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TRIPLE POSITIVE SOLUTIONS FOR A NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we investgate the existence of three positive solutions of a nonlinear fractional differential equations with multi-point and multi-strip boundary conditions. The existence result is obtained by using the Leggett-Williams fixed point theorem. An example is also given to illustrate our main results.

1. INTRODUCTION

Differential equations with fractional derivative have been used to model problems in many fields of science and technology as the mathematical modeling of systems, processes in the fields of physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, finance, etc. (see[3, 11, 15, 16, 17, 21, 25, 26, 28, 31, 36] and the references therein).

Several definitions of fractional derivative have been presented to the literature, amongst are; Riemann-Liouville, Caputo and Grunwald-Letnikov definitions, Atangana-Baleanu operator [4], Liouville-Caputo [22], Caputo-Fabrizio [9], the conformable derivative [18].

Many authors have studied the existence and the multiplicity of solutions of fractional boundary value problems by different approaches. We refer the reader to ([2, 5, 6, 10, 12]). Furthermore, the research in numerical approximations and analytical techniques for the solution of different boundary value problems for time-fractional equation has attracted by ([28, 34, 35, 37]).

Fractional-order multipoint or integral boundary value problems constitute a very interesting and important class of problems. They have been research topics from several authors ([1, 7, 13, 23, 29, 30, 32, 33]). It is worth mentioning that, in 2012, Cabada and Wang [8] investigate the existence of positive solutions of the following nonlinear fractional differential equations with integral boundary value

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conditions:

$$\begin{cases} {}^{C}D^{\alpha}u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, & u(1) = \lambda \int_0^1 u(s)\,ds, \end{cases} \tag{1.1}$$

where $2 < \alpha < 3$, $0 < \lambda < 2$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative and $f : [0,1] \times [0,\infty) \to [0,\infty)$ by using the Guo–Krasnoselskii fixed point theorem.

In 2014, Zhou and Jiang [38] studied the existence of positive solutions of the following problem:

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u'(0) - \beta u(\xi) = 0, \ u'(1) + \sum_{i=1}^{m-3} \gamma_i u(\eta_i) = 0, \end{cases} \tag{1.2}$$

where α is a real number with $1 < \alpha \le 2$, $0 \le \beta \le 1$, $0 \le \gamma_i \le 1$, $i = 1, 2, ..., m-3$, $0 \le \xi < \eta_1 < \eta_2 < ... < \eta_{m-3} \le 1$, and D_{0+}^{α} denotes the Caputo's derivative. They used the fixed point index theory and Krein-Rutman theorem.

In 2016, Guo et al.[14] investigate the existence of at least three positive solutions to the problem

$$\begin{cases} {}^{C}D_{0+}^{\alpha}u(t) + f(t, u(t), u'(t)) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, \ u'(1) = \sum_{j=1}^{\infty} \eta_j u \xi_j, \end{cases}$$

where $2 < \alpha \le 3$, $\eta_j \ge 0$, $0 < \xi_1 < \xi_2 ... < \xi_{j-1} < \xi_j < ... < 1$ $(j = 1, 2, ...)$ and ${}^{C}D_{0+}^{\alpha}$ is the standard Caputo derivative. They applying the Avery–Peterson's fixed point theorem to obtain the existence of multiple positive solutions .

Motivated and inspired by the works mentioned above, we are concerned with the existence of multiple positive solutions of the following nonlinear fractional differential equations with multi-stip conditions

$$\begin{cases} {}^{C}D_{0+}^{\alpha}u(t) + h(t) f(t, u(t)) = 0, & t \in (0,1), \\ u^{(i)}(0) = 0, \ i = 2, ..., n-1, \\ u'(0) = \sum_{i=1}^{m-2} b_i u'(\eta_i), \ u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)\,ds, \end{cases} \tag{1.3}$$

where ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivatives, $n-1 < \alpha \le n$, $n \ge 3$ is an integer. Using the Leggett-Williams fixed point theorem, we provide sufficient conditions for the existence of multiple (at least three) positive solutions for the above boundary value problems.

