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Weak Type Estimates Of Hardy Integral Operators On Morrey Spaces With Variable Exponent Lebesgue spaces

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Abstract: We show that when the infimum of the exponent function, Hardy integral operator is a bounded operator from the Morrey space with variable exponent to the weak Morrey space with variable exponent.

Keywords: Hardy integral operator, Morrey spaces, Weak Morrey spaces, Variable exponent.

1 Introduction

We show that when $\inf_{x\in R^n} p(x) = 1$ Hardy integral operator is a bounded linear operator from the Morrey space with variable exponent to the weak Morrey space with variable exponent. In this work, we obtain the weak type estimates for Hardy integral operators on Morrey spaces with variable exponents. We introduce the weak(w) Morrey spaces with variable exponent $M_{u,w}^{p(.)}$ (see Definition 2.2) and show that H is a bounded linear operator that maps $M_u^{p(.)}$ to $M_{u,w}^{p(.)}$. The weak Morrey spaces has applications on the study of Navier-Stokes equations, see [7, 10]. The duality of weak Morrey space is investigated in [11]. Furthermore, we also have the atomic decompositions of weak-Hardy Morrey spaces in [4].

2 Definitions and Auxillary Statements

For any $p(.) : R^n \longrightarrow [1,\infty]$, we define $p^+ = \sup_{x\in R^n} p(x)$ and $p^- = \inf_{x\in R^n} p(x)$ and also

$$R_\infty^{p(.)} = \{x \in R^n : p(x) = \infty\}.$$

And also any $x \in R^n$ and $r > 0$, write $B(x,r) = \{z : |z - x| < r\}$.
Define $\Psi = \{B(x,r) : x \in R^n, r > 0\}$.Furthermore we define

$$\Gamma_{log} = \{p(.) : R^n \longrightarrow [1,\infty] : \frac{1}{p(.)} \quad is \quad globally \quad log-Holder \quad continuous\}.$$

Definition 2.1. The weak Lebesgue space with variable exponent $L_w^{p(.)}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L_w^{p(.)}} = \sup_{\lambda>0} \lambda \|\chi_{\{x:|f(x)|>\lambda\}}\|_{L^{p(.)}}$$

We call $p(.)$ the exponent function of $L_w^{p(.)}$.

Lemma 1. *(See* [5]*) If* $p(.) : R^n \longrightarrow [1,\infty]$*, then* $\|.\|_{L_w^{p(.)}}$ *is a quasi-norm. We now recall some basic results for* $L^{p(.)}$*. For some details on the study of* $L^{p(.)}$*. the reader is referred to [2, 8]. For any exponent function* $p(.) : R^n \longrightarrow [1,\infty]$*, define* $p'(.)$ *by*

$$\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$$

with the convention that $\frac{1}{\infty} = 0$.

Lemma 2. *(See* [5]*) Let* $p(.) : R^n \longrightarrow [1,\infty]$*. For any Lebesgue measurable set* E *with* $|E| < \infty$*, we have*

$$\|\chi_E\|_{L^{p(.)}} = \|\chi_E\|_{L_w^{p(.)}}.$$

Theorem 1. *(See $[8, Theorem 4.3.8]$) Let $p(.): R^n \longrightarrow [1,\infty]$. If $p(.) \in \Gamma_{log}$ with $p^- > 1$, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(.)}$.*

Lemma 3. *(See $[5]$) Let $p(.): R^n \longrightarrow [1,\infty]$ be a globally log-Holder continuous with $1 \le p^- \le p^+ < \infty$. Then, there exists a constant $C > 0$ such that for any $B \in \psi$ we have*

$$|B| \le \|\chi_B\|_{L^{p(.)}} \|\chi_B\|_{L^{p'(.)}} \le C|B|.$$

Lemma 4. *(See $[8, Corollary 4.5.9]$) Let $p(.) \in \Gamma_{log}$. There exist constants $K, C > 0$ such that for any $B \in \psi$, we have*

$$K|B|^{\frac{1}{p_B}} \le \|\chi_B\|_{L^{p(.)}} \le C|B|^{\frac{1}{p_B}}.$$

Theorem 2. *(See $[1, Theorem 1.8(for \quad \alpha = 1)]$).Let $p(.): R^n \longrightarrow [1,\infty]$. Suppose that $p(.)$ is globally log-Holder continuous and satisfies $1 < p^- \le p^+ < n$. Define $q(.)$ by*

$$\frac{1}{p(.)} - \frac{1}{q(.)} = \frac{1}{n} \tag{1}$$

We have a constant $C > 0$ such that for any $f \in L^{p(.)}$,

$$\|Hf\|_{L^{q(.)}} \le C\|f\|_{L^{p(.)}}$$

We see that whenever $p(.)$ and $q(.)$ satisfy (1), we have

$$\frac{1}{p_B} - \frac{1}{q_B} = \frac{1}{n}, \qquad \forall B \in \psi \tag{2}$$

