frac{1}{t}\right)^{2\gamma+k} \times f_{n-k,k}^{\left(\gamma+k+\frac{1}{2},\gamma\right)}(\theta,\varphi)L_n^{\left(2\gamma+k+\frac{1}{2}\right)}\left(\Omega\left\{\left(\frac{x-a}{b-a}\right)\log(1/u),\left(\frac{1}{2}-\tfrac{(b-a)y}{4\sqrt{(x-a)(b-x)}}\right)\log(1/t);\theta,\varphi\right\}\right) \times d\varphi d\theta dudt$$

provided that $\mathrm{Re}(\gamma)>-\frac{1}{2}$ and integers $n\geq k\geq 0,$ and where

$$f_{m,n}^{(\alpha,\beta)}(\theta,\varphi)=e^{\{(m-n)\varphi i+(\alpha-\beta)\theta i\}} \quad (30)$$
$$\times\cos^{m+n}\varphi\cos^{\alpha+\beta}\theta$$

and

$$\Omega(x,y;\theta,\varphi)=\frac{\cos\theta}{\cos\varphi}\left[xe^{(\theta-\varphi)i}+ye^{-(\theta-\varphi)i}\right]. \quad (31)$$

Proof. Carlitz [10] gave an integral representation for the product of two Laguerre polynomials in the form

$$L_m^{(\alpha)}(x)L_n^{(\beta)}(y) \quad (32)$$
$$=\frac{2^{\alpha+\beta+m+n}\Gamma(\alpha+m+1)\Gamma(\beta+n+1)}{\pi^2\Gamma(\alpha+\beta+m+n+1)}$$
$$\times\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}f_{m,n}^{(\alpha,\beta)}(\theta,\varphi)$$
$$\times L_{m+n}^{(\alpha+\beta)}(\Omega\{x,y;\theta,\varphi\})d\varphi d\theta,$$
$$\left(\mathrm{Re}(\alpha+\beta)>-1 \quad , \quad m,\ n\in\mathbb{N}_0\right)$$

where $f_{m,n}^{(\alpha,\beta)}(\theta,\varphi)$ and $\Omega(x,y;\theta,\varphi)$ are defined by (30) and (31), respectively. In (32), we replace β by γ, n by k, y by $\left(\frac{y-a}{b-a}\right)\log(1/t)$, and multiply all of them by $(\log(1/t))^{2\gamma+k}$, and also integrate each side with respect to t over the interval $(0,1)$. As a result of (29), we obtain

$$L_m^{(\alpha)}(x)F_k^{(\gamma,\gamma)}(y;a,b,c)$$
$$=\frac{(c(a-b))^k 2^{\alpha+\gamma+m+k}\Gamma(\alpha+m+1)\Gamma(\gamma+k+1)}{\pi^2\Gamma(2\gamma+k+1)\Gamma(\alpha+\gamma+m+k+1)}$$
$$\times\int_0^1\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}(\log(1/t))^{2\gamma+k}f_{m,k}^{(\alpha,\gamma)}(\theta,\phi)$$
$$\times L_{m+k}^{(\alpha+\gamma)}\left(\Omega\left\{x,\left(\frac{y-a}{b-a}\right)\log(1/t);\theta,\phi\right\}\right)d\phi d\theta dt,$$

where $\mathrm{Re}(\gamma)>-1/2,\quad \mathrm{Re}(\alpha+\gamma)>-1,$ $m,\ k\in\mathbb{N}_0$. A further application of (29) to this formula gives the following integral representation

$$F_{n-k}^{(\gamma+k+\frac{1}{2},\gamma+k+\frac{1}{2})}(x;a,b,c)F_k^{(\gamma,\gamma)}(y;a,b,c)$$
$$=\frac{(c(a-b))^n 2^{2\gamma+n+k+\frac{1}{2}}\Gamma(\gamma+n+\frac{3}{2})}{\pi^2\Gamma(2\gamma+k+1)\Gamma(2\gamma+n+k+2)}$$
$$\times\frac{\Gamma(\gamma+k+1)}{\Gamma(2\gamma+n+k+\frac{3}{2})}$$
$$\times\int_0^1\int_0^1\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}(\log(1/t))^{2\gamma+k}(\log(1/u))^{2\gamma+n+k+1}$$
$$\times f_{n-k,k}^{(\gamma+k+\frac{1}{2},\gamma)}(\theta,\varphi)L_n^{(2\gamma+k+\frac{1}{2})}\left(\Omega\left\{\left(\frac{x-a}{b-a}\right)\log(1/u),\right.\right.$$
$$\left.\left.\left(\frac{y-a}{b-a}\right)\log(1/t);\theta,\varphi\right\}\right)d\varphi d\theta dt du$$

provided that $\mathrm{Re}(\gamma)>-1/2$, $n\geq k\geq 0.$ If we replace y by $\frac{a+b}{2}-\frac{(b-a)^2 y}{4\sqrt{(x-a)(b-x)}}$ in the last equality and use (12), we obtain the desired integral representation for $F_{n,k}^{\gamma}(x,y;a,b,c)$.