In the remainder, we assume the following conditions:

(H_1) $0 = \eta_0 < \eta_1 < \eta_2 ... < \eta_{m-2} < 1$, $a_i \ge 0$, $b_i \ge 0$, $(i = 1, ..., m-2)$, $0 \le \sum_{i=1}^{m-2} b_i < 1$ and $0 \le \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1}) < 1$, where $m > 2$ is an integer;

(H_2) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous;

(H_3) $h : (0,1) \to [0,+\infty)$ is continuous, and $h(t)$ does not identically vanish on any subinterval of $(0,1)$. Furthermore h satisfies $0 < \int_0^1 h(t)\,dt < +\infty$.

2. PRELIMINARIES

For the reader's convenience, we present some necessary definitions and relations for fractional-order derivatives and integrals, which can be found in [22, 28].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : (0,+\infty) \to \mathbb{R}$ is defined as*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$

provided the right side is pointwise defined on $(0,+\infty)$ *where* $\Gamma(\cdot)$ *is the Gamma function.*

Definition 2.2. *For a function* $f:[0,+\infty)\to\mathbb{R}$*, the Caputo derivative of fractional order* α *is defined as*

$$^{C}D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)\,ds,\quad n=[\alpha]+1,$$

where $[\alpha]$ *denotes the integer part of the real number* α*, provided the right side is pointwise defined on* $(0,+\infty)$.

Lemma 2.1. *Let* $\alpha>0$ *and* $u\in AC^{N}[0,1]$*. Then the fractional differential equation*

$$^{C}D^{\alpha}u(t)=0,$$

has a unique solution

$$u(t)=c_0+c_1t+c_2t^2+...+c_{N-1}t^{N-1},\quad c_i\in\mathbb{R},\ i=1,2,...,N,$$

where N *is the smallest integer greater than or equal to* α.

Remark 1. *The following property (Dirichlet's formula) of the fractional calculus is well known ([26] p.57)*

$$I^{\nu}I^{\mu}y(t)=I^{\nu+\mu}y(t),\quad t\in[0,1],\ y\in L(0,1),\ \nu+\mu\geq 1,$$

which has the form

$$\int_0^t (t-s)^{\nu-1}\left(\int_0^s (s-\tau)^{\mu-1}y(\tau)\,d\tau\right)ds=\frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}\int_0^t (t-s)^{\nu+\mu-1}y(s)\,ds$$

Definition 2.3. *Let* E *be a real Banach space. A nonempty convex closed set* $K\subset E$ *is said to be a cone provided that*
(i) $au\in K$ *for all* $u\in K$ *and all* $a\geq 0$*, and*
(ii) $u,-u\in K$ *implies* $u=0$.

Definition 2.4. *The map* α *is defined as a nonnegative continuous concave functional on a cone* K *of a real Banach space* E *provided that* $\alpha:K\to[0,+\infty)$ *is continuous and*

$$\alpha(tx+(1-t)y)\geq t\alpha(x)+(1-t)\alpha(y)$$

for all $x,y\in K$ *and* $0\leq t\leq 1$.

Let $0<a<b$ be given and let α be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\alpha,a,b)$ by

$$P_r=\{x\in K\mid \|x\|<r\}$$

and

$$P(\alpha,a,b)=\{x\in K\mid a\leq\alpha(x),\|x\|\leq b\}.$$

Theorem 2.2. [19] *Let* $A:\overline{P_c}\to\overline{P_c}$ *be a completely continuous operator and let* α *be a nonnegative continuous concave functional on* K *such that* $\alpha(x)\leq\|x\|$ *for all* $x\in\overline{P_c}$*. Suppose there exist* $0<a<b<d<c$ *such that*
(C_1) $\{x\in P(\alpha,b,d)\mid\alpha(x)>b\}\neq\emptyset$ *and* $\alpha(Ax)>a$ *for* $x\in P(\alpha,b,d)$,
(C_2) $\|Ax\|<a$ *for* $\|x\|\leq a$*, and*
(C_3) $\alpha(Ax)>b$ *for* $x\in P(\alpha,b,c)$ *with* $\|Ax\|>d$.
Then A *has at least three fixed points* x_1, x_2 *and* x_3 *in* $\overline{P_c}$ *such that*

$\|x_1\| < a$, $b < \alpha(x_2)$, and $\|x_3\| > a$ with $\alpha(x_3) < b$.