Theorem 3. *(See $[1, Theorem 1.8(for \quad \alpha = 1)]$).Let $p(.): R^n \longrightarrow [1,\infty]$. Suppose that $p(.)$ is globally log-Holder continuous and satisfies $1 \le p^- \le p^+ < n$. Let $q(.)$ be defined by (1). We have a constant $C > 0$ such that for any $f \in L^{p(.)}$,*

$$\|Hf\|_{L_w^{q(.)}} \le C\|f\|_{L^{p(.)}}.$$

Definition 2.2. Let $p(.): R^n \longrightarrow [1,\infty)$ and $u : R^n \times (0,\infty) \longrightarrow (0,\infty)$. The Morrey space with variable exponent $M_k^{p(.)}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{M_k^{p(.)}} = \sup_{B(x,r)\in\psi} \frac{1}{k(x,r)} \|f\chi_{B(x,r)}\|_{L^{p(.)}} < \infty$$

The weak Morrey space with variable exponent $M_{k,w}^{p(.)}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{M_{k,w}^{p(.)}} = \sup_{B(x,r)\in\psi} \frac{1}{k(x,r)} \|f\chi_{B(x,r)}\|_{L_w^{p(.)}} < \infty.$$

3 Main Result

Theorem 4. *Let $p(.): R^n \longrightarrow [1,\infty)$ and $k : R^n \times (0,\infty) \longrightarrow (0,\infty)$. Suppose that $p(.)$ is globally log-Holder continuous and satisfies $1 \le p^- \le p^+ < n$. Let $q(.)$ be defined by (1).If there exists a constant $C > 0$ such that for any $x \in R^n$ and $r > 0$, k satisfies*

$$\sum_{j=0}^{\infty} = \frac{\|\chi_{B(x,r)}\|_{L^{q(.)}}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{q(.)}}} k(x, 2^{j+1}r) \le Ck(x,r) \tag{3}$$

then we have a constant $C > 0$ such that for any $f \in M_k^{p(.)}$,

$$\|Hf\|_{M_{k,w}^{q(.)}} \le C\|f\|_{M_k^{p(.)}}.$$

Proof: Let $f \in M_k^{p(.)}$. For any $z \in R^n$ and $r > 0$, write $f_0 = \chi_{B(z,2r)}f$ and $f_j = \chi_{B(z,2^{j+1}r)/B(z,2^j r)}f$, $j \in N/\{0\}$. We have $f = \sum_{j=0}^{\infty} f_j$. In view of Theorem 2.7, we find that

$$\|\chi_{B(z,r)} Hf_0\|_{L_w^{q(.)}} \le C\|f_0\|_{L^{p(.)}} = C\|f\chi_{B(z,2r)}\|_{L^{p(.)}} \tag{4}$$

Notice that there exists a constant $C > 0$ such that for any $z \in R^n$ and $r > 0$,

$$\chi_{B(z,2r)} \le CM_{\chi_{B(x,r)}}$$

Moreover, whenever $p(.)$ is globally log-Holder continuous with $1 \le p^- \le p^+ < \infty$, then $q(.)$ is globally log-Holder continuous with $1 < p^- \le p^+ < \infty$. Therefore, Theorem 2.3 asserts that

$$\|\chi_{B(z,2r)}\|_{L^{q(.)}} \le C\|M_{\chi_{B(z,r)}}\| \le C\|\chi_{B(z,r)}\|_{L^{q(.)}}$$

for some $C > 0$. Consequently, (3) gives $k(z,2r) < Ck(z,r)$ for some $C > 0$ independent of z and r. As a result of the above inequality, (4) yields

$$\frac{1}{k(z,r)}\|\chi_{B(z,r)}Hf_0\|_{L_w^{q(.)}} \leq C\frac{1}{k(z,r)}\|\chi_{B(z,2r)}f\|_{L^{p(.)}}$$

$$\leq C\frac{1}{k(z,2r)}\|\chi_{B(z,2r)}f\|_{L^{p(.)}} \leq C\|f\|_{M_k^{p(.)}} \tag{5}$$