Theorem 6.3. For the polynomials $F_{n,k}^{\gamma}(x,y;a,b,c)$, we have

$$F_{n,k}^{\gamma}(x,y;a,b,c)$$
$$=\frac{2^{2\gamma+k+n+\frac{1}{2}}\Gamma(\gamma+n+\frac{3}{2})\Gamma(\gamma+k+1)}{\pi^2 n!\Gamma(2\gamma+k+\frac{3}{2})}$$
$$\times(c(a-b))^{n-k}((x-a)(b-x))^{k/2}$$
$$\times\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}f_{n-k,k}^{(\gamma+k+\frac{1}{2},\gamma)}(\theta,\varphi)F_1\left[-n,2\gamma+n+k+2,2\gamma+k+1;\ 2\gamma+k+\frac{3}{2};\right.$$
$$\left.\frac{x-a}{b-a}\frac{\cos\theta}{\cos\varphi}e^{(\theta-\varphi)i},\left(\frac{1}{2}-\frac{(b-a)y}{4\sqrt{(x-a)(b-x)}}\right)\frac{\cos\theta}{\cos\varphi}e^{-(\theta-\varphi)i}\right]d\varphi d\theta$$

which holds true for integers $n\geq k\geq 0,$ provided that $\mathrm{Re}(\gamma)>-\frac{1}{2}$ and where $f_{m,n}^{(\alpha,\gamma)}(\theta,\varphi)$ is given by (30) and F_1 is the first kind of Appell's double hypergeometric function.

Proof. By ([19, p.168, (7)]) and (1), the integral representation in above proof can be rewritten in the form

$$F_{n-k}^{\left(\gamma+k+\frac{1}{2},\gamma+k+\frac{1}{2}\right)}(x;a,b,c)L_k^{(\gamma)}(y)$$
$$= \frac{(c(a-b))^{n-k}2^{2\gamma+n+k+\frac{1}{2}}\Gamma\left(\gamma+n+\frac{3}{2}\right)\Gamma(\gamma+k+1)}{\pi^2 n!\Gamma\left(2\gamma+k+\frac{3}{2}\right)}$$
$$\times\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} f_{n-k,k}^{\left(\gamma+k+\frac{1}{2},\gamma\right)}(\theta,\phi)\,\Phi_1\Big[-n,2\gamma+n+k+2;2\gamma+k+\frac{3}{2};$$
$$\frac{x-a}{b-a}\frac{\cos\theta}{\cos\phi}e^{(\theta-\phi)i},y\frac{\cos\theta}{\cos\phi}e^{-(\theta-\phi)i}\Big]d\phi d\theta,$$

where Φ_1 denotes one of the Humbert's confluent hypergeometric functions of two variables given by (see [22])

$$\Phi_1[\alpha,\beta;\gamma;x,y] = \sum_{r,s=0}^{\infty}\frac{(\alpha)_{r+s}(\beta)_r}{(\gamma)_{r+s}}\frac{x^r}{r!}\frac{y^s}{s!},\ \ |x|<1.$$

Applying (29) to this formula and making necessary arrangements, we conclude that

$$F_{n-k}^{\left(\gamma+k+\frac{1}{2},\gamma+k+\frac{1}{2}\right)}(x;a,b,c)F_k^{(\gamma,\gamma)}\left(\frac{a+b}{2}-\frac{(b-a)^2y}{4\sqrt{(x-a)(b-x)}};a,b,c\right)$$
$$=\frac{\left(c(a-b)\right)^n 2^{2\gamma+n+k+\frac{1}{2}}\Gamma\left(\gamma+n+\frac{3}{2}\right)\Gamma\left(\gamma+k+1\right)}{\pi^2 n!\Gamma\left(2\gamma+k+\frac{3}{2}\right)}$$
$$\times\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} f_{n-k,k}^{\left(\gamma+k+\frac{1}{2},\gamma\right)}(\theta,\varphi)$$
$$\times F_1\Big[-n,2\gamma+n+k+2,2\gamma+k+1;2\gamma+k+\frac{3}{2};$$
$$\frac{x-a}{b-a}\frac{\cos\theta}{\cos\varphi}e^{(\theta-\varphi)i},\left(\frac{1}{2}-\frac{(b-a)y}{4\sqrt{(x-a)(b-x)}}\right)\frac{\cos\theta}{\cos\varphi}e^{-(\theta-\varphi)i}\Big]$$
$$\times d\varphi d\theta$$

where $\mathrm{Re}(\gamma)>-1/2$, $n\geq k\geq 0.$ From this formula and (12), the proof follows.

7. GENERATING FUNCTIONS FOR TWO-VARIABLE EJPs $F_{n,k}^{\gamma}(x,y;a,b,c)$

Fujiwara [13] shows that the EJPs $F_n^{(\alpha,\beta)}(x;a,b,c)$ are generated by

$$\sum_{n=0}^{\infty}F_n^{(\alpha,\beta)}(x;a,b,c)t^n \tag{33}$$
$$=\frac{2^{\alpha+\beta}}{\rho}(1+\delta t+\rho)^{-\alpha}(1-\delta t+\rho)^{-\beta},$$

where

$$\rho := \left\{1+2tX'(x)+\delta^2t^2\right\}^{1/2},\quad X(x) := c(x-a)(b-x),$$
$$\delta := c(b-a).$$