Lemma 2.3. *For $y \in C[0,1]$, the following boundary value problem*

$$\begin{cases} {}^{C}D_{0+}^{\alpha} u(t) + y(t) = 0, \quad t \in (0,1), \\ u^{(i)}(0) = 0, \ i = 2, ..., n-1, \\ u'(0) = \sum_{i=1}^{m-2} b_i u'(\eta_i), \ u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)\, ds \end{cases} \tag{2.1}$$

has the unique solution

$$u(t) = c_0 + c_1 t - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds, \tag{2.2}$$

where

$$\begin{aligned} c_0 = {} & \frac{\int_0^1 (1-s)^{\alpha-1} y(s)\, ds}{\left(1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})\right)\Gamma(\alpha)} - \frac{\sum_{i=1}^{m-2} a_i \left[\int_0^{\eta_i} (\eta_i - s)^{\alpha} - \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^{\alpha}\right] y(s)\, ds}{\left(1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})\right)\Gamma(\alpha+1)} \\ & + \frac{\left(1 - \sum_{i=1}^{m-2} a_i \frac{\eta_i^2 - \eta_{i-1}^2}{2}\right) \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\, ds}{\left(1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})\right)\left(1 - \sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}, \\ c_1 = {} & -\frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\, ds}{\left(1 - \sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}. \end{aligned} \tag{2.3}$$

Proof. In view of Definition 2.1 and Lemma 2.1, it is clear that equation 2.1 is equivalent to the integral form

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_0 + c_1 t + ... + c_{n-1} t^{n-1},$$

where $c_0, c_1, ..., c_{n-1} \in \mathbb{R}$ are arbitrary constants.

Next, using the initial conditions: $u^{(i)}(0) = 0$, $i = 2, ..., n-1$, we get

$$c_2 = c_3 = ... = c_{n-1} = 0,$$

that is,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_0 + c_1 t. \tag{2.4}$$

So we get

$$u'(t) = -\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} y(s)\, ds + c_1. \tag{2.5}$$

By $u'(0) = \sum_{i=1}^{m-2} b_i u'(\eta_i)$, we obtain

$$c_1 = -\frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\, ds}{\left(1 - \sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}. \tag{2.6}$$

Integrating the equation 2.4 from η_{i-1} to η_i for $0 \leq \eta_{i-1} \leq \eta_i \leq 1$, $i = 1, ..., m-2$, and using Remark 1, we get

$$\begin{aligned}
\int_{\eta_{i-1}}^{\eta_i} u(t)\,dt &= -\frac{1}{\Gamma(\alpha)} \int_{\eta_{i-1}}^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau)\,d\tau \right) ds + c_0 \int_{\eta_{i-1}}^{\eta_i} ds + c_1 \int_{\eta_{i-1}}^{\eta_i} s ds \\
&= -\frac{1}{\Gamma(\alpha)} \left[\int_0^{\eta_i} \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau)\,d\tau \right) ds + \int_{\eta_{i-1}}^0 \left(\int_0^s (s-\tau)^{\alpha-1} y(\tau)\,d\tau \right) ds \right] \\
&\quad + c_0 \int_{\eta_{i-1}}^{\eta_i} ds + c_1 \int_{\eta_{i-1}}^{\eta_i} s ds \\
&= -\frac{1}{\Gamma(\alpha+1)} \int_0^{\eta_i} (\eta_i - s)^{\alpha} y(s)\,ds + \frac{1}{\Gamma(\alpha+1)} \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^{\alpha} y(s)\,ds \\
&\quad + c_0 (\eta_i - \eta_{i-1}) + c_1 \frac{\eta_i^2 - \eta_{i-1}^2}{2}.
\end{aligned}$$

Then, by the condition $u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)\,ds$, we get

$$\begin{aligned}
-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\,ds + c_0 + c_1 &= -\frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^{m-2} a_i \int_0^{\eta_i} (\eta_i - s)^{\alpha} y(s)\,ds \\
&\quad + \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^{m-2} a_i \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^{\alpha} y(s)\,ds \\
&\quad + c_0 \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1}) + c_1 \sum_{i=1}^{m-2} a_i \frac{\eta_i^2 - \eta_{i-1}^2}{2}.
\end{aligned}$$

Which implies

$$\begin{aligned}
c_0 &= \frac{\int_0^1 (1-s)^{\alpha-1} y(s)\,ds}{\left(1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})\right) \Gamma(\alpha)} - \frac{\sum_{i=1}^{m-2} a_i \left[\int_0^{\eta_i} (\eta_i - s)^{\alpha} - \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^{\alpha} \right] y(s)\,ds}{\left(1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})\right) \Gamma(\alpha+1)} \\
&\quad + \frac{\left(1 - \sum_{i=1}^{m-2} a_i \frac{\eta_i^2 - \eta_{i-1}^2}{2}\right) \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-2} y(s)\,ds}{\left(1 - \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})\right) \left(1 - \sum_{i=1}^{m-2} b_i\right) \Gamma(\alpha-1)}.
\end{aligned}$$

