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Durrmeyer-Type Generalization of Mittag-Leffler Operators

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ABSTRACT

In this paper, we study Mittag-Leffler operators. We establish moments of these operators and estimate convergence results with the help of classical modulus of continuity. Also we give their A-statistical convergence property.
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Key words: Mittag-Leffler operators, A-statistical convergence, Gamma function, Beta function.

1. INTRODUCTION

In 1903, G.M. Mittag-Leffler [1] defined the Mittag-Leffler function by

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\cong(\alpha k+1)};(z\epsilon\mathbb{C},R(\alpha)>0).$$

In 1905, A. Wiman [2] gave the definition of two-index Mittag-Leffler function by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\cong(\alpha k+\beta)};(z,\beta\epsilon\mathbb{C},\qquad R(\alpha)>0).$$

Note that $E_{\alpha,1}(z)=E_\alpha(z)$.

M.A. Özarslan [12] investigated properties of the following Mittag-Leffler operators

$$L_n^{(\beta)}(f;x)=\frac{1}{E_{1,\beta}(\frac{nx}{b_n})}\sum_{k=0}^{\infty}f(\frac{kb_n}{n})\frac{(nx)^k}{b_n^k\cong(k+\beta)},\qquad(1.1)$$

Where $b_n$is a sequence of positive real numbers, $\beta>0$ is fixed, $n\epsilon\mathbb{N}$, $C[0,\infty)$ denotes the space of continuous functions defined on $[0,\infty)$, and

$$f\epsilon E:=\left\{f\epsilon C[0,\infty):\lim_{x\to\infty}\frac{|f(x)|}{1+x^2}\ is\ finite\right\}.$$

Recall that the Banach lattice E has the norm

$$\|f\|_*=sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}.$$

It is obvious that operators $L_n^{(\beta)}$ given by (1.1) are linear and positive. Furthermore, $L_n^{(1)}(f;x)=S_n(f;x)$, which are modified Szász-Mirakjan operators, can be easily seen.

Ozarslan [12] gave the following inequalities:

$$L_n^{(\beta)}(1;x)=1, \quad (1.2)$$

$$\left|L_n^{(\beta)}(t;x)-x\right|\leq\frac{|1-\beta|b_n}{n}, \quad (1.3)$$

$$\left|L_n^{(\beta)}(t^2;x)-x^2\right|\leq\frac{(2+|1-\beta|)b_n}{n}x$$
$$+\frac{(2|1-\beta|^2+|1-\beta|+|1-\beta||\beta-2|)b_n{}^2}{n^2}, \quad (1.4)$$

$$L_n^{(\beta)}((t-x)^2;x)\leq\frac{(4|1-\beta|+1)b_n}{n}x$$
$$+\frac{(2|1-\beta|^2+|1-\beta|+|1-\beta||\beta-2|)b_n{}^2}{n^2}. \quad (1.5)$$

Now, we define the operators $D_n^{(\beta)}$ by

$$D_n^{(\beta)}(f;x)=\sum_{k=0}^{\infty}\frac{(nx)^k}{E_{1,\beta}\left(\frac{nx}{b_n}\right)b_n^k\cong(k+\beta)}$$

$$\times\int_0^\infty\frac{t^k}{(1+t)^{n+k+1}}\frac{f(\frac{(n-1)b_nt}{n})}{B(n,k+1)}dt, \quad (1.6)$$

Where $f\epsilon C[0,\infty)$, $x\epsilon[0,\infty)$, b_n is a sequence of positive real numbers, $\beta>0$ is fixed, $n\in\mathbb{N}$ and $B(.,.)$.is the Beta function.

