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A Note on Two Classes of ξ-Conformally Flat Almost Kenmotsu Manifolds

Dibakar Dey[1*]

[1] *Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata - 700019, West Bengal, India.*
[*] *Corresponding Author E-mail: deydibakar3@gmail.com*

Abstract

The object of the present paper is to characterize ξ-conformally flat (k,μ)-almost Kenmotsu manifolds and $(k,\mu)'$-almost Kenmotsu manifolds. It is proved that a (k,μ)-almost Kenmotsu manifold is ξ-conformally flat if and only if the manifold is an Einstein manifold. Further it is shown that a $(2n+1)$-dimensional $(k,\mu)'$-almost Kenmotsu manifold is ξ-conformally flat if and only if it is conformally flat. As a consequence of the main results we obtain several corollaries. Finally, we give an example to verify our result.

Keywords: *Almost Kenmotsu manifold; Einstein manifold; Riemannian curvature tensor; Ricci tensor; Weyl conformal curvature tensor.*
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1. Introduction

Let M be a $(2n+1)$-dimensional Riemannian manifold with metric g and let $T(M)$ be the Lie algebra of differentiable vector fields in M. The Ricci operator Q of (M,g) is defined by

$$g(QX,Y) = S(X,Y), \tag{1.1}$$

where S denotes the Ricci tensor of type $(0,2)$ on M and $X,Y \in T(M)$. The Weyl conformal curvature tensor C is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad - g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.2}$$

for $X,Y,Z \in T(M)$, where R and r denote the Riemannian curvature tensor and the scalar curvature of M respectively.
In the present time the study of nullity distributions is a very interesting topic on almost contact metric manifolds. The notion of k-nullity distribution was introduced by Gray [9] and Tanno [14] in the study of Riemannian manifolds (M,g), which is defined for any $p \in M$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_p(M) : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \tag{1.3}$$

for any $X,Y \in T_p(M)$, where $T_p(M)$ denotes the tangent space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1,3)$. Blair, Koufogiorgos and Papantoniou [1] introduced the generalized notion of the k-nullity distribution, named the (k,μ)-nullity distribution on a contact metric manifold (M, ϕ, ξ, η, g), which is defined for any $p \in M$ and $k,\mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu) = \{Z \in T_p(M) : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &\quad + \mu[g(Y,Z)hX - g(X,Z)hY]\}, \end{aligned} \tag{1.4}$$

where $h = \frac{1}{2}£_\xi \phi$ and $£$ denotes the Lie differentiation.
In [4], Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution, another generalized notion of the k-nullity distribution, on an almost Kenmotsu manifold (M,ϕ,ξ,η,g), which is defined for any $p \in M$ and $k,\mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu)' = \{Z \in T_p(M) : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &\quad + \mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \end{aligned} \tag{1.5}$$

Email addresses: **deydibakar3@gmail.com (D. Dey)**

where $h' = h \circ \phi$.
A $(2n+1)$-dimensional differentiable manifold M is said to have a (ϕ,ξ,η)-structure or an almost contact structure, if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying ([2], [3]),

$$\phi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \tag{1.6}$$

where I denote the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (1.6) easily.
If a manifold M with a (ϕ,ξ,η)-structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X, $Y \in T(M)$, then M is said to be an almost contact metric manifold. The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any X, $Y \in T(M)$. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi,\phi] + 2d\eta \otimes \xi$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ [2]. Recently in ([4],[5],[6],[12],[13]), almost contact metric manifold such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and they are called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X, $Y \in T(M)$. It is well known [10] that a $(2n+1)$-dimensional Kenmotsu manifold M is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathscr{D}$ and defined by $\mathscr{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathscr{D}$ is an integrable distribution.
At each point $p \in M$, we have

$$T_p(M) = \phi(T_p(M)) \oplus \{\xi_p\},$$

where $\{\xi_p\}$ is 1-dimensional linear subspace of $T_p(M)$ generated by ξ_p. Then the Weyl conformal curvature tensor C is a map:

$$C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M)) \oplus \{\xi\}.$$

Three particular cases can be considered as follows :
(1) $C : T_p(M) \times T_p(M) \times T_p(M) \to \{\xi\}$, that is, the projection of the image of C in $\phi(T_p(M))$ is zero.
(2) $C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M))$, that is, the projection of the image of C in $\{\xi\}$ is zero.
(3) $C : T_p(M) \times T_p(M) \times T_p(M) \to \{\xi\}$, that is, when C is restricted to $\phi(T_p(M)) \times \phi(T_p(M))$, the projection of the image of C in $\phi(T_p(M))$ is zero, which is equivalent to $\phi^2 C(\phi X, \phi Y)\phi Z = 0$.