□

Remark 2. *i) Assume that (H_1) hold. Then, for $y \in C([0,1])$ and $y(t) \geq 0$ by (2.5) and (2.6), we obtain $u'(t) < 0$ and*

$$u''(t) = -\frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} y(s)\,ds < 0. \tag{2.7}$$

ii) If we assume that (H_1) hold, we have

$$0 \leq \sum_{i=1}^{m-2} a_i \left(\eta_i^2 - \eta_{i-1}^2\right) \leq \sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1}) < 1.$$

Lemma 2.4. *Let (H_1) satisfied. If $y(t) \in C[0,1]$ satisfying $y(t) \geq 0$, then the function u of (2.2) satisfies $u(t) \geq 0$.*

Proof. From Remark 2, $u(t)$ is concave and non-increasing on $[0,1]$. Then

$$\max_{0\le t\le 1} u(t) = u(0), \quad \min_{0\le t\le 1} u(t) = u(1). \tag{2.8}$$

From the concavity of u , we have

$$\frac{u(\eta_1)}{\eta_1} \ge \frac{u(\eta_2)}{\eta_2} \ge ... \ge \frac{u(\eta_{i-1})}{\eta_{i-1}} \ge \frac{u(\eta_i)}{\eta_i} \ge ... \ge \frac{u(1)}{1} \tag{2.9}$$

and

$$\int_{\eta_{i-1}}^{\eta_i} u(s)\,ds \ge \frac{1}{2}(\eta_i - \eta_{i-1})(u(\eta_i) + u(\eta_{i-1})), \tag{2.10}$$

where $\frac{1}{2}(\eta_i - \eta_{i-1})(u(\eta_i) + u(\eta_{i-1}))$ is the area of the trapezoid under the curve $u(t)$ from $t = \eta_{i-1}$ to $t = \eta_i$ for $i = 1, 2, ..., m-2$. Multiplying both sides of the inequality (2.10) with a_i and combining conditions (2.9), (2.10) and $u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)\,ds$, we get

$$\begin{aligned} u(1) &\ge \frac{1}{2}\sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})(u(\eta_i) + u(\eta_{i-1})) \\ &\ge \frac{1}{2}\sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})(\eta_i u(1) + \eta_{i-1} u(1)) \\ &= \frac{1}{2}\sum_{i=1}^{m-2} a_i \left(\eta_i^2 - \eta_{i-1}^2\right) u(1). \end{aligned}$$

If $u(1) < 0$, we get

$$2 \le \sum_{i=1}^{m-2} a_i \left(\eta_i^2 - \eta_{i-1}^2\right).$$

This contradicts the fact that $\sum_{i=1}^{m-1} a_i \left(\eta_i^2 - \eta_{i-1}^2\right) < 1$. Then $u(1) \ge 0$. Therefore, we get $u(t) \ge 0$ for $t \in [0,1]$. The proof is complete. □

Lemma 2.5. *Let (H_1) hold. If $y \in C([0,1])$ and $y \ge 0$, then the unique solution u of the problem (2.1) satisfies*

$$\min_{t\in[0,1]} u(t) \ge \gamma \|u\|,$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2} a_i (\eta_i - \eta_{i-1})(2 - \eta_i - \eta_{i-1})}{2 - \sum_{i=1}^{m-2} a_i \left(\eta_i^2 - \eta_{i-1}^2\right)}. \tag{2.11}$$

