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On The Duality Problem for the p-Compact Approximation Property and Its Inheritance to Subspaces

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Abstract

In this paper, for $1 < p < \infty$ we define the v_p and v_p^*-topologies on the space of bounded linear operators between Banach spaces, and by way of these topologies we introduce the properties v_p^*D and Bv_p^*D for the dual space E'. Under the assumption of the property v_p^*D on the dual space E', we obtain a solution of the duality problem for the p-CAP with $2 < p < \infty$. We show that, if M is a closed subspace of a Banach space E such that $M^\perp$ is complemented in the dual space E', then M has the p-CAP (respectively, BCAP) whenever E has the p-CAP (respectively, BCAP) and the dual space M' has the v_p^*D (respectively, Bv_p^*D).

Keywords: *complemented subspace, duality problem, p-compact approximation property*
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1. Introduction

As a stronger form of a relatively compact set Sinha and Karn [19] introduced a relatively p-compact set concept, which was motivated by the well-known Grothendieck's characterization of a relatively compact set [14]. Then it has appeared plenty of papers related to the relatively p-compact set concept in different directions. We mention [1], [2], [3], [8], [9], [11], [12], [13], [16], [18] and [20].

Let $1 \le p \le \infty$. A Banach space E is said to have the p-approximation property (in short, p-AP) if identity map I_E of E can be uniformly approximated by finite rank operators on p-compact sets, i.e., there is a net $(S_\alpha)_\alpha$ of finite rank operators on E such that $S_\alpha \to I_E$ uniformly on p-compact subsets of E [19]. If identity map I_E can be uniformly approximated by compact operators on p-compact subsets of E, i.e., there is a net $(S_\alpha)_\alpha$ of compact operators on E such that $S_\alpha \to I_E$ uniformly on p-compact subsets of E, then E is said to have the p-compact approximation property (in short, p-CAP) [8]. Note that every Banach space has the p-AP for $1 \le p \le 2$ [19, Theorem 6.4]. It is clear that every Banach space with the p-AP has the p-CAP, but the converse is not true in general. Choi and Kim [8, Theorem 5.2] constructed a Banach space having the the p-CAP, which fails to have the p-AP for every $p > 2$.

A Banach space E is said to have the p-weak approximation property (in short, p-WAP) if every compact operator from E to E can be uniformly approximated by finite rank operators on p-compact subsets of E, i.e., for each compact operator $S : E \longrightarrow E$ there is a net $(S_\alpha)_\alpha$ of finite rank operators on E such that $S_\alpha \to S$ uniformly on p-compact subsets of E [9]. Changjing and Xiaochun [9] show that a Banach space E has the p-AP if and only if E has both the p-CAP and p-WAP for $1 \le p \le \infty$. So, by [8, Theorem 5.2] there is a Banach space without the p-WAP for every $p > 2$.

Let $\lambda \ge 1$. A Banach space E is said to have the λ-bounded approximation property (in short, λ-BAP) if there is a net $(S_\alpha)_\alpha$ of finite rank operators on E such that $\|S_\alpha\| \le \lambda$ and $S_\alpha \to I_E$ uniformly on compact subsets of E. If E has the λ-BAP for some λ, then E is said to have the bounded approximation property (in short, BAP)[4], [17]. In this definition if the compact sets are replaced by p-compact sets for any $1 \le p < \infty$, then definition of the p-λ-bounded approximation property (in short, p-λ-BAP) is obtained. On the other hand, it is well known that in the definition of λ-BAP, instead of compact sets, it is enough to take finite sets only (see, e.g., [17, pp. 37]). Since each p-compact set is a compact set, then it follows that the p-λ-BAP is equivalent to the λ-BAP. That is, the p-λ-BAP is nothing more than the λ-BAP for any $1 \le p < \infty$.