By using the idea in [22] and (1), we can obtain the following generating functions for EJPs, respectively:

$$\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n}{(1+\alpha)_n}\left(c(a-b)\right)^{-n}F_n^{(\alpha,\beta)}(x;a,b,c)t^n$$
$$=(1-t)^{-1-\alpha-\beta} \tag{34}$$
$$\times{}_2F_1\left[\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta+2}{2};1+\alpha;\frac{4t(x-a)}{(a-b)(1-t)^2}\right],$$
$$(|t|<1)$$

$$\sum_{n=0}^{\infty}\binom{m+n}{n}\left(c(a-b)\right)^{-n}F_{m+n}^{(\alpha-n,\beta-n)}(x;a,b,c)t^n$$
$$=\left(1+\frac{(x-b)}{a-b}t\right)^{\alpha}\left(1+\frac{(x-a)}{a-b}t\right)^{\beta} \tag{35}$$
$$\times F_m^{(\alpha,\beta)}\left(x+\frac{t(x-a)(x-b)}{a-b};a,b,c\right),$$
$$\left(|t|<\min\left\{\left|\frac{a-b}{x-a}\right|,\left|\frac{a-b}{x-b}\right|\right\}\right),$$

for every non-negative integer m.

Theorem 7.1. For the two-variable analogue of EJPs $F_{n,k}^{\gamma}(x,y;a,b,c)$, we have

$$\sum_{n,k=0}^{\infty}\frac{(\lambda)_k}{(\mu)_k}F_{n+k,k}^{\gamma-k}(x,y;a,b,c)t^{n+k}$$
$$=\frac{2^{2\gamma+1}}{\rho}\left((1+\rho)^2-\delta^2t^2\right)^{-\gamma-\frac{1}{2}}$$
$$\times F_1\left[\lambda,-\gamma,-\gamma;\mu;\tfrac{(a-b)ty}{4}-\tfrac{t\sqrt{(x-a)(b-x)}}{2},\tfrac{(a-b)ty}{4}+\tfrac{t\sqrt{(x-a)(b-x)}}{2}\right]$$

where ρ , $X(x)$ and δ are mentioned above.

Proof. By (5) and (33), we may write

$$\sum_{n,k=0}^{\infty}\frac{(\lambda)_k}{(\mu)_k}F_{n+k,k}^{\gamma-k}(x,y;a,b,c)t^{n+k}$$
$$=\sum_{n=0}^{\infty}F_n^{\left(\gamma+\frac{1}{2},\gamma+\frac{1}{2}\right)}(x;a,b,c)t^n\sum_{k=0}^{\infty}\frac{(\lambda)_k}{(\mu)_k}\left(c(a-b)\right)^{-k}$$
$$\times F_k^{(\gamma-k,\gamma-k)}\left(\tfrac{a+b}{2}-\tfrac{(a-b)^2y}{4\sqrt{(x-a)(b-x)}};a,b,c\right)$$
$$\times\left(t\sqrt{(x-a)(b-x)}\right)^k$$
$$=\frac{2^{2\gamma+1}}{\rho}\left((1+\rho)^2-\delta^2t^2\right)^{-\gamma-\frac{1}{2}}$$
$$\times F_1\left[\lambda,-\gamma,-\gamma;\mu;\tfrac{(a-b)ty}{4}-\tfrac{t\sqrt{(x-a)(b-x)}}{2},\tfrac{(a-b)ty}{4}+\tfrac{t\sqrt{(x-a)(b-x)}}{2}\right]$$

The proof is completed.

Similarly, the next results can be easily obtained from (4), (5), (6) and (34), immediately.

Theorem 7.2. The polynomials $F_{n,k}^{\gamma}(x,y;a,b,c)$ are generated by

$$\sum_{n,k=0}^{\infty}\frac{(\sigma)_n(\delta)_n(\lambda)_k}{(\mu)_k\left(\gamma+\frac{3}{2}\right)_n^2}\left(c(a-b)\right)^{-n}$$
$$\times F_{n+k,k}^{\gamma-k}(x,y;a,b,c)t^{n+k}$$
$$=F_4\left[\sigma,\delta;\gamma+\tfrac{3}{2},\gamma+\tfrac{3}{2};\tfrac{(x-a)}{a-b}t,\tfrac{(x-b)}{a-b}t\right]$$
$$\times F_1\left[\lambda,-\gamma,-\gamma;\mu;\tfrac{(a-b)ty}{4}-\tfrac{t\sqrt{(x-a)(b-x)}}{2},\tfrac{(a-b)ty}{4}+\tfrac{t\sqrt{(x-a)(b-x)}}{2}\right].$$

Theorem 7.3. For the polynomials $F_{n,k}^{\gamma}(x,y;a,b,c)$, we have

$$\sum_{n,k=0}^{\infty}\frac{(2\gamma+2)_n}{\left(\gamma+\frac{3}{2}\right)_n}\left(c(a-b)\right)^{-n}F_{n+k,k}^{\gamma-k}(x,y;a,b,c)t^ns^k$$
$$=(1-t)^{-2\gamma-2}\,{}_2F_1\left(\gamma+1,\gamma+\frac{3}{2};\gamma+\frac{3}{2};\frac{4t(x-a)}{(a-b)(1-t)^2}\right)$$
$$\times\left(1-\tfrac{(a-b)ys}{4}+\tfrac{s\sqrt{(x-a)(b-x)}}{2}\right)^{\gamma}$$
$$\times\left(1-\tfrac{(a-b)ys}{4}-\tfrac{s\sqrt{(x-a)(b-x)}}{2}\right)^{\gamma}$$

