



Classifications of K -Contact Semi-Riemannian Manifolds

Chiranjib Dey^{1*} and Uday Chand De²

¹Dhamla Jr. High School, Vill-Dhamla, P.O.-Kedarpur, Dist-Hooghly, Pin-712406, West Bengal, India

²Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol- 700019, West Bengal, India

*Corresponding author E-mail: dey9chiranjib@gmail.com

Abstract

The object of the present paper is to characterize a K -contact semi-Riemannian manifold satisfying certain curvature conditions. We study Ricci semi-symmetric K -contact semi-Riemannian manifolds and obtain an equivalent condition. Next we prove that a K -contact semi-Riemannian manifold is of harmonic conformal curvature tensor if and only if the manifold is an Einstein manifold. Also we study ξ -conformally flat K -contact semi-Riemannian manifolds. Finally, we characterize conformally semisymmetric Lorentzian K -contact manifolds.

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1. Introduction

Let M^{2n+1} be a contact manifold with contact form η , associated vector field ξ and $(1, 1)$ tensor field ϕ ([1],[2]). A semi-Riemannian metric g is said to be an associated metric if it satisfies: $\eta(X) = \varepsilon g(X, \xi)$, where $\varepsilon = \pm 1$. Then M is said to be a contact semi-Riemannian (or Pseudo-Riemannian) manifold and (M, ϕ, ξ, η, g) is called a semi-Riemannian (or Pseudo-Riemannian) structure.

Takahashi [10] introduced the concept of contact semi-Riemannian manifolds. Semi-Riemannian contact metric manifolds have been studied by Calvaruso and Perrone ([3],[4]) and Perrone ([7],[8]).

A contact semi-Riemannian manifold is said to be K -contact if ξ is a Killing vector field and Sasakian if it is normal.

A semi-Riemannian manifold is said to be Ricci semi-symmetric if $R(X, Y).S = 0$, where $R(X, Y)$ is the curvature operator and S denotes the Ricci tensor. In [9] Tanno studied Ricci symmetric ($\nabla S = 0$) K -contact Riemannian manifolds. Also Yildiz and Ata[11] studied K -contact Riemannian manifolds. The paper is organized as follows:

After preliminaries in section 3, we extend the Tanno's result for semi-Riemannian case.

In a recent paper [7], Perrone showed that a conformally flat K -contact semi-Riemannian manifold is Sasakian. In section 4 we generalize the result for a K -contact semi-Riemannian manifold of harmonic conformal curvature tensor. In section 5, we have shown that any ξ -conformally flat semi-Riemannian K -contact manifold is Sasakian. Finally, we consider conformally semisymmetric Lorentzian K -contact manifolds.

2. Preliminaries

A contact semi-Riemannian manifold is Sasakian if it is normal, that is,

$$[\phi X, \phi X] + 2d\eta(X, X)\xi = 0. \tag{2.1}$$

This condition is equivalent to

$$(\nabla_X \phi)Y = g(X, Y)\xi - \varepsilon\eta(Y)X. \tag{2.2}$$

In a contact semi-Riemannian manifold the following relations hold ([7],[8]):

$$\eta(\xi) = 1, \varepsilon g(X, \xi) = \eta(X), \tag{2.3}$$

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon \eta(X) \eta(Y), \quad (2.5)$$

$$\nabla_X \xi = \varepsilon \phi X, \quad (2.6)$$

where ∇ is the Levi-Civita connection.

Also in a K -contact semi-Riemannian manifold ξ satisfies the following relation (see [6]):

$$Q\xi = 2n\varepsilon\xi, \quad (2.7)$$

that is, ξ is an eigen vector of the Ricci operator Q defined by [7]

$$S(\xi, X) = g(Q\xi, X) = 2n\varepsilon g(\xi, X) = 2n\varepsilon \eta(X), \quad (2.8)$$

where S denotes the Ricci tensor.

Also the curvature tensor R satisfies

$$R(X, \xi)\xi = \phi^2 X. \quad (2.9)$$

Further, since ξ is Killing in a K -contact manifold, S and r remain invariant under it.

Consequently,

$$\mathcal{L}_\xi S = 0 \quad (2.10)$$

and

$$\mathcal{L}_\xi r = 0 \quad (2.11)$$

where \mathcal{L} denotes the Lie derivation.