A Banach space E is said to have the λ-bounded compact approximation property (in short, λ-BCAP) if there is a net $(S_\alpha)_\alpha$ of compact operators on E such that $\|S_\alpha\| \le \lambda$ and $S_\alpha \to I_E$ uniformly on compact subsets of E. If E has the λ-BCAP for some λ, then E is said to have the bounded approximation property (in short, BCAP) [4]. In this definition if the compact sets are replaced by p-compact sets for any

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$1 \leq p < \infty$, then definition of the p-λ-bounded compact approximation property (in short, p-λ-BCAP) is obtained. But as similar to the above, the p-λ-BCAP is equivalent to the λ-BCAP for any $1 \leq p < \infty$.
In this paper, we get some characterizations of the λ-BAP (respectively, λ-CAP) and the p-CAP. Also, for $1 < p < \infty$ we define the v_p and v_p^*-topologies on the space of bounded linear operators from a Banach space E to E and from the dual space E' to E', respectively. By means of these topologies we introduce the properties v_p^*D and Bv_p^*D for the dual space E'. Under the assumption of the property v_p^*D on the dual space E', we get a solution of the duality problem for the p-CAP, that is, for $2 < p < \infty$ if the dual space E' has the p-CAP and the v_p^*D, then so does E . If M is a closed subspace of a Banach space E such that $M^\perp$ is complemented in the dual space E', then we show that M has the p-AP whenever E has the p-AP, and also we show that M has the p-CAP (respectively, BCAP) whenever E has the p-CAP (respectively, BCAP) and the dual space M' has the v_p^*D (respectively, Bv_p^*D).

2. Notation and preliminaries

The symbols E and F will always denote complex Banach spaces. Let M be a subset of E. The symbol I_M will denote the identity mapping on M, and for any topology τ on E, $\overline{M}^\tau$ will denote the τ-closure of M in E. The symbol B_E represents the closed unit ball of E. The Banach space of all linear continuous operators from E to F with usual operator norm $\|,\|$ is denoted by $L(E,F)$. When $F = \mathbb{C}$ we write E' instead of $L(E,\mathbb{C})$. An operator T in $L(E,F)$ is called compact if $T(B_E)$ is a relatively compact subset of F. The subspace of all compact (respectively, finite rank) operators of $L(E,F)$ is denoted by $K(E,F)$ (respectively, $F(E,F)$). Let $\lambda \geq 1$. The space of all compact (respectively, finite rank) operators with the norm $\leq \lambda$ is denoted by $K^\lambda(E,E)$ (respectively, $F^\lambda(E,E)$). The space of all compact (respectively, finite rank) and *weak**-to-*weak** continuous operators with the norm $\leq \lambda$ is denoted by $K^\lambda_{w^*}(E',E')$ (respectively, $F^\lambda_{w^*}(E',E')$). Let $1 \leq p < \infty$. The symbol $l_p(E)$ (respectively, $l_\infty(E)$) will denote Banach space of all sequences $(x_n)_{n=1}^\infty$ in E with $\sum_{n=1}^\infty \|x_n\|^p < \infty$ (respectively, $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$). The notation $c_0(E)$ will denote Banach space of all null sequences $(x_n)_{n=1}^\infty$ in E. Then a subset K of E is said to be relatively p-compact if there exists a sequence $(x_n)_{n=1}^\infty \in l_p(E)$ $(1 \leq p < \infty)$ $((x_n)_{n=1}^\infty \in c_0(E)$ if $p = \infty)$ such that $K \subset \{\sum_{n=1}^\infty \alpha_n x_n : (\alpha_n)_{n=1}^\infty \in B_{l_q}\}$, where $\frac{1}{p} + \frac{1}{q} = 1$ [19]. Note that the relatively ∞-compact sets are the relatively compact sets and also the relatively p-compact sets are relatively compact [19]. A relatively p-compact and closed set will be called p-compact.
Throughout the paper the notations τ and τ_p denote the topologies of uniform convergence on the compact subsets and p-compact subsets, respectively. Recall that the τ and τ_p are locally convex topologies by generated the family of seminorms [8], [19]. Choi and Kim [8, Proposition 2.2] proved that $(L(E,F),\tau_p)$ is complete for any $1 \leq p \leq \infty$, and gave a representation of the dual space $(L(E,F),\tau_p)'$ for $1 < p < \infty$ [8, Theorem 2.5].