where $|t|<1.$

Theorem 7.4. The polynomials $F_{n,k}^{\gamma}(x,y;a,b,c)$ are generated by

(A)

$$\sum_{n,p=0}^{\infty}\binom{m+p}{p}\left(c(a-b)\right)^mF_{n+m+p,m+p}^{\gamma-p}(x,y;a,b,c)$$
$$\times\left((x-a)(b-x)\right)^{-\frac{m+p}{2}}t^{n+p}$$

$$=\frac{2^{2\gamma+2m+1}}{\rho}(1+\delta t+\rho)^{-\gamma-m-\frac{1}{2}}(1-\delta t+\rho)^{-\gamma-m-\frac{1}{2}}$$

$$\times\left(1+\frac{t}{2}-\frac{(a-b)yt}{4\sqrt{(x-a)(b-x)}}\right)^{\gamma}$$

$$\times\left(1-\frac{t}{2}-\frac{(a-b)yt}{4\sqrt{(x-a)(b-x)}}\right)^{\gamma}$$

$$\times F_m^{(\gamma,\gamma)}\left(u+\frac{t(u-a)(u-b)}{a-b};a,b,c\right),$$

(B)

$$\sum_{n,p=0}^{\infty}\binom{m+p}{p}\frac{(\sigma)_n(\delta)_n}{\left(\gamma+\frac{3}{2}\right)_n^2}(c(a-b))^{m-n}$$

$$\times F^{\gamma-m-p}_{n+m+p,m+p}\left(x,y;a,b,c\right)$$
$$\times\left(\left(x-a\right)\left(b-x\right)\right)^{-\frac{m+p}{2}}\ t^n s^p$$

$$=F_4\left[\sigma,\delta;\gamma+\frac{3}{2},\gamma+\frac{3}{2};\frac{\left(x-a\right)}{a-b}t,\frac{\left(x-b\right)}{a-b}t\right]$$
$$\times\left(1+\frac{s}{2}-\frac{\left(a-b\right)ys}{4\sqrt{\left(x-a\right)\left(b-x\right)}}\right)^{\gamma-m}$$
$$\times\left(1-\frac{s}{2}-\frac{\left(a-b\right)ys}{4\sqrt{\left(x-a\right)\left(b-x\right)}}\right)^{\gamma-m}$$
$$\times F_m^{(\gamma-m,\gamma-m)}\left(u+\frac{s\left(u-a\right)\left(u-b\right)}{a-b};a,b,c\right),$$

where ρ , $X\left(x\right)$, δ are stated before and

$u=\frac{a+b}{2}-\frac{(a-b)^2y}{4\sqrt{(x-a)(b-x)}}$.

Proof . (A) Use (33) and (35).

(B) Taking $\alpha\to\gamma-m,\quad\beta\to\gamma-m$ in equality (35) and using (6) and (35), we get

$$\sum_{n,p=0}^{\infty}\binom{m+p}{p}\frac{(\sigma)_n(\delta)_n}{\left(\gamma+\frac{3}{2}\right)_n^2}\left(c\left(a-b\right)\right)^{m-n}$$
$$\times F^{\gamma-m-p}_{n+m+p,m+p}\left(x,y;a,b,c\right)$$
$$\times\left(\left(x-a\right)\left(b-x\right)\right)^{-\frac{m+p}{2}}\ t^n s^p$$
$$=\sum_{n=0}^{\infty}\frac{(\sigma)_n(\delta)_n}{\left(\gamma+\frac{3}{2}\right)_n^2}\left(c\left(a-b\right)\right)^{-n}$$
$$\times F_n^{\left(\gamma+\frac{1}{2},\gamma+\frac{1}{2}\right)}\left(x;a,b,c\right)t^n$$
$$\times\sum_{p=0}^{\infty}\binom{m+p}{p}\left(c\left(a-b\right)\right)^{-p}$$
$$\times F_{m+p}^{(\gamma-m-p,\gamma-m-p)}\left(\frac{a+b}{2}-\frac{(a-b)^2y}{4\sqrt{(x-a)(b-x)}};a,b,c\right)\ s^p$$

$$=F_4\left[\sigma,\delta;\gamma+\tfrac{3}{2},\gamma+\tfrac{3}{2};\tfrac{(x-a)}{a-b}t,\tfrac{(x-b)}{a-b}t\right]$$
$$\times\left(1+\tfrac{s}{2}-\tfrac{(a-b)ys}{4\sqrt{(x-a)(b-x)}}\right)^{\gamma-m}$$
$$\times\left(1-\tfrac{s}{2}-\tfrac{(a-b)ys}{4\sqrt{(x-a)(b-x)}}\right)^{\gamma-m}$$
$$\times F_m^{(\gamma-m,\gamma-m)}\left(u+\frac{s\left(u-a\right)\left(u-b\right)}{a-b};a,b,c\right)$$

where $u=\frac{a+b}{2}-\frac{(a-b)^2y}{4\sqrt{(x-a)(b-x)}}$. The proof is completed.