We state the following:

Lemma 2.1. ([7],[8]) *Let (M, η, g, ξ, ϕ) be a K -contact semi-Riemannian manifold. Then M is Sasakian if and only if the curvature tensor R satisfies*

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X. \quad (2.12)$$

Definition 2.2. *A contact semi-Riemannian manifold is said to be η -Einstein if*

$$S(X, Y) = a g(X, Y) + b \eta(X) \eta(Y), \quad (2.13)$$

where a, b are smooth functions.

Remark 2.3. *The basic difference between the Riemannian and semi-Riemannian geometry is the existence of a null vector; that is, a vector v satisfying $g(v, v) = 0$, where g is the metric tensor. The signature of the metric g of a Riemannian manifold is $(+, +, +, \dots, +, +)$ and of a semi-Riemannian manifold is $(-, -, -, \dots, +, +)$. There are differences of the basic results of K -contact Riemannian and semi-Riemannian manifolds due the presence of ε .*

3. Ricci semisymmetric K -contact semi-Riemannian manifolds

It is known that [9] a K -contact Riemannian metric manifold is an Einstein manifold if and only if it is Ricci-symmetric. In this section, we prove a similar result for the semi-Riemannian case.

Theorem 3.1. *A K -contact semi-Riemannian manifold is Ricci semi-symmetric if and only if the manifold is an Einstein manifold.*

Proof. Let M be a Ricci semi-symmetric K -contact semi-Riemannian manifold. Then

$$R.S = 0,$$

which implies

$$S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0. \quad (3.1)$$

Putting $Y = U = \xi$ in (3.1) we obtain

$$S(R(X, \xi)\xi, V) + S(\xi, R(X, \xi)V) = 0. \quad (3.2)$$

Using (2.4) and (2.9) in (3.2) yields

$$S(-X + \eta(X)\xi, V) + S(\xi, R(X, \xi)V) = 0,$$

which implies

$$-S(X, V) + \eta(X)S(\xi, V) + S(\xi, R(X, \xi)V) = 0. \quad (3.3)$$

Interchanging X and V in (3.3) we obtain

$$-S(V, X) + \eta(V)S(\xi, X) + S(\xi, R(V, \xi)X) = 0. \quad (3.4)$$

Subtracting (3.3) from (3.4) it follows that

$$\eta(X)S(\xi, V) - \eta(V)S(\xi, X) + S(\xi, R(X, \xi)V) + S(R(V, \xi)\xi, X) = 0. \quad (3.5)$$

Putting $V = \xi$ in (3.5) we obtain

$$\eta(X)S(\xi, \xi) - \eta(\xi)S(\xi, X) - S(\xi, R(\xi, X)\xi) = 0.$$

Using (2.3) and (2.9) in the above equation yields

$$\eta(X)S(\xi, \xi) - S(\xi, X) - S(\xi, X) + \eta(X)S(\xi, \xi) = 0,$$

which implies

$$S(X, \xi) = 2n\eta(X). \quad (3.6)$$

Consequently from (3.2) we get

$$-S(X, V) + 2n\eta(X)\eta(V) + 2n\epsilon g(R(X, \xi)V, \xi) = 0. \quad (3.7)$$

It follows that

$$S(X, V) + 2n\eta(X)\eta(V) - 2n\epsilon g(-X + \eta(X)\xi, V) = 0,$$

which in terms implies that

$$S(X, V) = 2n\epsilon g(X, V), \quad \text{where } \epsilon = g(X_i, X_i).$$

Converse part follows easily. \square

In view of Theorem 3.1 and the fact that Ricci symmetric ($\nabla S = 0$) implies Ricci-semisymmetric ($R.S = 0$), we may conclude the following:

Theorem 3.2. *In a K-contact semi-Riemannian manifold M the following conditions are equivalent*

- (i) *M is an Einstein manifold*
- (ii) *M is Ricci-symmetric*
- (iii) *M is Ricci semi-symmetric.*

4. K-Contact Semi-Riemannian Manifolds With Harmonic Conformal Curvature Tensor

This section deals with a K-contact Semi-Riemannian manifold with $\text{div}C = 0$.