Next, for any $j \geq 1$, we have that for any $x \in B(z,r)$

$$|Hf_j(x)| \leq C2^{-j(n-1)}r^{-n+1}\int_{B(z,2^{j+1}r)}|f(y)|dy.$$

The Holder inequality for $L^{p(.)}$ gives

$$\chi_{B(z,r)}(x)|Hf_j(x)|$$

$$\leq C2^{-j(n-1)}r^{-n+1}\chi_{B(z,r)}(x) \times \|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(.)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(.)}} \tag{6}$$

Since $p(.)$ is globally log-Holder continuous with $1 \leq p^- \leq p^+ < \infty$. Lemma 2.4 ensures that

$$D_j \leq C2^{-j(n-1)}r^{-n+1}2^{n(j+1)}r^n\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(.)}}} \leq C2^jr\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(.)}}}$$

Lemma 2.5 and (2) show that

$$K\frac{|B(z,2^{j+1}r)|^{\frac{1}{n}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(.)}}} \leq \frac{1}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}} \leq C\frac{|B(z,2^{j+1}r)|^{\frac{1}{n}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(.)}}}$$

for some $C, K > 0$ independent of z and r.
Since $|B(z,2^{j+1}r)|^{\frac{1}{n}} = C2^jr$, where $C > 0$ is a constant independent of z and $r > 0$, we obtain

$$D_j \leq C\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}}.$$

Consequently,

$$\chi_{B(z,r)}(x)\sum_{j=1}^{\infty}|Hf_j(x)| \leq C\chi_{B(z,r)}(x)\sum_{j=1}^{\infty}\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}}.$$

By applying the quasi-norm $\|.\|_{L_w^{q(.)}}$ on both sides of the above inequality, we get

$$\|\chi_{B(z,r)}(x)\sum_{j=1}^{\infty}|Hf_j(x)|\|_{L_w^{q(.)}} \leq C\|\chi_{B(z,r)}\|_{L_{w,k}^{q(.)}}\sum_{j=1}^{\infty}\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}}$$

$$\leq \sum_{j=1}^{\infty}\frac{\|\chi_{B(z,r)}f\|_{L_w^{q(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}}k(z,2^{j+1}r)\|f\|_{M_k^{p(.)}}.$$

Lemma 2.2 gives

$$\frac{1}{k(y,r)}\|\chi_{B(z,r)}(x)\sum_{j=1}^{\infty}|Hf_j(x)|\|_{L_w^{q(.)}} \leq \sum_{j=1}^{\infty}\frac{k(z,2^{j+1}r)}{k(z,r)}\frac{\|\chi_{B(z,r)}f\|_{L_w^{q(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}}\|f\|_{M_k^{p(.)}}$$

$$\leq \sum_{j=1}^{\infty}\frac{k(z,2^{j+1}r)}{k(z,r)}\frac{\|\chi_{B(z,r)}f\|_{L^{q(.)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(.)}}}\|f\|_{M_k^{p(.)}}.$$

Therefore, (3) and (5) yield

$$\frac{1}{k(z,r)}\|\chi_{B(z,r)}(x)Hf\|_{L_w^{q(.)}}$$

$$C(\frac{1}{k(z,r)}\|\chi_{B(z,r)}(x)Hf_0\|_{L_w^{q(.)}} + \frac{1}{k(z,r)}\|\chi_{B(z,r)}(x)\sum_{j=1}^{\infty}|Hf_j|\|_{L_w^{q(.)}}) \leq C\|f\|_{M_k^{p(.)}}.$$

for some $C > 0$ independent of $B(z,r) \in \psi$. By taking the supremum over $z \in R^n$ and $r > 0$, we obtain

$$\|Hf\|_{M_{k,w}^{q(.)}} \leq C\|f\|_{M_k^{p(.)}}.$$

Thus the proof of Theorem 3.1 is completed.

□

The reader is referred to $([6], pp.366 - 367)$ for some examples of k that satisfies (3) and the relation between (3) with the conditions imposed on k for the results obtained in $[3, 9]$.

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