Proof. From Remark 2, u is concave and nonincreasing on $[0,1]$. This implies that

$$\|u\| = u(0), \qquad \min_{t\in[0,1]} u(t) = u(1)$$

and

$$u(0) \le u(1) + \frac{u(1) - u(t)}{1-t}(0-1)$$

or

$$u(0)(1-t) \le u(1)(1-t) + u(t) - u(1). \tag{2.12}$$

By integrating the both sides of the inequality (2.12) from $t=\eta_{i-1}$ to $t=\eta_i$, we have

$$u(0)\int_{\eta_{i-1}}^{\eta_i}(1-t)\,dt \le u(1)\int_{\eta_{i-1}}^{\eta_i}(1-t)\,dt+\int_{\eta_{i-1}}^{\eta_i}u(t)\,dt-u(1)\int_{\eta_{i-1}}^{\eta_i}dt$$

and by the condition $u(1)=\sum_{i=1}^{m-2}a_i\int_{\eta_{i-1}}^{\eta_i}u(s)\,ds$, we get

$$\begin{aligned} u(0) &\le u(1)\left[1+\frac{1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})}{\sum_{i=1}^{m-2}a_i\left(\eta_i-\eta_{i-1}-\frac{1}{2}\left(\eta_i^2-\eta_{i-1}^2\right)\right)}\right] \\ &\le u(1)\left[\frac{2-\sum_{i=1}^{m-2}a_i\left(\eta_i^2-\eta_{i-1}^2\right)}{\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})(2-(\eta_i+\eta_{i-1}))}\right]. \end{aligned}$$

Thus

$$\min_{t\in[0,1]}u(t)\ge\frac{\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})(2-\eta_i-\eta_{i-1})}{2-\sum_{i=1}^{m-2}a_i\left(\eta_i^2-\eta_{i-1}^2\right)}u(0).$$

□

Let $E=C([0,1])$ be a Banach space of all continuous real functions on $[0,1]$ equipped with the norm $\|u\|=\max_{t\in[0,1]}|u(t)|$ for $u\in E$, and define

$$K=\{u\in E \mid u \textit{ is nonnegative concave and nonincreasing on } [0,1]\}.$$

It is obvious that K is a cone.

Define the operator $A:E\to E$ as follows:

$$\begin{aligned} Au(t)=&\frac{\int_0^1(1-s)^{\alpha-1}h(s)f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)} \\ &-\frac{\sum_{i=1}^{m-2}a_i\left[\int_0^{\eta_i}(\eta_i-s)^{\alpha}-\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}\right]h(s)f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)} \\ &+\frac{\left(1-\sum_{i=1}^{m-2}a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2}b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2}h(s)f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2}b_i\right)\Gamma(\alpha-1)} \\ &-\frac{\sum_{i=1}^{m-2}b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2}h(s)f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2}b_i\right)\Gamma(\alpha-1)}t \\ &-\frac{\int_0^t(t-s)^{\alpha-1}h(s)f(s,u(s))\,ds}{\Gamma(\alpha)}. \end{aligned} \tag{2.13}$$

Then u is a solution of the boundary value problem (1.3) if and only if it is a fixed point of the operator A.

Lemma 2.6. *Assume that $(H_1)-(H_3)$ hold. Then the operator $A:E\to E$ is completely continuous.*

Proof. Let $u\in K$, then $Au(t)\ge 0$, $(Au)'(t)\le 0$ and $(Au)''(t)\le 0$, $0\le t\le 1$, consequently, $A:K\to K$. In view of continuity of $h(t)$ and $f(t,u)$, we get A is continuous.

Take $N \subset K$ be bounded, that is, there exists a positive constant l for any $u \in N$, such that $\|u\| \leq l$. Let $L = \max_{t\in[0,1],u\in[0,l]} f(t,u) + 1$, then, for any $u \in N$, we have

$$
\begin{aligned}
Au(t) &\leq \frac{\int_0^1 (1-s)^{\alpha-1} h(s) f(s,u(s))\, ds}{\left(1-\sum_{i=1}^{m-2} a_i (\eta_i-\eta_{i-1})\right)\Gamma(\alpha)} \\
&\quad + \frac{\left(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\, ds}{\left(1-\sum_{i=1}^{m-2} a_i (\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} \\
&\leq L\left[\frac{\int_0^1 h(s)\, ds}{\left(1-\sum_{i=1}^{m-2} a_i (\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}\right. \\
&\quad \left. + \frac{\left(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\eta_i^{\alpha-2} \int_0^{\eta_i} h(s)\, ds}{\left(1-\sum_{i=1}^{m-2} a_i (\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}\right].
\end{aligned}
$$