Definition 4.1. *The conformal curvature tensor is given by*

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y],$$

where Q is the Ricci operator and r is the scalar curvature.

It is known that ([5],[6])

$$(\text{div}C)(X, Y)Z = \frac{2(n-1)}{2n-1} \{(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) - \frac{1}{4n} [g(Y, Z)dr(X) - g(X, Z)dr(Y)]\},$$

where 'div' denotes divergence.

Definition 4.2. *A Riemannian or a semi-Riemannian manifold is said to be of harmonic conformal curvature tensor if*

$$(\text{div}C)(X, Y)Z = 0,$$

where 'div' denotes divergence.

We prove the following:

Theorem 4.3. *A K-contact semi-Riemannian manifold is of harmonic conformal curvature if and only if the manifold is an Einstein manifold.*

Proof. Let M be a semi-Riemannian K -contact manifold satisfying $divC = 0$. It is known that $divC = 0$ implies

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{1}{4n} [g(Y, Z)dr(X) - g(X, Z)dr(Y)]. \tag{4.1}$$

From (2.10) and (2.11) it follows that

$$(\nabla_\xi S)(Y, Z) = -S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi). \tag{4.2}$$

$$dr(\xi) = 0. \tag{4.3}$$

Consequently putting $X = \xi$ in (4.1) we get

$$(\nabla_\xi S)(Y, Z) - (\nabla_Y S)(\xi, Z) = \frac{1}{4n} [g(Y, Z)dr(\xi) - g(\xi, Z)dr(Y)]. \tag{4.4}$$

Thus from (4.2) and (4.3) it follows that

$$-S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi) - (\nabla_Y S)(\xi, Z) = -\frac{1}{4n} g(\xi, Z)dr(Y). \tag{4.5}$$

That is,

$$-S(Y, \nabla_Z \xi) - \nabla_Y S(\xi, Z) + S(\xi, \nabla_Y Z) = -\frac{1}{4n} \varepsilon \eta(Z)dr(Y). \tag{4.6}$$

Using (2.6) and (2.8) in (4.6) we get,

$$\varepsilon S(Y, \phi Z) + 2n \nabla_Y \eta(Z) - 2n \eta(\nabla_Y Z) = \frac{\varepsilon}{4n} \eta(Z)dr(Y), \tag{4.7}$$

which implies

$$\varepsilon S(Y, \phi Z) + 2n(\nabla_Y \eta)Z = \frac{\varepsilon}{4n} \eta(Z)dr(Y). \tag{4.8}$$

Hence

$$S(Y, \phi Z) + 2n\varepsilon(\nabla_Y \eta)(Z) = \frac{1}{4n} \eta(Z)dr(Y). \tag{4.9}$$

We note that

$$\begin{aligned} (\nabla_Y \eta)(Z) &= \nabla_Y \eta(Z) - \eta(\nabla_Y Z) \\ &= \varepsilon \nabla_Y g(Z, \xi) - \varepsilon g(\nabla_Y Z, \xi) \\ &= \varepsilon g(Z, \nabla_Y \xi) \\ &= \varepsilon g(Z, \varepsilon \phi Y) \\ &= g(Z, \phi Y). \end{aligned} \tag{4.10}$$

So from (4.10) it follows that

$$S(Y, \phi Z) + 2n\varepsilon g(Z, \phi Y) = \frac{1}{4n} \eta(Z)dr(Y). \tag{4.11}$$

Putting $Z = \phi Z$ in (4.11) we get

$$S(Y, \phi^2 Z) + 2n\varepsilon g(\phi Z, \phi Y) = \frac{1}{4n} \eta(\phi Z)dr(Y). \tag{4.12}$$

Again using (2.4) and (2.5) in (4.12) we get

$$S(Y, -Z + \eta(Z)\xi) + 2n\varepsilon g(Z, Y) - 2n\eta(Z)\eta(Y) = 0. \tag{4.13}$$

That is,

$$-S(Y, Z) + \eta(Z)S(Y, \xi) + 2n\varepsilon g(Z, Y) - 2n\eta(Z)\eta(Y) = 0. \tag{4.14}$$

Using (2.8) in (4.14), we obtain

$$S(Y, Z) = 2n\varepsilon g(Y, Z), \quad \text{where } \varepsilon = g(X_i, X_i). \tag{4.15}$$

Conversely, if the K -contact manifold is an Einstein manifold, then it can be easily seen that the conformal curvature tensor is harmonic.