If we get $a=-1,\ b=1$ and $c=1/2$ in the generating functions given above, we have various generating functions for the polynomials ${}_2P_{n,k}^{\gamma}\left(x,y\right)$ given by (10). For $a=-1,\ b=1$ and $c=1/2,$ some generating functions given above are reduced to the following equalities given by Malave and Bhonsle [17]. From Theorem 7.1 and 7.2, we have the following

Corollary 7.1. For the polynomials ${}_2P_{n,k}^{\gamma}\left(x,y\right),$ the following generating functions are satisfied:

$$\sum_{n,k=0}^{\infty}\frac{(\lambda)_k}{(\mu)_k}\ {}_2P^{\gamma-k}_{n+k,k}\left(x,y\right)t^{n+k}$$
$$=\frac{2^{2\gamma+1}}{\rho}\left(\left(1+\rho\right)^2-t^2\right)^{-\gamma-\frac{1}{2}}$$
$$\times F_1\left[\lambda,-\gamma,-\gamma;\mu;-\tfrac{ty}{2}-\tfrac{t\sqrt{1-x^2}}{2},\right.$$
$$\left.-\tfrac{ty}{2}+\tfrac{t\sqrt{1-x^2}}{2}\right]$$

where

$$\rho=\left\{1-2xt+t^2\right\}^{1/2}$$

and

$$\sum_{n,k=0}^{\infty}\frac{(\sigma)_n(\delta)_n(\lambda)_k}{(\mu)_k\left(\gamma+\frac{3}{2}\right)_n^2}P^{\gamma-k}_{n+k,k}\left(x,y;a,b,c\right)t^{n+k}$$
$$=F_4\left[\sigma,\delta;\gamma+\tfrac{3}{2},\gamma+\tfrac{3}{2};\tfrac{(x-1)}{2}t,\tfrac{(x+1)}{2}t\right]$$
$$\times F_1\left[\lambda,-\gamma,-\gamma;\mu;-\tfrac{ty}{2}-\tfrac{t\sqrt{1-x^2}}{2},\right.$$
$$\left.-\tfrac{ty}{2}+\tfrac{t\sqrt{1-x^2}}{2}\right].$$

8. BILINEAR AND BILATERAL GENERATING FUNCTIONS

In this section, we derive several families of bilinear and bilateral generating functions for the two-variable analogue of EJPs $F_{n,k}^{\gamma}\left(x,y;a,b,c\right)$ given by (12).

We begin by stating the following theorem.

Theorem 8.1. Corresponding to an identically non-vanishing function $\Omega_{\mu}(y_1,...,y_s)$ of s complex variables $y_1,...,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let

$$\Lambda_{\mu,\nu}(y_1,...,y_s;z) \qquad (36)$$
$$:=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,...,y_s)z^k$$

$$(a_k\neq 0,\quad \mu,\nu\in\mathbb{C}).$$

Then we have

$$\sum_{n_1,n_2=0}^{\infty}\sum_{k=0}^{[n_1/p]}\frac{a_k(\lambda)_{n_2}}{(\mu)_{n_2}}F_{n_1+n_2-pk,n_2}^{\gamma-n_2}\left(x_1,x_2;a,b,c\right) \qquad (37)$$
$$\times\Omega_{\mu+\nu k}(y_1,...,y_s)\eta^k t^{n_1+n_2-pk}$$
$$=\Lambda_{\mu,\nu}(y_1,...,y_s;\eta)\,\frac{2^{2\gamma+1}}{\rho}\left(\left(1+\rho\right)^2-\delta^2t^2\right)^{-\gamma-\frac{1}{2}}$$
$$\times F_1\left[\lambda,-\gamma,-\gamma;\mu;\frac{(a-b)tx_2}{4}-\frac{t\sqrt{(x_1-a)(b-x_1)}}{2},\right.$$
$$\left.\frac{(a-b)tx_2}{4}+\frac{t\sqrt{(x_1-a)(b-x_1)}}{2}\right]$$

provided that each member of (37) exists. Where

$$\rho=\left\{1+2tX'(x_1)+\delta^2t^2\right\}^{1/2},\ X(x_1)=c(x_1-a)(b-x_1)$$
$$\delta=c(b-a).$$

Proof. For convenience, let S denote the first member of the assertion (37) of Theorem 8.1. Straightforward calculations give

$$S=\sum_{n_1,n_2=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_k(\lambda)_{n_2}}{(\mu)_{n_2}}F_{n_1+n_2,n_2}^{\gamma-n_2}(x_1,x_2;a,b,c)$$
$$\times\Omega_{\mu+\nu k}(y_1,\ldots,y_s)\eta^k t^{n_1+n_2}$$
$$=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,\ldots,y_s)\eta^k$$
$$\times\sum_{n_1,n_2=0}^{\infty}\frac{(\lambda)_{n_2}}{(\mu)_{n_2}}F_{n_1+n_2,n_2}^{\gamma-n_2}(x_1,x_2;a,b,c)\,t^{n_1+n_2}.$$

If we use Theorem 7.1, then the proof of Theorem 8.1 is completed.