Hence, $A(N)$ is uniformly bounded. Now, we will prove that $A(N)$ is equicontinuous. For each $u \in N$, $0 \leq \tau_1 < \tau_2 \leq 1$, we have

$$
\begin{aligned}
&|(Au)(\tau_2) - (Au)(\tau_1)| \\
&= \left| \frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\, ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}\tau_2 + \frac{\int_0^{\tau_2} (\tau_2-s)^{\alpha-1} h(s) f(s,u(s))\, ds}{\Gamma(\alpha)} \right. \\
&\quad \left. - \frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\, ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}\tau_1 - \frac{\int_0^{\tau_1} (\tau_1-s)^{\alpha-1} h(s) f(s,u(s))\, ds}{\Gamma(\alpha)} \right| \\
&\leq \frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\, ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}(\tau_2-\tau_1) \\
&\quad + \left| \frac{\int_0^{\tau_2} (\tau_2-s)^{\alpha-1} h(s) f(s,u(s))\, ds}{\Gamma(\alpha)} - \frac{\int_0^{\tau_1} (\tau_1-s)^{\alpha-1} h(s) f(s,u(s))\, ds}{\Gamma(\alpha)} \right| \\
&\leq \frac{\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\, ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}(\tau_2-\tau_1)
\end{aligned}
$$

$$+\left|\frac{\int_0^{\tau_1}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]h(s)\,f(s,u(s))\,ds}{\Gamma(\alpha)}\right|$$
$$+\left|\frac{\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}h(s)\,f(s,u(s))\,ds}{\Gamma(\alpha)}\right|$$
$$\leq \frac{L\sum_{i=1}^{m-2}b_i\eta_i^{\alpha-2}\int_0^{\eta_i}h(s)\,ds}{\left(1-\sum_{i=1}^{m-2}b_i\right)\Gamma(\alpha-1)}(\tau_2-\tau_1)$$
$$+\frac{L\int_0^{\tau_1}h(s)\,ds}{\Gamma(\alpha)}\left(\tau_2^{\alpha-1}-\tau_1^{\alpha-1}\right)+\frac{L\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds}{\Gamma(\alpha)}(\tau_2-\tau_1)^{\alpha-1}.$$
$$\leq L\left(\frac{\sum_{i=1}^{m-2}b_i\eta_i^{\alpha-2}\int_0^{\eta_i}h(s)\,ds}{\left(1-\sum_{i=1}^{m-2}b_i\right)\Gamma(\alpha-1)}(\tau_2-\tau_1)+\frac{\int_0^{\tau_1}h(s)\,ds}{\Gamma(\alpha)}\left(\tau_2^{\alpha-1}-\tau_1^{\alpha-1}\right).\right.$$
$$\left.+\frac{\int_{\tau_1}^{\tau_2}h(s)\,ds}{\Gamma(\alpha)}(\tau_2-\tau_1)^{\alpha-1}\right).$$

Therefore, $A(N)$ is equicontinuous. Applying the Arzela -Ascoli theorem, we conclude that A is a completely continuous operator. The proof is completed. □

3. MAIN RESULTS

In this section, we discuss the existence of triple positive solutions of the Problem (1.3). We define the nonnegative continuous concave functional on K by

$$\alpha(u)=\min_{0\leq t\leq 1}u(t).$$

It is obvious that, for each $u\in K$, $\alpha(u)\leq\|u\|$. For convenience, we use the following notation. Let

$$M=\frac{\int_0^1 h(s)\,ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}+\frac{\sum_{i=1}^{m-2}a_i\eta_{i-1}^{\alpha}\int_0^{\eta_{i-1}}h(s)\,ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)}$$
$$+\frac{\left(1-\sum_{i=1}^{m-2}a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2}b_i\eta_i^{\alpha-2}\int_0^{\eta_i}h(s)\,ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2}b_i\right)\Gamma(\alpha-1)},$$

$$m=\frac{\int_0^1(1-s)^{\alpha-1}h(s)\,ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}-\frac{\sum_{i=1}^{m-2}a_i\left[\int_0^{\eta_i}(\eta_i-s)^{\alpha}-\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}\right]h(s)\,ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)}$$
$$+\frac{\left(1-\sum_{i=1}^{m-2}a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2}b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2}h(s)\,ds}{\left(1-\sum_{i=1}^{m-2}a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2}b_i\right)\Gamma(\alpha-1)}$$
$$-\frac{\sum_{i=1}^{m-2}b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2}h(s)\,ds}{\left(1-\sum_{i=1}^{m-2}b_i\right)\Gamma(\alpha-1)}-\frac{\int_0^1(1-s)^{\alpha-1}h(s)\,ds}{\Gamma(\alpha)}.$$