This completes the proof. \square

Since $\nabla C = 0$ (conformally symmetric) implies $divC = 0$, therefore we can state the following:

Corollary 4.4. *A conformally symmetric K -contact semi-Riemannian manifold is an Einstein manifold.*

Note: A conformally symmetric K -contact Riemannian manifold has been studied by Zhen [13].

5. ξ -Conformally Flatness

ξ -conformally flat K -contact manifolds have been studied by Zhen et al. Since at each point $p \in M^n$ the tangent space $T_p(M^n)$ can be decomposed into the direct sum $T_p(M^n) = \phi(T_p(M^n)) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the one-dimensional linear subspace of $T_p(M^n)$ generated by ξ_p , the conformal curvature tensor C is a map

$$C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \rightarrow \phi(T_p(M^n)) \oplus \{\xi_p\}$$

Definition 5.1. [12] A K -contact semi-Riemannian manifold is said to be ξ conformally flat if the projection of the image of C onto $\{\xi_p\}$ is zero, that is, $C(X, Y)\xi = 0$, where C is the conformal curvature tensor.

Proposition 5.2. A ξ -conformally flat K -contact semi-Riemannian manifold is an η -Einstein manifold.

Proof. Let M^{2n+1} be a ξ -conformally flat K -contact semi-Riemannian manifold. Then $C(X, Y)\xi = 0$ implies that

$$R(X, Y)\xi - \frac{1}{(2n-1)}[S(Y, \xi)X - S(X, \xi)Y + g(Y, \xi)QX - g(X, \xi)QY] + \frac{r}{2n(2n-1)}[g(Y, \xi)X - g(X, \xi)Y] = 0. \quad (5.1)$$

Putting $Y = \xi$ in (5.1) and using (2.3), (2.8) and (2.9) we obtain,

$$-X + \eta(X)\xi = \frac{1}{2n-1}[2nX - 2n\eta(X)\xi + \varepsilon QX - 2n\eta(X)\xi] - \frac{r\varepsilon}{2n(2n-1)}[X - \eta(X)\xi],$$

which implies

$$QX = \frac{1}{\varepsilon}\{1 - 4n + \frac{r\varepsilon}{2n}\}X + \frac{1}{\varepsilon}\{6n - 1 - \frac{r\varepsilon}{2n}\}\eta(X)\xi. \quad (5.2)$$

That is,

$$S(X, Y) = a g(X, Y) + b \eta(X)\eta(Y), \quad (5.3)$$

where

$$a = \frac{1}{\varepsilon}\{1 - 4n + \frac{r\varepsilon}{2n}\} \quad (5.4)$$

and

$$b = \frac{1}{\varepsilon}\{6n - 1 - \frac{r\varepsilon}{2n}\}. \quad (5.5)$$

Consequently M is η -Einstein. \square

Theorem 5.3. A ξ -conformally flat K -contact manifold is Sasakian.

Proof. In view of (5.3) and (2.3) we obtain the following:

$$QX = aX + b\eta(X)\xi, \quad (5.6)$$

$$S(X, \xi) = a g(X, \xi) + b\varepsilon\eta(X) \quad (5.7)$$

and

$$r = (2n+1)a + b. \quad (5.8)$$

From (5.4) and (5.5), it follows that

$$a + b = \frac{2n}{\varepsilon} \quad (5.9)$$

Using (2.3), (5.6), (5.7) and (5.8) in (5.1) we have

$$R(X, Y)\xi = \left[\frac{a}{2n\varepsilon} - \frac{b}{2n(2n-1)\varepsilon} + \frac{b\varepsilon}{2n-1} \right] \{\eta(Y)X - \eta(X)Y\}. \quad (5.10)$$

Using (5.9) in (5.10) we obtain

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X. \quad (5.11)$$

Therefore from Lemma 2.1 we conclude the theorem. \square

Since conformally flatness implies ξ -conformally flat, therefore we obtain the following:

Corollary 5.4. A conformally flat K -contact semi-Riemannian manifold is Sasakian.