Lemma 1. For each $x\geq 0$ and $n\in\mathbb{N}$, we have

$$D_n^{(\beta)}(1;x)=1, \quad (1.7)$$

$$\left|D_n^{(\beta)}(t;x)-x\right|\leq\frac{(|1-\beta|+1)b_n}{n}x, \quad (1.8)$$

$$\left|D_n^{(\beta)}(t^2;x)-x^2\right|\leq\frac{x^2}{n-2}+(\frac{n-1}{n-2})\frac{2(2+|1-\beta|)b_n}{n}x$$
$$+(\frac{n-1}{n-2})\frac{(2|1-\beta|^2+|1-\beta|+|1-\beta||\beta-2|)b_n{}^2}{n^2}, \quad (1.9)$$

$$D_n^{(\beta)}((t-x)^2;x)\leq\frac{x^2}{n-2}$$
$$+\frac{2b_n}{n}\left[\left(\frac{n-1}{n-2}\right)(2+|1-\beta|)+|1-\beta|+1\right]x$$
$$+\left(\frac{n-1}{n-2}\right)\frac{(2|1-\beta|^2+|1-\beta|+|1-\beta||\beta-2|)b_n{}^2}{n^2} \quad (1.10)$$

Proof From definition of the two-index Mittag-Leffler function, we see that

$$D_n^{(\beta)}(1;x)=1.$$

From the operators $D_n^{(\beta)}$ given by (1.6), for n>1 we get

$$D_n^{(\beta)}(t;x)=\sum_{k=0}^{\infty}\frac{(nx)^k}{E_{1,\beta}\left(\frac{nx}{b_n}\right)b_n^k\cong(k+\beta)}$$

$$\times\frac{(n-1)b_n}{n}\frac{B(n-1,k+2)}{B(n,k+1)}$$

$$=\sum_{k=0}^{\infty}\frac{(nx)^k}{E_{1,\beta}\left(\frac{nx}{b_n}\right)b_n^k\cong(k+\beta)}(\frac{k+1}{n}b_n)$$

$$=L_n^{(\beta)}(t;x)+\frac{b_n}{n}L_n^{(\beta)}(1;x).$$

Using (1.2) and (1.3), we have

$$\left|D_n^{(\beta)}(t;x)-x\right|\leq\left|L_n^{(\beta)}(t;x)-x\right|+\frac{b_n}{n}$$

$$\leq\frac{(|1-\beta|+1)b_n}{n}.$$

From (1.6), for $n>2$ we get

$$D_n^{(\beta)}(t^2;x)=\sum_{k=0}^{\infty}\frac{(nx)^k}{E_{1,\beta}\left(\frac{nx}{b_n}\right)b_n^k\cong(k+\beta)}$$

$$\times\left[\frac{(n-1)b_n}{n}\right]^2\frac{B(n-2,k+3)}{B(n,k+1)}$$

$$=\sum_{k=0}^{\infty}\frac{(nx)^k}{E_{1,\beta}\left(\frac{nx}{b_n}\right)b_n^k\cong(k+\beta)}\frac{n-1}{n-2}\left(\frac{b_n}{n}\right)^2(k^2+3k$$
$$+2)$$

$$=\left(\frac{n-1}{n-2}\right)\left[L_n^{(\beta)}(t^2;x)+\frac{3b_n}{n}L_n^{(\beta)}(t;x)\right.$$
$$\left.+\frac{2b_n{}^2}{n^2}L_n^{(\beta)}(1;x)\right].$$

Using (1.2)-(1.4), we have

$$\left|D_n^{(\beta)}(t^2;x)-x^2\right|\leq\left(\frac{n-1}{n-2}\right)\left|L_n^{(\beta)}(t^2;x)-x^2\right|$$
$$+\frac{3b_n}{n}\left(\frac{n-1}{n-2}\right)\left|L_n^{(\beta)}(t;x)-x\right|$$
$$+\frac{x^2}{n-2}+\frac{3b_n}{n}\left(\frac{n-1}{n-2}\right)x$$
$$+\frac{2b_n{}^2}{n^2}\left(\frac{n-1}{n-2}\right)$$

$$\leq\left(\frac{n-1}{n-2}\right)\left[\frac{(2|1-\beta|+1)b_n}{n}x\right.$$
$$\left.+\frac{(2(1-\beta)^2+|1-\beta|+|1-\beta||\beta-2|)b_n^2}{n^2}\right]$$
$$+\frac{3b_n}{n}\left(\frac{n-1}{n-2}\right)\frac{|1-\beta|b_n}{n}+\frac{x^2}{n-2}+\frac{3b_n}{n}\left(\frac{n-1}{n-2}\right)x$$
$$+\frac{2b_n{}^2}{n^2}\left(\frac{n-1}{n-2}\right).$$