Theorem 3.1. *Suppose that the conditions $(H_1)-(H_3)$ hold. In addition, assume there exist non-negative numbers a, b and c such that $0<a<b<\gamma c$, and $f(t,u)$ satisfies the following growth conditions:*

(H_4) $f(t,u) \leq \frac{c}{M}$, *for all* $(t,u) \in [0,1] \times [0,c]$,
(H_5) $f(t,u) \leq \frac{a}{M}$, *for all* $(t,u) \in [0,1] \times [0,a]$,
(H_6) $f(t,u) > \frac{b}{m}$, *for all* $(t,u) \in [0,1] \times \left[b, \frac{b}{\gamma}\right]$.
Then the boundary value problems (1.3) have at least three positive solutions u_1, u_2 *and* u_3 *such that*
$\|u_1\| < a,\ b < \alpha(u_2),\ \|u_3\| > a,\ with\ \alpha(u_3) < b.$

Proof. From Lemma 2.6, the operator $A : K \to K$ is completely continuous. Now, we prove that $A : \overline{P_c} \to \overline{P_c}$. For $u \in \overline{P_c}$, we have $\|Au\| = Au(0)$. Then

$$\begin{aligned}
Au(0) &= \frac{\int_0^1 (1-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)} \\
&\quad - \frac{\sum_{i=1}^{m-2} a_i \left[\int_0^{\eta_i} (\eta_i - s)^{\alpha} - \int_0^{\eta_{i-1}} (\eta_{i-1}-s)^{\alpha}\right] h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)} \\
&\quad + \frac{\left(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} \\
&\leq \frac{\int_0^1 h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)} + \frac{\sum_{i=1}^{m-2} a_i \eta_{-1}^{\alpha} \int_0^{\eta_{i-1}} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)} \\
&\quad + \frac{\left(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i \eta_i^{\alpha-2} \int_0^{\eta_i} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} \\
&\leq \frac{c}{M}\left(\frac{\int_0^1 h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}\right. \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i \eta_{-1}^{\alpha} \int_0^{\eta_{i-1}} h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)} \\
&\quad \left. + \frac{\left(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i \eta_i^{\alpha-2} \int_0^{\eta_i} h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)}\right) \\
&\leq c.
\end{aligned}$$

Thus, $\|Au\| \leq c$. Consequently, $A : \overline{P_c} \to \overline{P_c}$.

In a completely analogous manner, the condition (H_5) implies that the condition (C_2) of Theorem 2.2 is satisfied for A.

Now, we show that condition (C_1) of Theorem 2.2 is satisfied. Since $\alpha\left(\frac{b}{\gamma}\right) = \frac{b}{\gamma} > b$, then $\left\{u \in P\left(\alpha, b, \frac{b}{\gamma}\right) \mid \alpha(u) > b\right\} \neq \emptyset$. If $u \in P\left(\alpha, b, \frac{b}{\gamma}\right)$, then $b \leq u(s) \leq \frac{b}{\gamma}$, $s \in [0,1]$.

By condition (H_6), we get

$$
\begin{aligned}
\alpha\left((Au)(t)\right) &= \min_{0\le t\le 1}\left((Au)(t)\right) = (Au)(1) \\
&= \frac{\int_0^1 (1-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)} \\
&\quad - \frac{\sum_{i=1}^{m-2} a_i\left[\int_0^{\eta_i}(\eta_i-s)^{\alpha} - \int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}\right] h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)} \\
&\quad + \frac{\left(1-\sum_{i=1}^{m-2} a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} \\
&\quad - \frac{\sum_{i=1}^{m-2} b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2} h(s) f(s,u(s))\,ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} - \frac{\int_0^1 (1-s)^{\alpha-1} h(s) f(s,u(s))\,ds}{\Gamma(\alpha)} \\
&\ge \frac{b}{m}\left(\frac{\int_0^1 (1-s)^{\alpha-1} h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}\right. \\
&\quad - \frac{\sum_{i=1}^{m-2} a_i\left[\int_0^{\eta_i}(\eta_i-s)^{\alpha} - \int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}\right] h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)} \\
&\quad + \frac{\left(1-\sum_{i=1}^{m-2} a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2} h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} \\
&\quad \left.- \frac{\sum_{i=1}^{m-2} b_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-2} h(s)\,ds}{\left(1-\sum_{i=1}^{m-2} b_i\right)\Gamma(\alpha-1)} - \frac{\int_0^1 (1-s)^{\alpha-1} h(s)\,ds}{\Gamma(\alpha)}\right) \\
&\ge b.
\end{aligned}
$$

Therefore, condition (C_1) of Theorem 2.2 is satisfied.