Definition 1.1. *[17] A contact metric manifold (M,ϕ,ξ,η,g) is said to be ξ-conformally flat if the linear operator $C(X,Y)$ is an endomorphism of $\phi(T(M))$, that is, if*

$$C(X,Y)\phi(T(M)) \subset \phi(T(M)).$$

Then it is immediately follows that

Proposition 1.2. *[17] On a contact metric manifold (M,ϕ,ξ,η,g), the following conditions are equivalent.*

(a) M *is ξ-conformally flat,*
(b) $\eta(C(X,Y)Z) = 0$,
(c) $\phi^2 C(X,Y)Z = -C(X,Y)Z$,
(d) $C(X,Y)\xi = 0$,

where $X, Y, Z \in T(M)$.

Almost Kenmotsu manifolds have been studied by several authors such as Dileo and Pastore ([4], [5], [6]), De and Mandal ([7], [8], [11]) and many others. In the present paper we like to study ξ-conformally flat almost Kenmotsu manifolds with (k,μ) and $(k,\mu)'$-nullity distributions respectively.
The paper is organized as follows:

In Section 2, we give a brief account on almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity distribution and ξ belonging to the $(k,\mu)'$-nullity distribution. Section 3 deals with ξ-conformally flat almost Kenmotsu manifolds with the characteristic vector field ξ belonging to the (k,μ)-nullity distribution. As a consequence of the main result we obtain several corollaries. Section 4 is devoted to study ξ-conformally flat almost Kenmotsu manifolds with the characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution. Finally, we present an example to verify our results.

2. Almost Kenmotsu manifolds

Let M be a $(2n+1)$-dimensional almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot,\xi)\xi$ on M. The tensor fields l and h are symmetric operators and satisfy the following relations [12]:

$$h\xi = 0, \ l\xi = 0, \ tr(h) = 0, \ tr(h\phi) = 0, \ h\phi + \phi h = 0, \tag{2.1}$$

$$\nabla_X \xi = X - \eta(X)\xi - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{2.2}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.3}$$

$$R(X,Y)\xi = \eta(X)(Y-\phi hY) - \eta(Y)(X-\phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.4}$$

for any vector fields $X,Y \in T(M)$. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anti-commuting with ϕ and $h'\xi = 0$. Also it is clear that ([4], [16])

$$h = 0 \Leftrightarrow h' = 0, \;\; h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{2.5}$$

3. ξ belonging to the (k,μ)-nullity distribution

In this section we study ξ-conformally flat almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity distribution.
From (1.4) we obtain

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{3.1}$$

where $k,\mu \in \mathbb{R}$. Before proving our main results in this section we first state the following:

Lemma 3.1. *[4] Let M be an almost Kenmotsu manifold of dimension $(2n+1)$. Suppose that the characteristic vector field ξ belonging to the (k,μ)-nullity distribution. Then $k=-1$, $h=0$ and M is locally a warped product of an open interval and an almost Kähler manifold.*

In view of Lemma 3.1 it follows from (3.1),

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{3.2}$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{3.3}$$

$$S(X,\xi) = -2n\eta(X), \tag{3.4}$$

$$Q\xi = -2n\xi, \tag{3.5}$$

for any vector fields $X,Y \in T(M)$.