If we simplify the above inequality, then we obtain

$$\left|D_n^{(\beta)}(t^2;x)-x^2\right| \leq \frac{x^2}{n-2} + \left(\frac{n-1}{n-2}\right)\frac{(2|1-\beta|+4)b_n}{n}x + \frac{{b_n}^2}{n^2}\left(\frac{n-1}{n-2}\right)(2(1-\beta)^2 + 4|1-\beta| + |1-\beta||\beta-2| + 2)\,.$$

On the other hand, we can give the second moment as

$$D_n^{(\beta)}((t-x)^2;x) \leq \left|D_n^{(\beta)}(t^2;x)-x^2\right| + 2x\left|D_n^{(\beta)}(t;x)-x\right|$$

$$\leq \frac{2b_n}{n}\left[\left(\frac{n-1}{n-2}\right)(|1-\beta|+2)+|1-\beta|+1\right]x$$

$$+\frac{x^2}{n-2}+\frac{{b_n}^2}{n^2}\left(\frac{n-1}{n-2}\right)(2(1-\beta)^2+4|1-\beta| + |1-\beta||\beta-2|+2).$$

2. RATE OF CONVERGENCE

We start with the following lemma, which proves that $D_n^{(\beta)}$ maps E into itself.

Lemma 2. Let $\left(\frac{b_n}{n}\right)$ be a bounded sequence of positive numbers and $\beta > 0$ be fixed. Then there exists a constant $M(\beta)$ such that, for $\omega(x) = \frac{1}{1+x^2}$, we have

$$\omega(x)D_n^{(\beta)}\left(\frac{1}{\omega};x\right) \leq M(\beta)$$

holds for all $x \in [0,\infty)$ and $n \in \mathbb{N}$. Furthermore, for all $f \in E$, we have

$$\left\|D_n^{(\beta)}(f)\right\|_* \leq M(\beta)\|f\|_*.$$

Proof. From (1.7) and (1.9), we have

$$\omega(x)D_n^{(\beta)}\left(\frac{1}{\omega};x\right) = \frac{1}{1+x^2}\left[D_n^{(\beta)}(1;x)+D_n^{(\beta)}(t^2;x)\right]$$

$$\leq \frac{1}{1+x^2}\left\{1 + \frac{n-1}{n-2}\left[\frac{(2|1-\beta|+1)b_n}{n}x + \frac{3{b_n}^2}{n^2}|1-\beta| + x^2 + \frac{3b_n}{n}x + \frac{2{b_n}^2}{n^2} + \frac{(2|1-\beta|^2+|1-\beta|+|1-\beta||\beta-2|)b_n{}^2}{n^2}\right]\right\}$$

$$\leq 1 + \frac{n-1}{n-2}\left[\frac{(2|1-\beta|+1)b_n}{2n} + \frac{3{b_n}^2}{n^2}|1-\beta| + 1 + \frac{3b_n}{2n} + \frac{2{b_n}^2}{n^2} + \frac{(2|1-\beta|^2+|1-\beta|+|1-\beta||\beta-2|)b_n{}^2}{n^2}\right]$$

$$= \frac{2n-3}{n-2} + \frac{n-1}{n-2}\frac{b_n}{n}(|1-\beta|+2) + \frac{n-1}{n-2}\frac{{b_n}^2}{n^2}(2|1-\beta|^2+4|1-\beta| + |1-\beta||\beta-2|+2) = M(\beta).$$

On the other hand, we obtain

$$\omega(x)\left|D_n^{(\beta)}(f;x)\right| \leq \omega(x)\left|D_n^{(\beta)}\left(\omega\frac{f}{\omega};x\right)\right| \leq \|f\|_*\omega(x)D_n^{(\beta)}\left(\frac{1}{\omega};x\right) \leq M(\beta)\|f\|_*.$$

Taking supremum on both sides of above inequality, we easily prove the results.