Theorem 2.1. *[8, Theorem 2.5] Let $1 < p < \infty$. Then*

$$(L(E,F),\tau_p)' = \{f : f(S) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j y_j'(Sx_n),\ (y_j')_{j=1}^\infty \subset F',\ (x_n)_{n=1}^\infty \in l_p(E) \text{ and } z_j = (\lambda_n^j)_{n=1}^\infty \in l_q \text{ for each } j \in \mathbb{N} \text{ satisfying } \sum_{j=1}^\infty \|z_j\|_q \|y_j'\| < \infty\}.$$

Changjing and Xiaochun [9] obtained the following characterization of the p-WAP.

Theorem 2.2. *[9] Let E be a Banach space and let $2 < p < \infty$. E has the p-WAP if and only if for every $(x_n)_{n=1}^\infty \in l_p(E)$, $(x_j')_{j=1}^\infty \subset E'$ and $z_j = (\lambda_n^j)_{n=1}^\infty \in l_q$ for every $j \in \mathbb{N}$ with $\sum_{j=1}^\infty \|z_j\|_q \|x_j'\| < \infty$ and $\sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j x_j'(Sx_n) = 0$ for all $S \in F(E,E)$, we have $\sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j x_j'(Sx_n) = 0$ for all $S \in K(E,E)$.*

3. Characterizations of the λ-BAP (respectively, λ-CAP) and the p-CAP

In this section, we will obtain some characterizations of the λ-BAP (respectively, λ-CAP) and the p-CAP. A characterization for the λ-BAP is given by Çalışkan [10]. The following proposition gives another characterization of the λ-BAP (respectively, λ-CAP) and it can be proved easily by using Theorem 2.1.

Proposition 3.1. *Let E be a Banach space and let $\lambda \geq 1$ and $1 < p < \infty$. Then the following are equivalent.*
(a) E has the λ-BAP (respectively, λ-CAP).
(b) For every $c > 0$, every $(x_n)_{n=1}^\infty \in l_p(E)$, $(x_j')_{j=1}^\infty \subset E'$ and $z_j = (\lambda_n^j)_{n=1}^\infty \in l_q$ for each $j \in \mathbb{N}$ with $\sum_{j=1}^\infty \|z_j\|_q \|x_j'\| < \infty$, and satisfying $\left|\sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j x_j'(Sx_n)\right| \leq c$ for every $S \in F^\lambda(E,E)$ (respectively, $S \in K^\lambda(E,E)$), we have $\left|\sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j x_j'(x_n)\right| \leq c$.

Proof. $(a) \Rightarrow (b)$ Assume that E has the λ-BAP. Let $c > 0$, $(x_n)_{n=1}^\infty \subset l_p(E)$, $(x_j')_{j=1}^\infty \subset E'$ and $z_j = (\lambda_n^j)_{n=1}^\infty \in l_q$ for each $j \in \mathbb{N}$ with $\sum_{j=1}^\infty \|z_j\|_q \|x_j'\| < \infty$, such that $\left|\sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j x_j'(Sx_n)\right| \leq c$ for every $S \in F^\lambda(E,E)$. We will show that $\left|\sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j x_j'(x_n)\right| \leq c$, or equivalently, by Theorem 2.1, for a given $\varphi \in (L(E,E),\tau_p)'$ with $|\varphi(S)| \leq c$ for every $S \in F^\lambda(E,E)$, we will show that $|\varphi(I_E)| \leq c$. Indeed, since by hypothesis $I_E \in \overline{F^\lambda(E,E)}^{\tau_p}$, there exists a net $(S_\alpha)_\alpha \subset F^\lambda(E,E)$ such that $S_\alpha \xrightarrow{\tau_p} I_E$. Hence $\varphi(S_\alpha) \longrightarrow \varphi(I_E)$. Since $|\varphi(S_\alpha)| \leq c$ for all α, then $|\varphi(I_E)| = \lim_\alpha |\varphi(S_\alpha)| \leq c$, or $\left|\sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j x_j'(x_n)\right| \leq c$.