Let us consider the manifold M be ξ-conformally flat, that is,

$$C(X,Y)\xi = 0, \tag{3.6}$$

for any vector fields $X,Y \in T(M)$.
From (1.2) and (3.6), we have

$$\begin{aligned} R(X,Y)\xi &= \frac{1}{2n-1}[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY] \\ &\quad - \frac{r}{2n(2n-1)}[g(Y,\xi)X - g(X,\xi)Y]. \end{aligned} \tag{3.7}$$

Using (3.2) and (3.4), we have from (3.7)

$$\begin{aligned} \eta(X)Y - \eta(Y)X &= \frac{1}{2n-1}[-2n\eta(Y)X + 2n\eta(X)Y + \eta(Y)QX - \eta(X)QY] \\ &\quad - \frac{r}{2n(2n-1)}[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{3.8}$$

Simplifying the above equation, we have

$$\eta(Y)QX - \eta(X)QY = (1+\frac{r}{2n})[\eta(Y)X - \eta(X)Y]. \tag{3.9}$$

Putting $Y=\xi$ in (3.9) and using (3.5), yields

$$QX = (1+\frac{r}{2n})X - (1+2n+\frac{r}{2n})\eta(X)\xi. \tag{3.10}$$

Taking inner product of (3.10) with Y, we get

$$S(X,Y) = (1+\frac{r}{2n})g(X,Y) - (1+2n+\frac{r}{2n})\eta(X)\eta(Y). \tag{3.11}$$

In [4], Dileo and Pastore prove that in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the sectional curvature $K(X,\xi) = -1$. From this we get in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the scalar curvature $r = -2n(2n+1)$.
Thus from (3.11), we obtain

$$S(X,Y) = -2ng(X,Y), \tag{3.12}$$

which implies that the manifold is an Einstein manifold.
Conversely, suppose that the manifold is Einstein. Then we have

$$S(X,Y) = -2ng(X,Y). \tag{3.13}$$

From above, we get

$$QX = -2nX. \tag{3.14}$$

Now putting $Z = \xi$ in (1.2) we obtain

$$\begin{aligned} C(X,Y)\xi &= R(X,Y)\xi - \frac{1}{2n-1}[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX \\ &\quad -g(X,\xi)QY] + \frac{r}{2n(2n-1)}[g(Y,\xi)X - g(X,\xi)Y]. \end{aligned} \tag{3.15}$$

With the help of (3.4), (3.13) and (3.14), the relation (3.15) reduces to

$$C(X,Y)\xi = R(X,Y)\xi + (\eta(Y)X - \eta(X)Y). \tag{3.16}$$

Using (3.2) in the foregoing equation, we obtain

$$C(X,Y)\xi = 0. \tag{3.17}$$

Hence we can state the following:

Theorem 3.2. *An almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution is ξ-conformally flat if and only if the manifold is an Einstein manifold.*

Since conformally flatness implies ξ-conformally flat, hence we obtain the following:

Corollary 3.3. *A conformally flat almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution is an Einstein manifold.*

From (1.2), we get for a conformally flat manifold

$$\begin{aligned} R(X,Y)Z &= \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.18}$$

for $X,Y,Z \in T(M)$, where R and r denote the Riemannian curvature tensor and the scalar curvature of M respectively.
Now using (3.12) in the above expression we get

$$R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y]. \tag{3.19}$$

Conversely, if the manifold is of constant curvature -1, then obviously the manifold is conformally flat.
Thus we arrive to the following:

Corollary 3.4. *An almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution is conformally flat if and only if it is of constant curvature -1.*

The above corollary has been proved by De and Mandal [7].

4. ξ belonging to the $(k,\mu)'$-nullity distribution

In this section we study ξ-conformally flat almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution. Let $X \in \mathscr{D}$ be the eigen vector of h' corresponding to the eigen value λ. Then from (2.5) it is clear that $\lambda^2 = -(k+1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigen spaces related to the non-zero eigen value λ and $-\lambda$ of h', respectively. Before presenting our main theorem we recall some results:

Lemma 4.1. *(Prop. 4.1 and Prop. 4.3 of [4]) Let (M,ϕ,ξ,η,g) be a $(2n+1)$-dimensional almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and Spec$(h') = \{0,\lambda,-\lambda\}$, with 0 as simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi]\oplus[\lambda]'$ and $[\xi]\oplus[-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvature are given by the following:*

(a) $K(X,\xi) = k - 2\lambda$ *if* $X \in [\lambda]'$ *and*
$K(X,\xi) = k + 2\lambda$ *if* $X \in [-\lambda]'$,
(b) $K(X,Y) = k - 2\lambda$ *if* $X,Y \in [\lambda]'$;
$K(X,Y) = k + 2\lambda$ *if* $X,Y \in [-\lambda]'$ *and*
$K(X,Y) = -(k+2)$ *if* $X \in [\lambda]'$, $Y \in [-\lambda]'$,
(c) M^{2n+1} *has constant negative scalar curvature* $r = 2n(k-2n)$.