Theorem 8.2. Corresponding to an identically non-vanishing function $\Phi_{\mu}(y_1,y_2)$ of complex variables y_1,y_2 and of complex order μ, let

$$\Xi_{\mu,\nu_1,\nu_2}(y_1,y_2;z,w) \qquad (38)$$
$$:=\sum_{k_1,k_2=0}^{\infty}b_{k_1,k_2}\,\Phi_{k_1\nu_1+k_2\nu_2+\mu,k_2}(y_1,y_2)z^{k_1}w^{k_2}$$
$$(b_{k_1,k_2}\neq 0\ ,\ \mu,\nu_1,\nu_2\in\mathbb{C}).$$

and

$$\Theta_{n_1,n_2,p,l}^{\mu,\nu_1,\nu_2}(x_1,x_2;y_1,y_2;\zeta,\xi) \qquad (39)$$
$$:=\sum_{k_1=0}^{[n_1/p]}\sum_{k_2=0}^{[n_2/l]}\frac{\left(2\gamma+2\right)_{n_1-pk_1}b_{k_1,k_2}}{\left(\gamma+\frac{3}{2}\right)_{n_1-pk_1}}\left(c\left(a-b\right)\right)^{-n_1+pk_1}$$
$$\times F_{n_1+n_2-(pk_1+lk_2),n_2-lk_2}^{\gamma-n_2+lk_2}\left(x_1,x_2;a,b,c\right)$$
$$\times\Phi_{k_1\nu_1+k_2\nu_2+\mu,k_2}(y_1,y_2)\,\zeta^{k_1}\xi^{k_2}$$

where $n_1,n_2,p,l\in\mathbb{N}$. Then, we have

$$\sum_{n_1,n_2=0}^{\infty}\Theta_{n_1,n_2,p,l}^{\mu,\nu_1,\nu_2}\left(x_1,x_2;y_1,y_2;\frac{\eta}{t^p},\frac{\upsilon}{s^l}\right)t^{n_1}s^{n_2} \quad (40)$$

$$=\Xi_{\mu,\nu_1,\nu_2}(y_1,y_2;\eta,\upsilon)\left(1-t\right)^{-2\gamma-2}$$
$$\times\,{}_2F_1\left(\gamma+1,\gamma+\frac{3}{2};\gamma+\frac{3}{2};\frac{4t\left(x_1-a\right)}{(a-b)(1-t)^2}\right)$$
$$\times\left(1-\tfrac{(a-b)x_2s}{4}+\tfrac{s\sqrt{(x_1-a)(b-x_1)}}{2}\right)^{\gamma}$$
$$\times\left(1-\tfrac{(a-b)x_2s}{4}-\tfrac{s\sqrt{(x_1-a)(b-x_1)}}{2}\right)^{\gamma}$$

provided that each member of (40) exists.

Proof. For convenience, let S denote the first member of the assertion (40) of Theorem 8.2. Then, upon substituting for the polynomials

$$\Theta_{n_1,n_2,p,l}^{\mu,\nu_1,\nu_2}\left(x_1,x_2;y_1,y_2;\frac{\eta}{t^p},\frac{\upsilon}{s^l}\right)$$

from the definition (39) into the left-hand side of (40), we obtain

$$S=\sum_{n_1,n_2=0}^{\infty}\sum_{k_1=0}^{[n_1/p]}\sum_{k_2=0}^{[n_2/l]}\frac{(2\gamma+2)_{n_1-pk_1}\,b_{k_1,k_2}}{\left(\gamma+\frac{3}{2}\right)_{n_1-pk_1}} \quad (41)$$
$$\times\left(c\left(a-b\right)\right)^{-n_1+pk_1}$$
$$\times F_{n_1+n_2-(pk_1+lk_2),n_2-lk_2}^{\gamma-n_2+lk_2}\left(x_1,x_2;a,b,c\right)$$
$$\times\Phi_{k_1\nu_1+k_2\nu_2+\mu,k_2}(y_1,y_2)$$
$$\times\eta^{k_1}\upsilon^{k_2}t^{n_1-pk_1}s^{n_2-k_2l}.$$

Upon inverting the order of summation in (41), if we replace n_1 by n_1+pk_1 and n_2 by n_2+lk_2 and then we use Theorem 7.3, we can write

$$S=\sum_{n_1,n_2=0}^{\infty}\sum_{k_1,k_2=0}^{\infty}\frac{(2\gamma+2)_{n_1}\,b_{k_1,k_2}}{\left(\gamma+\frac{3}{2}\right)_{n_1}}$$
$$\times\left(c\left(a-b\right)\right)^{-n_1}F_{n_1+n_2,n_2}^{\gamma-n_2}\left(x_1,x_2;a,b,c\right)$$
$$\times\Phi_{k_1\nu_1+k_2\nu_2+\mu,k_2}(y_1,y_2)\eta^{k_1}\upsilon^{k_2}t^{n_1}s^{n_2}$$
$$=\sum_{n_1,n_2=0}^{\infty}\frac{(2\gamma+2)_{n_1}}{\left(\gamma+\frac{3}{2}\right)_{n_1}}\left(c\left(a-b\right)\right)^{-n_1}$$
$$\times F_{n_1+n_2,n_2}^{\gamma-n_2}\left(x_1,x_2;a,b,c\right)t^{n_1}s^{n_2}$$
$$\times\sum_{k_1,k_2=0}^{\infty}b_{k_1,k_2}\,\Phi_{k_1\nu_1+k_2\nu_2+\mu,k_2}(y_1,y_2)\eta^{k_1}\upsilon^{k_2}$$
$$=\Xi_{\mu,\nu_1,\nu_2}(y_1,y_2;\eta,\upsilon)\left(1-t\right)^{-2\gamma-2}$$
$$\times{}_2F_1\left(\gamma+1,\gamma+\frac{3}{2};\gamma+\frac{3}{2};\frac{4t\left(x_1-a\right)}{(a-b)(1-t)^2}\right)$$
$$\times\left(1-\tfrac{(a-b)x_2s}{4}+\tfrac{s\sqrt{(x_1-a)(b-x_1)}}{2}\right)^{\gamma}$$
$$\times\left(1-\tfrac{(a-b)x_2s}{4}-\tfrac{s\sqrt{(x_1-a)(b-x_1)}}{2}\right)^{\gamma},$$

which completes the proof.