$(b) \Rightarrow (a)$ By Theorem 2.1, (b) says that for every $\varphi \in (L(E,E),\tau_p)'$ with $|\varphi(S)| \leq c$ for every $S \in F^{\lambda}(E,E)$, we have $|\varphi(I_E)| \leq c$. We assume, for a contradiction, that $I_E \in (L(E,E),\tau_p) \backslash \overline{F^{\lambda}(E,E)}^{\tau_p}$. Then, by Hahn-Banach separation theorem there exists a $\psi \in (L(E,E),\tau_p)'$ such that $|\psi(I_E)| > \sup\limits_{S \in F^{\lambda}(E,E)} |\psi(S)|$. If we define a functional ϕ by $\phi(S) := \frac{c\psi(S)}{\sup\limits_{S\in F^{\lambda}(E,E)} |\psi(S)|}$ for all $S \in L(E,E)$, then $\phi \in (L(E,E),\tau_p)'$ and $\sup\limits_{S\in F^{\lambda}(E,E)} |\phi(S)| = c$. But $\phi(I_E)| = \frac{c|\psi(I_E)|}{\sup\limits_{S\in F^{\lambda}(E,E)} |\psi(S)|} > c$, which is a contradiction. Thus, the proof for λ-BAP is completed.
The proof for the λ-CAP can be done as similar. □

By using the standard methods and Theorem 2.1 we obtain the following characterization for the p-CAP.

Proposition 3.2. *Let E be a Banach space and let $2 < p < \infty$. Then the following are equivalent.*
(a) E has the p-CAP.
(b) K(E, E) is τ_p-dense in $L(E,E)$.
(c) K(F, E) is τ_p-dense in $L(F,E)$ for every Banach space F.
(d) K(E, F) is τ_p-dense in $L(E,F)$ for every Banach space F.
(e) For every $(x_n)_{n=1}^{\infty} \in l_p(E)$, $(x_j^{'})_{j=1}^{\infty} \subset E^{'}$ and $z_j = (\lambda_n^j)_{n=1}^{\infty} \in l_q$ for each $j \in \mathbb{N}$ with $\sum\limits_{j=1}^{\infty} \|z_j\|_q \|x_j^{'}\| < \infty$, and satisfying $\sum\limits_{j=1}^{\infty}\sum\limits_{n=1}^{\infty} \lambda_n^j x_j^{'}(Sx_n) = 0$ for every $S \in K(E,E)$, we have $\sum\limits_{j=1}^{\infty}\sum\limits_{n=1}^{\infty} \lambda_n^j x_j^{'}(x_n) = 0$.

Proof. It is easy to show that $(a) \Leftrightarrow (b)$, $(b) \Leftrightarrow (c)$ and $(a) \Leftrightarrow (d)$. The proof of $(a) \Leftrightarrow (e)$ can be follow from the proof of [8, Theorem 5.1]. □

4. Some topologies on the space of linear operators

Let $1 < p < \infty$. In this section, by defining two topologies (ν_p and ν_p^*-topologies) on the space of bounded linear operators, we introduce the properties ν_p^*D and Bν_p^*D for the dual space $E^{'}$. We show that E has the p-CAP whenever the dual space $E^{'}$ has the p-CAP and the ν_p^*D $(2 < p < \infty)$. Later, we show that if M is a complemented subspace of a Banach space E, then the pair (E,M) have the three space property for the p-CAP (respectively, p-AP). If M is a closed subspace of a Banach space E such that $M^{\perp}$ is complemented in the dual space $E^{'}$, then we show that M has the p-AP whenever E has the p-AP, and also we show that M has the p-CAP (respectively, BCAP) whenever E has the p-CAP (respectively, BCAP) and the dual space $M^{'}$ has the ν_p^*D (respectively, Bν_p^*D).