Recall that the usual modulus of continuity of f on the closed interval $[0,B]$ is defined by

$$\omega_B(f,\delta) = sup\{|f(t)-f(x)|: x,t \in [0,B], |t-x| \leq \delta\}.$$

It is well known that, for a function $f \in C[0,B]$, we have $\lim_{\delta\to 0}\omega_B(f,\delta)=0$.

Now, we acquire the rate of convergence of the operators $D_n^{(\beta)}f$ to f, for all $f \in C[0,B]$.

Theorem 1. Let $\beta > 0$ be fixed, $\left(\frac{b_n}{n}\right)$ be a bounded sequence of positive numbers with $\frac{b_n}{n} \to 0$ as $n \to \infty$, $f \in C[0,B]$, and $\omega_{B+1}(f,\delta)$ $(B>0)$ be modulus of continuity of f on the finite interval $[0,B+1] \subset [0,\infty)$. Then

$$\left\|D_n^{(\beta)}(f;x)-f(x)\right\|_{C[0,B]} \leq M_f(\beta,B)\delta_n^2(\beta,B) + 2\omega_{B+1}\big(f,\delta_n(\beta,B)\big)$$

where

$$\delta_n(\beta,B) = \left\{\frac{B^2}{n-2} + \frac{2b_n}{n}\left[\left(\frac{n-1}{n-2}\right)(|1-\beta| + 2) + |1-\beta|+1\right]B + \frac{{b_n}^2}{n^2}\left(\frac{n-1}{n-2}\right)[2|1-\beta|^2+4|1-\beta| + |1-\beta||\beta-2|]+2\right\}^{1/2}$$

and $M_f(\beta,B)$ is an absolute constant depending on $f,\ \beta$ and B.

Proof. Let $\beta>0$ be fixed. For $x\in[0,B]$ and $t\le B+1$, we have

$$|f(t)-f(x)|\le\omega_{B+1}(f,|t-x|)\le\left(1+\frac{|t-x|}{\delta}\right)\omega_{B+1}(f,\delta)$$

where $\delta>0$. On the other hand, for $x\in[0,B]$ and $t>B+1$, we get $t-x>1$, we can write

$$|f(t)-f(x)|\le A_f(1+x^2+t^2)\le A_f(2+3x^2+2(t-x)^2)\le 6A_f(1+B^2)(t-x)^2.$$

By the above two inequlities, for $x\in[0,B]$ and $t\ge 0$, we get

$$|f(t)-f(x)|\le 6A_f(1+B^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{B+1}(f,\delta).$$

Using Cauchy-Schwarz inequality and (1.10), we have

$$\left|D_n^{(\beta)}(f;x)-f(x)\right|\le 6A_f(1+B^2)D_n^{(\beta)}((t-x)^2;x)+\omega_{B+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{D_n^{(\beta)}((t-x)^2;x)}\right)$$

$$\le M_f(\beta,B)\delta_n^2(\beta,B)+2\omega_{B+1}\big(f,\delta_n(\beta,B)\big),$$

where

$$\delta_n(\beta,B)=\left\{\frac{B^2}{n-2}+\frac{2b_n}{n}\left[\left(\frac{n-1}{n-2}\right)(|1-\beta|+2)+|1-\beta|+1\right]B+\frac{{b_n}^2}{n^2}\left(\frac{n-1}{n-2}\right)[2|1-\beta|^2+4|1-\beta|+|1-\beta||\beta-2|]+2\right\}^{1/2}$$

and $M_f(\beta,B)=6(1+B^2)A_f$. So, we have the desired results.

3. A-STATISTICAL CONVERGENCE

Recently, some authors deal with A-statistically convergence of linear positive operators $[8,10,11]$.