By expressing the multivariable function

$$\Omega_{\mu+\nu k}(y_1,...,y_s)\ (k\in\mathbb{N}_0,\ s\in\mathbb{N})$$

in terms of simpler function of one and more variables, we can give further applications of Theorem 8.1. For example, if we set

$$s=r \text{ and } \Omega_{\mu+\nu k}(y_1,\ldots,y_r)=h_{\mu+\nu k}^{(\alpha_1,\ldots,\alpha_r)}(y_1,\ldots,y_r)$$

in Theorem 8.1, where a multivariable extension of the Lagrange-Hermite polynomials

$$h_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)$$

are defined by means of the generating function ([7])

$$\prod_{j=1}^{r}\left\{(1-x_jt^j)^{-\alpha_j}\right\} \quad (42)$$
$$=\sum_{n=0}^{\infty}h_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,...,x_r)t^n,$$

$$\left(\alpha_j \in \mathbb{C}\ (j=1,...,r)\ ;\ |t| < \min_{j\in\{1,...,r\}}\left\{\left|x_j\right|^{-1/j}\right\}\right),$$

then we obtain the following result which provides a class of bilateral generating functions for the multivariable Lagrange-Hermite polynomials and for the polynomials $F_{n,k}^{\gamma}\left(x,y;a,b,c\right)$ defined by (12).

Corollary 8.1. If

$$\Lambda_{\mu,\nu}(y_1,...,y_r;z) := \sum_{k=0}^{\infty} a_k h_{\mu+\nu k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)z^k$$

where $a_k \neq 0,\ \nu,\mu\in\mathbb{C};$ then we have

$$\sum_{n_1,n_2=0}^{\infty}\sum_{k=0}^{[n_1/p]}\frac{a_k\left(\lambda\right)_{n_2}}{\left(\mu\right)_{n_2}}F_{n_1+n_2-pk,n_2}^{\gamma-n_2}\left(x_1,x_2;a,b,c\right) \times h_{\mu+\nu k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)\eta^k t^{n_1+n_2-pk}$$
$$=\Lambda_{\mu,\nu}(y_1,...,y_r;\eta)\,\frac{2^{2\gamma+1}}{\rho}\left(\left(1+\rho\right)^2-\delta^2t^2\right)^{-\gamma-\frac{1}{2}} \times F_1\left[\lambda,-\gamma,-\gamma;\mu;\tfrac{(a-b)tx_2}{4}-\tfrac{t\sqrt{(x_1-a)(b-x_1)}}{2},\ \tfrac{(a-b)tx_2}{4}+\tfrac{t\sqrt{(x_1-a)(b-x_1)}}{2}\right] \qquad (43)$$

provided that each member of (43) exists.

Remark 8.1. Using the generating relation (42) for the multivariable Lagrange-Hermite polynomials and taking $a_k=1,\ \mu=0,\ \nu=1,$ we have

$$\sum_{n_1,n_2=0}^{\infty}\sum_{k=0}^{[n_1/p]}\frac{\left(\lambda\right)_{n_2}}{\left(\mu\right)_{n_2}}F_{n_1+n_2-pk,n_2}^{\gamma-n_2}\left(x_1,x_2;a,b,c\right) \times h_k^{(\gamma_1,...,\gamma_r)}(y_1,...,y_r)\eta^k t^{n_1+n_2-pk}$$
$$=\prod_{j=1}^{r}\left\{\left(1-y_j\eta^j\right)^{-\gamma_j}\right\}\frac{2^{2\gamma+1}}{\rho}\left(\left(1+\rho\right)^2-\delta^2t^2\right)^{-\gamma-\frac{1}{2}} \times F_1\left[\lambda,-\gamma,-\gamma;\mu;\tfrac{(a-b)tx_2}{4}-\tfrac{t\sqrt{(x_1-a)(b-x_1)}}{2},\ \tfrac{(a-b)tx_2}{4}+\tfrac{t\sqrt{(x_1-a)(b-x_1)}}{2}\right]$$

where

$$|\eta| < \min\left\{|y_1|^{-1},\ldots,|y_r|^{-1/r}\right\}.$$

Choosing $\Phi_{k_1\nu_1+k_2\nu_2+\mu,k_2}(y_1,y_2)=F_{k_1\nu_1+k_2\nu_2+\mu,k_2}^{\alpha-k_2}(y_1,y_2),$ $(\mu,\nu_1,\nu_2\in\mathbb{N}_0),$ in Theorem 8.2, we obtain the following class of bilinear generating function for the polynomials $F_{n,k}^{\gamma}\left(x,y;a,b,c\right).$