Definition 4.1. *(See [6, Definition 2.3] Let $1 < p < \infty$. For a net $(S_\alpha)_\alpha$ and an operator S in $L(E,E)$ it is said to be the net $(S_\alpha)_\alpha$ converges to S according to the ν_p-topology, or $S_\alpha \xrightarrow{\nu_p} S$ iff*

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \lambda_n^j (x_j^{'})(S_\alpha x_n) \to \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \lambda_n^j (x_j^{'})(Sx_n)$$

for every $(x_n)_{n=1}^{\infty} \in l_p(E)$, $(x_j^{'})_{j=1}^{\infty} \subset E^{'}$ and $z_j = (\lambda_n^j)_{n=1}^{\infty} \in l_q$ for each $j \in \mathbb{N}$ satisfying $\sum\limits_{j=1}^{\infty} \|z_j\|_q \|x_j^{'}\| < \infty$.

By Theorem 2.1 we can see that the τ_p-topology on the space $L(E,E)$ is stronger than the ν_p-topology.
By using Theorem 2.2, Proposition 3.1, Proposition 3.2 (e), Definition 4.1 and standard methods, we get easily the following characterizations.

- Let $2 < p < \infty$. E Banach space has the p-AP iff $I_E \in \overline{F(E,E)}^{\nu_p}$.
- Let $1 < p < \infty$. E Banach space has the λ-BAP iff $I_E \in \overline{F^{\lambda}(E,E)}^{\nu_p}$.
- Let $2 < p < \infty$. E Banach space has the p-CAP iff $I_E \in \overline{K(E,E)}^{\nu_p}$.
- Let $1 < p < \infty$. E Banach space has the λ-CAP iff $I_E \in \overline{K^{\lambda}(E,E)}^{\nu_p}$.
- Let $2 < p < \infty$. E Banach space has the p-WAP iff $K(E,E) \subset \overline{F(E,E)}^{\nu_p}$.

Definition 4.2. *(See [6, Definition 2.4]) Let $1 < p < \infty$. For a net $(T_\alpha)_\alpha$ and an operator T in $L(E^{'},E^{'})$ it is said to be the net $(T_\alpha)_\alpha$ converges to T according to the ν_p^*-topology, or $T_\alpha \xrightarrow{\nu_p^*} T$ iff*

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \lambda_n^j (T_\alpha x_j^{'})(x_n) \to \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \lambda_n^j (T x_j^{'})(x_n)$$

for every $(x_n)_{n=1}^{\infty} \in l_p(E)$, $(x_j^{'})_{j=1}^{\infty} \subset E^{'}$ and $z_j = (\lambda_n^j)_{n=1}^{\infty} \in l_q$ for each $j \in \mathbb{N}$ satisfying $\sum\limits_{j=1}^{\infty} \|z_j\|_q \|x_j^{'}\| < \infty$.

Remark 4.3. *For any $1 < p < \infty$, on the space $L(E^{'},E^{'})$ the ν_p^*-topology is weaker than the ν_p-topology. If E is a reflexive Banach space, then these topologies coincide. Also, we denote that for S and a net $(S_\alpha)_\alpha$ in $L(E,E)$*

$$S_\alpha \xrightarrow{\nu_p} S \text{ iff } S_\alpha^{'} \xrightarrow{\nu_p^*} S^{'}.$$

Choi and Kim [6, Definiton 2.5] introduced the properties weak* density (in short, W*D) and bounded weak* density (in short, BW*D) for compact operators on the dual space $E^{'}$. Similar to these properties we introduce the following notions.

Definition 4.4. *Let E be a Banach space and let $1<p<\infty$.*
(a) If $K(E',E')\subset\overline{K_{w^*}(E',E')}^{v_p^*}$, then E' is said to have the v_p^*D.
(b) If $K^1(E',E')\subset\overline{K_{w^*}^{\lambda}(E',E')}^{v_p^*}$ for some $\lambda>0$, then E' is said to have the Bv_p^*D.

It is well known that the τ-topology is stronger than the τ_p-topology [8]. By this property and Remark 4.3, we obtain the following lemma due to Lindenstrauss and Tzafriri [17] and Choi and Kim [7], which will be used in the proofs of Proposition 4.7 and Theorem 4.12.