Lemma 4.2. *(Lemma 3 of [15]) Let (M,ϕ,ξ,η,g) be a $(2n+1)$-dimensional almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution. If $h'\neq 0$, then the Ricci operator Q of M is given by*

$$Q=-2nid+2n(k+1)\eta\otimes\xi-2nh'. \tag{4.1}$$

Moreover, the scalar curvature of M is $2n(k-2n)$.

From (1.5), we have

$$R(X,Y)\xi=k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)h'X-\eta(X)h'Y], \tag{4.2}$$

where $k,\mu\in\mathbb{R}$. Also we get from (4.2)

$$R(\xi,X)Y=k[g(X,Y)\xi-\eta(Y)X]+\mu[g(h'X,Y)\xi-\eta(Y)h'X]. \tag{4.3}$$

Contracting (4.2) over X, we have

$$S(Y,\xi)=2nk\eta(Y). \tag{4.4}$$

Moreover in an almost Kenmotsu manifold with $(k,\mu)'$-nullity distribution, we have

$$\nabla_X\xi=X-\eta(X)\xi+h'X, \tag{4.5}$$

$$(\nabla_X\eta)Y=g(X,Y)-\eta(X)\eta(Y)+g(h'X,Y). \tag{4.6}$$

Let us consider the manifold M be ξ-conformally flat, that is,

$$C(X,Y)\xi=0, \tag{4.7}$$

for any vector fields $X,Y\in T(M)$.
From (1.2) and (4.7), we have

$$\begin{aligned} R(X,Y)\xi &= \frac{1}{2n-1}[S(Y,\xi)X-S(X,\xi)Y+g(Y,\xi)QX-g(X,\xi)QY] \\ &\quad -\frac{r}{2n(2n-1)}[g(Y,\xi)X-g(X,\xi)Y]. \end{aligned} \tag{4.8}$$

Using (4.2) and (4.4), we get from (4.8)

$$\begin{aligned} & k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)h'X-\eta(X)h'Y] \\ = & \frac{1}{2n-1}[2nk\eta(Y)X-2nk\eta(X)Y+\eta(Y)QX-\eta(X)QY] \\ & -\frac{r}{2n(2n-1)}[\eta(Y)X-\eta(X)Y]. \end{aligned} \tag{4.9}$$

Using (4.1), we get from the foregoing equation

$$(\mu+\frac{2n}{2n-1})[\eta(Y)h'X-\eta(X)h'Y]=0. \tag{4.10}$$

Putting $Y=\xi$ in (4.10), we obtain

$$(\mu+\frac{2n}{2n-1})h'X=0. \tag{4.11}$$

Since $h'\neq 0$, we have

$$\mu+\frac{2n}{2n-1}=0. \tag{4.12}$$

From Lemma 4.1, we have $\mu=-2$. Using the value of μ in (4.12), we get $n=1$.
Hence we obtain the following:

Proposition 4.3. *A $(2n+1)$-dimensional ξ-conformally flat almost Kenmotsu manifold with $(k,\mu)'$-nullity distribution reduces to a 3-dimensional almost Kenmotsu manifold.*

Since a 3-dimensional Riemannian manifold is conformally flat, therefore ξ-conformally flat almost Kenmotsu manifold with $(k,\mu)'$-nullity distribution is conformally flat. Conversely, conformally flatness implies ξ-conformally flat. Hence, we obtain the following:

Theorem 4.4. *A $(2n+1)$-dimensional almost Kenmotsu manifold with $(k,\mu)'$-nullity distribution is ξ-conformally flat if and only if it is conformally flat.*