The above corollary has been proved by Perrone in his paper [7].

6. Conformally semisymmetric Lorentzian K -contact manifolds

The Ricci tensor S_L of η -Einstein Lorentzian K -contact structure (η, g_L) is given by [7]

$$S_L = \left(\frac{r_L}{2n} + 1\right)g_L + \left(\frac{r_L}{2n} + 2n + 1\right)\eta \otimes \eta, \quad (6.1)$$

where the scalar curvature $r_L = r + 4n$ is a constant when $n > 1$, and g_L is Einstein if and only if $r_L = -2n(2n + 1)$. From (6.1)

$$S_L(X, Y) = \left(\frac{r_L}{2n} + 1\right)g_L(X, Y) + \left(\frac{r_L}{2n} + 2n + 1\right)\eta(X)\eta(Y) = Ag_L(X, Y) + B\eta(X)\eta(Y), \quad (6.2)$$

where $A = \left(\frac{r_L}{2n} + 1\right)$, $B = \left(\frac{r_L}{2n} + 2n + 1\right)$ are constants.

The conformal curvature tensor C is given by

$$C_L(X, Y)Z = R_L(X, Y)Z - \frac{1}{2n-1}[g_L(Y, Z)Q_L(X) - g_L(X, Z)Q_L(Y) + S_L(Y, Z)X - S_L(X, Z)Y] + \frac{r_L}{2n(2n-1)}[g_L(Y, Z)X - g_L(X, Z)Y], \quad (6.3)$$

where Q_L is the Ricci operator defined by $g_L(QX, Y) = S_L(X, Y)$, for all vector fields X, Y . Using (6.2) in (6.3) we get

$$\begin{aligned} C_L(X, Y)Z &= R_L(X, Y)Z - \frac{1}{2n-1}[g_L(Y, Z)\left\{\left(\frac{r_L}{2n} + 1\right)X \right. \\ &\quad \left. + \left(\frac{r_L}{2n} + 1 + 2n\right)\eta(X)\xi\right\} \\ &\quad - g_L(X, Z)\left\{\left(\frac{r_L}{2n} + 1\right)Y + \left(\frac{r_L}{2n} + 1 + 2n\right)\eta(Y)\xi\right\} \\ &\quad + \left\{\left(\frac{r_L}{2n} + 1\right)g_L(Y, Z) + \left(\frac{r_L}{2n} + 1 + 2n\right)\eta(Y)\eta(Z)\right\}X \\ &\quad - \left\{\left(\frac{r_L}{2n} + 1\right)g_L(X, Z) + \left(\frac{r_L}{2n} + 1 + 2n\right)\eta(X)\eta(Z)\right\}Y] \\ &\quad + \frac{r_L}{2n(2n-1)}[g_L(Y, Z)X - g_L(X, Z)Y], \end{aligned} \quad (6.4)$$

from which it follows that

$$\begin{aligned} C_L(X, Y)Z &= R_L(X, Y)Z - \frac{1}{2n-1}[2\left(\frac{r_L}{2n} + 1\right)g_L(Y, Z)X \\ &\quad - 2\left(\frac{r_L}{2n} + 1\right)g_L(X, Z)Y \\ &\quad + \left(\frac{r_L}{2n} + 2n + 1\right)g_L(Y, Z)\eta(X)\xi - \left(\frac{r_L}{2n} + 1 + 2n\right)g_L(X, Z)\eta(Y)\xi \\ &\quad + \left(\frac{r_L}{2n} + 1 + 2n\right)\eta(Y)\eta(Z)X - \left(\frac{r_L}{2n} + 1 + 2n\right)\eta(X)\eta(Z)Y] \\ &\quad + \frac{r_L}{2n(2n-1)}[g_L(Y, Z)X - g_L(X, Z)Y]. \end{aligned} \quad (6.5)$$

The above equation implies

$$\begin{aligned} C_L(X, Y)Z &= R_L(X, Y)Z - \left\{\frac{2}{2n-1}\left(\frac{r_L}{2n} + 1\right) \right. \\ &\quad \left. - \frac{r_L}{2n(2n-1)}\right\}[g_L(Y, Z)X - g_L(X, Z)Y] \\ &\quad - \frac{1}{2n-1}\left(\frac{r_L}{2n} + 2n + 1\right)[g_L(Y, Z)\eta(X)\xi - g_L(X, Z)\eta(Y)\xi \\ &\quad + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \quad (6.6)$$