Lemma 4.5. *(See [17, Lemma 1.e.17], [7, Lemma 3.11]) Let E be a Banach space and let $1<p<\infty$. Then the following are satisfied.*
(a) $F(E',E')\subset\overline{F_{w^*}(E',E')}^{\tau_p}\subset\overline{F_{w^*}(E',E')}^{v_p^*}$.
(b) $F^{\lambda}(E',E')\subset\overline{F_{w^*}^{\lambda}(E',E')}^{\tau_p}\subset\overline{F_{w^*}^{\lambda}(E',E')}^{v_p^*}$ for all $\lambda>0$.

Remark 4.6. *Let $2<p<\infty$. Choi and Kim [8, Theorem 2.7] showed that if the dual E' of a Banach space E has p-AP, then E has the p-AP. The proof of this theorem can be shortened by using Remark 4.3 and Lemma 4.5. Actually, if E' has p-AP, then $I_{E'}\in\overline{F(E',E')}^{\tau_p}$. By Lemma 4.5 (a), $I_{E'}\in\overline{F_{w^*}(E',E')}^{v_p^*}$. Therefore, by Remark 4.3 $I_E\in\overline{F(E,E)}^{v_p}$ which shows that E has the p-AP.*

By modification [6, Proposition 2.7] we get the following proposition.

Proposition 4.7. *For a Banach space E, we have the following statements.*
(a) If E' is reflexive, then E' has the v_p^*D and Bv_p^*D. But, the conserve is not true in general.
(b) If E' has the p-WAP, then E' has the v_p^*D.
(c) If E' has the BAP, then E' has the Bv_p^*D.

Proof. Since the proof is similar to the proof of [6, Proposition 2.7], it is omitted. □

The duality problem for the CAP are not resolved yet (see [4, Problem 8.5]), but Choi and Kim [6, Theorem 3.1] have solved this problem under the extra assumption. However, the duality problem for the p-AP has a positive solution with $2<p<\infty$ [8, Theorem 2.7]. We will show in the following theorem that under extra assumption on the dual space, the duality problem for the p-CAP has a positive solution with $2<p<\infty$.

Theorem 4.8. *E has the p-CAP whenever the dual space E' has the p-CAP and the v_p^*D.*

Proof. Suppose that the dual space E' has the p-CAP and the v_p^*D, then

$$I_{E'}\in\overline{K(E',E')}^{v_p}\ \text{ and }\ K(E',E')\subset\overline{K_{w^*}(E',E')}^{v_p^*}.$$

By Remark 4.3, since the v_p-topology is stronger than the v_p^*-topology on the $L(E',E')$, we have $I_{E'}\in\overline{K_{w^*}(E',E')}^{v_p^*}$. Thus $I_E\in\overline{K(E,E)}^{v_p}$. This shows that E has the p-CAP. □

As a result of Proposition 4.7 (a) and Theorem 4.8, we can say that the duality problem of the p-CAP for reflexive Banach spaces has a positive solution.

Corollary 4.9. *Let E be a reflexive Banach space and let $2<p<\infty$. If E' has the p-CAP, then E has the p-CAP.*

The following theorem will be important in order to show that existence of a Banach space without the Bv_p^*D.

Theorem 4.10. *E has the BCAP whenever the dual space E' has the BCAP and the Bv_p^*D.*

Proof. If the dual space E' has the BCAP and the Bv_p^*D, then

$$I_{E'}\in\overline{K^{\lambda}(E',E')}^{v_p}\ \text{ and }\ K^1(E',E')\subset\overline{K_{w^*}^{\mu}(E',E')}^{v_p^*}$$

for some λ and $\mu>0$. On the other hand, $K^{\lambda}(E',E')\subset\overline{K_{w^*}^{\lambda\mu}(E',E')}^{v_p^*}$. Since $I_{E'}\in\overline{K^{\lambda}(E',E')}^{v_p^*}$, we have $I_{E'}\in\overline{K_{w^*}^{\lambda\mu}(E',E')}^{v_p^*}$. Thus, by Remark 4.3 we obtain $I_E\in\overline{K^{\lambda\mu}(E,E)}^{v_p}$, which proves that E has the BCAP. □

It is well known that there exists a Banach space E such that E has not the BCAP whenever the dual space E' has the BCAP [5, Theorem 2.5]. So, by Theorem 4.10 E cannot have the Bv_p^*D. However, it is not known whether every the dual space E' has the v_p^*D or not.
By a modification [6, Proposition 4.1] we get the solution of there space problems for p-CAP (respectively, p-AP) in terms of complemented subspace of a Banach space.