For the condition (C_3) of the Theorem 2.2, we can verify it easily under our assumptions using Lemma 2.5. Here

$$\alpha(Au) = \min_{0\le t\le 1}(Au)(t) \ge \gamma\frac{b}{\gamma} = b$$

as long as if $u \in P(\alpha,b,c)$, with $\|Au\| > \frac{b}{\gamma}$.

Therefore, the condition (C_3) of Theorem 2.2 is satisfied. By Theorem 2.2, there exist three positive solutions u_1, u_2 and u_3 such that $\|u_1\| < a$, $b < \alpha(u_2(t))$ and $\|u_3\| > a$, with $\alpha(u_3(t)) < b$. □

4. EXAMPLE

Consider the boundary value problem

$$
\begin{cases}
D_{0+}^{2,5}u(t) + (1-t) f(t,u(t)) = 0, \quad t\in(0,1), \\
u''(0) = 0, \\
u'(0) = 0,1u'(0,4) + 0,02u'(0,6) + 0,05u'(0,8), \\
u(1) = 0,01\int_0^{0,4} u(s)\,ds + 0,02\int_{0,4}^{0,6} u(s)\,ds + 0,4\int_{0,6}^{0,8} u(s)\,ds
\end{cases}
\tag{4.1}
$$

where

$$f(t,u) = \begin{cases} e^{-\frac{t}{8}} \left(\frac{u^3}{144} + 3 + \ln(4u+3) \right), \ 0 \le t \le 1, \ 0 \le u \le 3, \\ e^{-\frac{t}{8}} \left(\frac{51}{16} + \ln 15 + 25\sqrt{u-3} \right), \ 0 \le t \le 1, \ 3 < u \le 150, \\ e^{-\frac{t}{8}} \left(\frac{51}{16} + \ln 15 + 25\sqrt{147} + \sqrt{u-150} \right), \ 0 \le t \le 1, \ 3 < u \le 150. \end{cases}$$

To show the problem (4.1) has at least three positive solutions, we apply Theorem 3.1 with $\alpha = 2.5$, $m = 5$, $b_1 = 0.1$, $b_2 = 0.02$, $b_3 = 0.05$, $a_1 = 0.01$, $a_2 = 0.02$, $a_3 = 0.4$, $\eta_1 = 0.4$, $\eta_2 = 0.6$, $\eta_3 = 0.8$.

Then, by direct calculations, we can obtain that

$$1 - \sum_{i=1}^{3} b_i = 0.83, \quad 1 - \sum_{i=1}^{3} a_i (\eta_i - \eta_{i-1}) = 0.912, \quad 1 - \sum_{i=1}^{3} a_i \left(\eta_i^2 - \eta_{i-1}^2 \right) = 0.9412,$$

$$\gamma = 0.0310242 \quad , M = 0.495731 \quad , m = 0.16194.$$

If we choose $a = 3$, $b = 4$ and $c = 160$, we obtain

$$f(t,u) \le 312.166719 \le \frac{c}{M} \approx 322.7557, \ 0 \le t \le 1, \ 0 \le u \le 160,$$

$$f(t,u) \le 5.896 \le \frac{a}{M} \approx 6.052, \ 0 \le t \le 1, \ 0 \le u \le 3,$$

$$f(t,u) \ge 27.2652274 \ge \frac{b}{m} \approx 24.7005 \ 0 \le t \le 1, \ 4 \le u \le 128.931608.$$

Thus by Theorem 3.1 the problem (1.3) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$\|u_1\| < 3, \quad 4 < \alpha(u_2(t)), \ and \, \|u_3\| > 3, \ with \ \alpha(u_3(t)) < 4.$$

5. CONCLUSION

In this paper, some results on the existence and multiplicity of solutions for a nonlinear higher order fractional differential equation involving the left Caputo fractional derivative with both multi-point and multi-strip boundary conditions are obtained. Under sufficient conditions, we have applied the Leggett-Williams fixed point theorem to obtain the existence of at least three positive solutions. An example is given to show the applicability of our results.

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