Putting $X = Z = \xi$ in (6.6) we infer that

$$C_L(\xi, Y)\xi = R_L(\xi, Y)\xi - \left\{\frac{2}{2n-1}\left(\frac{r_L}{2n} + 1\right) - \frac{r_L}{2n(2n-1)}\right\}[\eta(Y)\xi + Y] - \frac{1}{2n-1}\left(\frac{r_L}{2n} + 2n + 1\right)[\eta(Y)\xi - Y]. \quad (6.7)$$

As we are interested to study conformally semisymmetric Lorentzian K -contact manifolds, therefore

$$(R_L(X, \xi) \cdot C_L)(U, V)W = 0. \quad (6.8)$$

This implies

$$R_L(X, \xi)C_L(U, V)W - C_L(R_L(X, \xi)U, V)W - C_L(U, R_L(X, \xi)V)W - C_L(U, V)R_L(X, \xi)W = 0,$$

for all vector fields U, V, X, W . Putting $U = W = \xi$ in (6.8) we get

$$R_L(X, \xi)C_L(\xi, V)\xi - C_L(R_L(X, \xi)\xi, V)\xi - C_L(\xi, R_L(X, \xi)V)\xi - C_L(\xi, V)R_L(X, \xi)\xi = 0. \quad (6.9)$$

From (6.7) we get

$$C_L(\xi, V)\xi = a\{V - \eta(V)\xi\}, \quad (6.10)$$

where $a = \frac{2r_L + 2n(2n+3)}{2n(2n-1)}$ = constant. Hence

$$R_L(X, \xi)C_L(\xi, V)\xi = (1-a)\{R_L(X, \xi)V + \eta(V)X - \eta(X)\eta(V)\xi\}. \quad (6.11)$$

Similarly,

$$C_L(R_L(X, \xi)\xi, V)\xi = -R_L(X, V)\xi + (2a-1)\{\eta(V)X - \eta(X)V\} \quad (6.12)$$

$$C_L(\xi, R_L(X, \xi)V)\xi = (1-a)\{R_L(X, \xi)V - g(V, X)\xi + \eta(X)\eta(V)\xi\} \quad (6.13)$$

and

$$C_L(\xi, V)R_L(X, \xi)\xi = -R_L(\xi, V)X + a\{g_L(V, X)\xi - \eta(X)V\} + (1-a)\{\eta(X)V - \eta(X)\eta(V)\xi\}. \quad (6.14)$$

Using (6.11), (6.12), (6.13), (6.14) in (6.9) we obtain

$$R_L(X, V)\xi + R_L(\xi, V)X - a\{2\eta(V)X - 3\eta(X)V + 2g_L(X, V)\xi - \eta(X)\eta(V)\xi\} + \{\eta(V)X - \eta(X)V + g_L(X, V)\xi - \eta(X)\eta(V)\xi\} = 0. \quad (6.15)$$

Interchanging X and V in (6.15) we get

$$R_L(V, X)\xi + R_L(\xi, X)V - a\{2\eta(X)V - 3\eta(V)X + 2g_L(V, X)\xi - \eta(V)\eta(X)\xi\} + \{\eta(X)V - \eta(V)X + g_L(V, X)\xi - \eta(V)\eta(X)\xi\} = 0. \quad (6.16)$$

Substraction (6.16) from (6.15) we get

$$R_L(X, V)\xi = \frac{5a-2}{3}[\eta(V)X - \eta(X)V].$$

It is clear that for $a = 1$, $R_L(X, V)\xi = \eta(V)X - \eta(X)V$ and hence Weyl Conformally semi-symmetric Lorentzian η -Einstein K -contact manifold is a Sasakian manifold. Again $a = 1$ is equivalent to $r_L = 4n$. Thus in view of the above we can state the following:

Theorem 6.1. *A conformally semisymmetric Lorentzian K -contact η -Einstein manifold is a Sasakian manifold, provided $r_L = 4n$.*

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