Proposition 4.11. *Let E be a Banach space and M be a closed subspace of E. If M is complemented in E, then the pair (E,M) have the there space property for the p-CAP (respectively, p-AP).*

Proof. Let M be a complemented subspace of E. Then there exists an onto projection $P_1 : E \to M$. Let $i_1 : M \hookrightarrow E$ be the inclusion mapping. First we will show that M has the p-CAP whenever E has the p-CAP. Since E has the p-CAP, there exists $(S_\alpha)_\alpha \subset K(E,E)$ such that $S_\alpha \xrightarrow{\nu_p} I_E$. Let us define $T_\alpha := P_1 S_\alpha i_1$, so that $(T_\alpha)_\alpha \subset K(M,M)$. If $(m_n)_{n=1}^\infty \in l_p(M)$, $(m_j^{'})_{j=1}^\infty \subset M^{'}$ and $z_j = (\lambda_n^j)_{n=1}^\infty \in l_q$ for each $j \in \mathbb{N}$ with $\sum_{j=1}^\infty \|z_j\|_q \|m_j^{'}\| < \infty$, then

$$\sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j m_j^{'}(T_\alpha m_n) \to \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j (m_j^{'} P_1)(i_1 m_n) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j m_j^{'}(m_n).$$

Since $\sum_{j=1}^\infty \|z_j\|_q \|m_j^{'} P_1\| < \infty$ and $(i_1 m_n)_{n=1}^\infty \in l_p(M)$, thus $T_\alpha \xrightarrow{\nu_p} I_M$ and M has the p-CAP.

Now, we will show that E/M has the p-CAP whenever E has the p-CAP. Since M is a complemented subspace, there is a closed subpace N of E such that N is complementary of M and the spaces E/M and N are isomorphic. By the above argument, we know that every complemented subspace of E has the p-CAP. Thus since N has the p-CAP, E/M has the p-CAP.

Finally, we will show that E has the p-CAP whenever the spaces M and E/M have the p-CAP. Note that E is the direct sum of M and N (where, the spaces E/M and N are isomorphic). Hence there is an onto projection $P_2 : E \to N$ and an inclusion $i_2 : N \hookrightarrow E$. Let K be a given p-compact subset of E and let $\varepsilon > 0$. Since M and N have the p-CAP, there exist $R_1 \in K(M,M)$ and $R_2 \in K(N,N)$ such that

$$\|R_1 P_1 x - P_1 x\| < \varepsilon \text{ and } \|R_2 P_2 x - P_2 x\| < \varepsilon$$

for all $x \in K$. Let $Tx := i_1 R_1 P_1 x + i_2 R_2 P_2 x$ for all $x \in E$. Thus $T \in K(E,E)$ and

$$\|Tx - x\| = \|i_1(R_1 P_1 x - P_1 x) + i_2(R_2 P_2 x - P_2 x)\| < 2\varepsilon$$

for all $x \in K$. Then E has the p-CAP. □

Now let M be a closed subspace of E. It is known that if M is a complemented subspace of E, then so is $M^\perp$ in $E^{'}$. But the converse, in general, is not true. So if we change the hypothesis of Proposition 4.11 with $M^\perp$ is complemented in $E^{'}$, by a modification [6, Theorem 4.2] we get the following proposition, which gives conditions for the subspace M to have the p-AP, the p-CAP and the BCAP.

Theorem 4.12. *Let E be a Banach space with a closed subspace M such that $M^\perp$ is complemented in $E^{'}$.*
(a) M has the p-AP whenever E has the p-AP.
(b) M has the p-CAP whenever E has the p-CAP and $M^{'}$ has the ν_p^*D.
(c) M has the BCAP whenever E has the BCAP and $M^{'}$ has the Bν_p^*D.