We recall concepts of A-statistical convergence. Let $A=(a_{jk})$ be a non-negative regular summability matrix. The A-density of a subset K of $\mathbb{N}$ is given by

$$\delta_A(K)=\lim_j\sum_{k\in K}a_{j,k},$$

provided that limit exists $(see\ [5])$. A sequence $x=(x_n)$ is said to be A-statistically convergent to l and denoted by $st_A-lim x=l$ if for every $\varepsilon>0$,

$$\lim_{k\to\infty}\sum_{n:|x_k-l\ge\varepsilon|}a_{k,n}=0$$

or $\delta_A\{n\in\mathbb{N}:|x_k-l|\ge\varepsilon\}=0$ $(see\ [4,9])$.

In the special case of $A=C_1$, Cesáro matrix of order one, the A-statistically convergence reduces to statistical convergence $[3,7]$. If we choose $A=I$, the identity matrix, then A-statistically convergence reduces to ordinary convergence. Kolk $[6]$ proved that in the case of $\lim_j\max_n|a_{j,n}|=0$, A-statistical convergence is stronger than ordinary convergence.

Assuming that $(b_n)_{n\in\mathbb{N}}$ is a sequence satisfying

$$st_A-\lim_n\frac{b_n}{n}=0.$$

Then we see

$$st_A-\lim_n\frac{{b_n}^2}{n^2}=0.$$

For an example, take $A=C_1$ and define

$$b_n:=\begin{cases}n: & n=m^2\ \ (m\in\mathbb{N})\\ n^{\frac{1}{3}}: & otherwise\end{cases}.\qquad(3.1)$$

Then we easily see that $st_A-\lim_n\frac{b_n}{n}=st_A-\lim_n\frac{{b_n}^2}{n^2}=0$ (see [12]).

Theorem 2. Let $A=(a_{jk})$ be a non-negative regular summability matrix and $\beta>0$ be fixed. If $st_A-\lim_n\frac{b_n}{n}=0$, then

$$st_A-\left\|D_n^{(\beta)}(f;x)-f(x)\right\|_{C[0,B]}=0$$

holds for every $f\in E$.

Proof. Given $r>0$ choose $\varepsilon>0$ such that $\varepsilon<r$. For fixed $\beta>0$, define the following sets:

$$\mathrm{U}:=\{\mathrm{n}:\delta_n(\beta,B)\ge \mathrm{r}\},$$

$$U_1:=\left\{\mathrm{n}:\frac{\beta^2}{n-2}\ge\frac{\mathrm{r}-\varepsilon}{3}\right\},$$

$$U_2 := \left\{ \mathrm{n}: 2\mathrm{B}\frac{b_n}{n}\left[(|1-\beta|+2)\frac{\mathrm{n}-1}{\mathrm{n}-2}+|1-\beta|+1\right] \geq \frac{\mathrm{r}-\varepsilon}{3} \right\},$$

$$U_3 := \left\{ \mathrm{n}: \left(\frac{n-1}{n-2}\right)[2(1-\beta)^2+4|1-\beta| + |1-\beta||\beta-2|+2]\frac{{b_n}^2}{n^2} \geq \frac{\mathrm{r}-\varepsilon}{3} \right\}.$$

Then $U \subset U_1 \cup U_2 \cup U_3$ can be seen. So, we can get

$$\sum_{k\in U} a_{jk} \leq \sum_{k\in U_1} a_{jk} + \sum_{k\in U_2} a_{jk} + \sum_{k\in U_3} a_{jk}.$$

For $j \to \infty$ in the above inequality and $st_A - \lim_n \frac{b_n}{n} = 0$, we have $\lim_j \sum_{k\in U} a_{jk} = 0$.

So this shows that $st_A - \lim_n \delta_n(\beta, B) = 0$ which implies

$$st_A - \lim_n \omega_{B+1}\big(f, \delta_n(\beta, B)\big) = 0$$

due to the right continuity of $\omega_{B+1}(f,.)$ at zero. Using the previous theorem, we get the desired result.

Remark 1. Note that choosing the sequence $(b_n)_{n\in\mathbb{N}}$ as in (3.1), the statistical approximation results in the last theorem works, but its classical case does not work since $\left(\frac{b_n}{n}\right)_{n\in\mathbb{N}}$ is not convergent in the ordinary sense.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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