5. Example of a 5-dimensional almost Kenmotsu manifolds

In this section, we construct an example of an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$, which is of constant curvature and is conformally flat. We consider 5-dimensional manifold $M = \{(x,y,z,u,v) \in \mathbb{R}^5\}$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. Let ξ, e_2, e_3, e_4, e_5 be five vector fields in $\mathbb{R}^5$ which satisfies [4]

$$[\xi, e_2] = -2e_2,\ [\xi, e_3] = -2e_3,\ [\xi, e_4] = 0,\ [\xi, e_5] = 0,$$

$[e_i, e_j] = 0$, where $i, j = 2,3,4,5$.
Let g be the Riemannian metric defined by

$$g(\xi,\xi) = g(e_2,e_2) = g(e_3,e_3) = g(e_4,e_4) = g(e_5,e_5) = 1$$

and $g(\xi, e_i) = g(e_i, e_j) = 0$ for $i \neq j$; $i, j = 2,3,4,5$.
Let η be the 1-form defined by $\eta(Z) = g(Z,\xi)$, for any $Z \in T(M)$.
Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(\xi) = 0,\ \phi(e_2) = e_4,\ \phi(e_3) = e_5,\ \phi(e_4) = -e_2,\ \phi(e_5) = -e_3.$$

Using the linearity of ϕ and g, we have
$\eta(\xi) = 1,\ \phi^2(Z) = -Z + \eta(Z)\xi,\ g(\phi Z, \phi U) = g(Z,U) - \eta(Z)\eta(U)$,
for any $Z, U \in T(M)$.
Moreover, $h'\xi = 0$, $h'e_2 = e_2$, $h'e_3 = e_3$, $h'e_4 = -e_4$, $h'e_5 = -e_5$.
The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

Using Koszul's formula we get the following:

$$\nabla_\xi \xi = 0,\ \nabla_\xi e_2 = 0,\ \nabla_\xi e_3 = 0,\ \nabla_\xi e_4 = 0,\ \nabla_\xi e_5 = \xi,$$

$$\nabla_{e_2} \xi = 2e_2,\ \nabla_{e_2} e_2 = -2\xi,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = 0,$$

$$\nabla_{e_3} \xi = 2e_3,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -2\xi,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = 0,$$

$$\nabla_{e_4} \xi = 0,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = 0,\ \nabla_{e_4} e_5 = 0,$$

$$\nabla_{e_5} \xi = 0,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = 0.$$

In view of the above relations we have

$$\nabla_X \xi = -\phi^2 X + h'X,$$

for any $X \in T(M)$. Therefore, the structure (ϕ, ξ, η, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, so that M is an almost Kenmotsu manifold.
By the above results, we can easily obtain the components of the curvature tensor R as follows:

$$R(\xi,e_2)\xi = 4e_2,\ R(\xi,e_2)e_2 = -4\xi,\ R(\xi,e_3)\xi = 4e_3,\ R(\xi,e_3)e_3 = -4\xi,$$

$$R(\xi,e_4)\xi = R(\xi,e_4)e_4 = R(\xi,e_5)\xi = R(\xi,e_5)e_5 = 0,$$

$$R(e_2,e_3)e_2 = 4e_3,\ R(e_2,e_3)e_3 = -4e_2,\ R(e_2,e_4)e_2 = R(e_2,e_4)e_4 = 0,$$

$$R(e_2,e_5)e_2 = R(e_2,e_5)e_5 = R(e_3,e_4)e_3 = R(e_3,e_4)e_4 = 0,$$

$$R(e_3,e_5)e_3 = R(e_3,e_5)e_5 = R(e_4,e_5)e_4 = R(e_4,e_5)e_5 = 0.$$

With the help of the expressions of the curvature tensor we conclude that the characteristic vector field ξ belongs to the $(k,\mu)'$-nullity distribution, with $k = -2$ and $\mu = -2$.
Using the expressions of the curvature tensor R we have

$$R(X,Y)Z = -4[g(Y,Z)X - g(X,Z)Y].$$

From the above equation we obtain

$$S(Y,Z) = -16g(Y,Z), \text{ which implies } r = -80.$$

Now using these values in the expression of the conformal curvature tensor C we get, $C(X,Y)Z = 0$. Hence Theorem 4.1 is verified.

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