Proof. Since $M^\perp$ is a complemented subspace of $E^{'}$, there exists an onto projection $P : E^{'} \to M^\perp$. Let $i : M \hookrightarrow E$ be the inclusion mapping. Define the bounded linear operator U from $M^{'}$ to $E^{'}$ by the formula $U(m^{'}) = x^{'} - Px^{'}$, where $x^{'} \in E^{'}$ with $x^{'}(m) = m^{'}(m)$ for all $m \in M$. Note that $(Um^{'})m = m^{'}(m)$ for all $m^{'} \in M^{'}$ (see, [15, Lemma 3.6]).

(a) Since E has the p-AP, there exists a net $(S_\alpha)_\alpha$ in $F(E,E)$ such that $S_\alpha \xrightarrow{\nu_p} I_E$. By Remark 4.3 $S_\alpha^{'} \xrightarrow{\nu_p^*} I_E^{'}$. On the other hand, $i^{'} S_\alpha^{'} U \in F(M^{'},M^{'})$ and if $(m_n)_{n=1}^\infty \in l_p(M)$, $(m_j^{'})_{j=1}^\infty \subset M^{'}$ and $z_j = (\lambda_n^j)_{n=1}^\infty \in l_q$ for each $j \in \mathbb{N}$ satisfying $\sum_{j=1}^\infty \|z_j\|_q \|m_j^{'}\| < \infty$, then $\sum_{j=1}^\infty \|z_j\|_q \|Um_j^{'}\| < \infty$ and since $S_\alpha^{'} \xrightarrow{\nu_p^*} I_E^{'}$, we have

$$\sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j (i^{'} S_\alpha^{'} U m_j^{'})(m_n) \to \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j (I_E^{'} U m_j^{'})(m_n) = \sum_{j=1}^\infty \sum_{n=1}^\infty \lambda_n^j m_j^{'}(m_n).$$

Thus $I_M^{'} \in \overline{F(M^{'},M^{'})}^{\nu_p^*}$. By Lemma 4.5 (a), we have $I_M^{'} \in \overline{F_{w^*}(M^{'},M^{'})}^{\nu_p^*}$. Hence by Remark 4.3 $I_M \in \overline{F(M,M)}^{\nu_p}$. This proves that M has the p-AP.

(b) Suppose that E has the p-CAP and $M^{'}$ has the ν_p^*D. Then there exists a net $(S_\alpha)_\alpha$ in $K(E,E)$ such that $S_\alpha \xrightarrow{\nu_p} I_E$. By Remark 4.3, $S_\alpha^{'} \xrightarrow{\nu_p^*} I_E^{'}$. On the other hand, $i^{'} S_\alpha^{'} U \in K(M^{'},M^{'})$. Thus, similar to (a) we get that $I_M^{'} \in \overline{K(M^{'},M^{'})}^{\nu_p^*}$. By hypothesis, since $M^{'}$ has the ν_p^*D, $I_M^{'} \in \overline{K_{w^*}(M^{'},M^{'})}^{\nu_p^*}$, and hence $I_M \in \overline{K(M,M)}^{\nu_p}$, which shows that M has the p-CAP.

(c) Suppose that E has the BCAP and $M^{'}$ has the Bν_p^*D. Then $I_E \in \overline{K^\lambda(E,E)}^{\nu_p}$ and $K^1(M^{'},M^{'}) \subset \overline{K_{w^*}^\mu(M^{'},M^{'})}^{\nu_p^*}$ for some $\mu > 0$. Hence, by the method given in the proof of (b) we get $i^{'} S_\alpha^{'} U \in K(M^{'},M^{'})$ such that

$$\|i^{'} S_\alpha^{'} U\| \le \lambda \|U\|.$$

Then $I_M^{'} \in \overline{K_{w^*}^{\mu\lambda\|U\|}(M^{'},M^{'})}^{\nu_p^*}$, or equivalently $I_M \in \overline{K^{\mu\lambda\|U\|}(M,M)}^{\nu_p}$. This proves that M has the BCAP. □

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