Corollary 8.2. If

$$\Xi_{\mu,\nu_1,\nu_2}(y_1,y_2;z,w) := \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2}\,F_{k_1\nu_1+k_2\nu_2+\mu,k_2}^{\alpha-k_2}(y_1,y_2)z^{k_1}w^{k_2}$$

$$(b_{k_1,k_2}\neq 0\ ,\ \mu,\nu_1,\nu_2\in\mathbb{N}_0).$$

and

$$\Theta_{n_1,n_2,p,l}^{\mu,\nu_1,\nu_2}(x_1,x_2;y_1,y_2;\zeta,\xi) := \sum_{k_1=0}^{[n_1/p]}\sum_{k_2=0}^{[n_2/l]}\frac{\left(2\gamma+2\right)_{n_1-pk_1}b_{k_1,k_2}}{\left(\gamma+\frac{3}{2}\right)_{n_1-pk_1}}\left(c\left(a-b\right)\right)^{-n_1+pk_1} \times F_{n_1+n_2-(pk_1+lk_2),n_2-lk_2}^{\gamma-n_2+lk_2}\left(x_1,x_2;a,b,c\right) \times F_{k_1\nu_1+k_2\nu_2+\mu,k_2}^{\alpha-k_2}(y_1,y_2)\,\zeta^{k_1}\xi^{k_2}$$

where $n_1,n_2,p,l\in\mathbb{N}$. Then, we have

$$\sum_{n_1,n_2=0}^{\infty}\Theta_{n_1,n_2,p,l}^{\mu,\nu_1,\nu_2}\left(x_1,x_2;y_1,y_2;\frac{\eta}{t^p},\frac{\upsilon}{s^l}\right)t^{n_1}s^{n_2}$$
$$=\Xi_{\mu,\nu_1,\nu_2}(y_1,y_2;\eta,\upsilon)\left(1-t\right)^{-2\gamma-2} \times {}_2F_1\left(\gamma+1,\gamma+\frac{3}{2};\gamma+\frac{3}{2};\frac{4t\left(x_1-a\right)}{\left(a-b\right)\left(1-t\right)^2}\right) \times\left(1-\tfrac{(a-b)x_2s}{4}+\tfrac{s\sqrt{(x_1-a)(b-x_1)}}{2}\right)^{\gamma} \times\left(1-\tfrac{(a-b)x_2s}{4}-\tfrac{s\sqrt{(x_1-a)(b-x_1)}}{2}\right)^{\gamma}.$$

Remark 8.2. Using Theorem 7.3 and taking $b_{k_1,k_2}=\frac{(2\alpha+2)_{k_1}\left(c(a-b)\right)^{-k_1}}{\left(\alpha+\frac{3}{2}\right)_{k_1}},\ \mu=0,\ \nu_1=1,\nu_2=1,$ we have

$$\sum_{n_1,n_2=0}^{\infty}\sum_{k_1=0}^{[n_1/p]}\sum_{k_2=0}^{[n_2/l]}\frac{(2\alpha+2)_{k_1}(2\gamma+2)_{n_1-pk_1}}{\left(\alpha+\frac{3}{2}\right)_{k_1}\left(\gamma+\frac{3}{2}\right)_{n_1-pk_1}}$$
$$\times\left(c(a-b)\right)^{-n_1+pk_1-k_1}$$
$$\times F^{\gamma-n_2+lk_2}_{n_1+n_2-(pk_1+lk_2),n_2-lk_2}(x_1,x_2;a,b,c)$$
$$\times F^{\alpha-k_2}_{k_1+k_2,k_2}(y_1,y_2)\,\eta^{k_1}\upsilon^{k_2}t^{n_1-pk_1}s^{n_2-k_2l}$$
$$=(1-t)^{-2\gamma-2}\;(1-\eta)^{-2\alpha-2}$$
$$\times{}_2F_1\left(\gamma+1,\gamma+\frac{3}{2};\gamma+\frac{3}{2};\frac{4t(x_1-a)}{(a-b)(1-t)^2}\right)$$
$$\times\left(1-\frac{(a-b)x_2s}{4}+\frac{s\sqrt{(x_1-a)(b-x_1)}}{2}\right)^{\gamma}$$
$$\times\left(1-\frac{(a-b)x_2s}{4}-\frac{s\sqrt{(x_1-a)(b-x_1)}}{2}\right)^{\gamma}$$
$$\times{}_2F_1\left(\alpha+1,\alpha+\frac{3}{2};\alpha+\frac{3}{2};\frac{4\eta(y_1-a)}{(a-b)(1-\eta)^2}\right)$$
$$\times\left(1-\frac{(a-b)y_2\upsilon}{4}+\frac{\upsilon\sqrt{(y_1-a)(b-y_1)}}{2}\right)^{\alpha}$$
$$\times\left(1-\frac{(a-b)y_2\upsilon}{4}-\frac{\upsilon\sqrt{(y_1-a)(b-y_1)}}{2}\right)^{\alpha}.$$

Furthermore, for every suitable choice of the coefficients $a_k\ (k\in\mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_s),\quad (s\in\mathbb{N})$, is expressed as an appropriate product of several simpler functions, the assertions of Theorems 8.1 and 8.2 can be applied in order to derive various families of multilinear and multilateral generating functions for the polynomials $F^{\gamma}_{n,k}(x,y;a,b,c)$.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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