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On $\mathscr{I}_2$-Convergence and $\mathscr{I}_2^*$-Convergence of Double Sequences in Fuzzy Normed Spaces

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Abstract

In this paper first, we investigate some properties of $\mathscr{I}_2$-convergence in fuzzy normed spaces. After, we study some relationships between $\mathscr{I}_2$-convergence and $\mathscr{I}_2^*$-convergence of double sequences in fuzzy normed spaces.

Keywords: *Double sequences, $\mathscr{I}_2$-convergence, $\mathscr{I}_2$-Cauchy, Fuzzy normed space.*
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1. Introduction and Background

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [11] and Schoenberg [30]. A lot of developments have been made in this area after the various studies of researchers [21, 25]. The idea of $\mathscr{I}$-convergence was introduced by Kostyrko et al. [15] as a generalization of statistical convergence which is based on the structure of the ideal $\mathscr{I}$ of subset of the set of natural numbers $\mathbb{N}$. Das et al. [3] introduced the concept of $\mathscr{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. A lot of developments have been made in this area after the works of [4, 16, 27, 31].
The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka [18] and proved some basic theorems for sequences of fuzzy numbers. Nanda [23] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers are a complete metric space. Şençimen and Pehlivan [29] introduced the notions of statistically convergent sequence and statistically Cauchy sequence in a fuzzy normed linear space. Hazarika [13] studied the concepts of $\mathscr{I}$-convergence, $\mathscr{I}^*$-convergence and $\mathscr{I}$-Cauchy sequence in a fuzzy normed linear space. Dündar and Talo [8, 9] introduced the concepts of $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy sequence for double sequences of fuzzy numbers and studied some properties and relations of them. Hazarika and Kumar [14] introduced the notion of $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy double sequences in a fuzzy normed linear space. A lot of developments have been made in this area after the various studies of researchers [17, 20, 32, 26].

Now, we recall the concept of ideal, convergence, statistical convergence, ideal convergence of sequence, double sequence and fuzzy normed and some basic definitions (see [1, 2, 5, 6, 7, 8, 10, 11, 12, 19, 20, 21, 22, 24, 25, 28, 29]).
Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0,1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh [33], a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u : X \to [0,1]$. The function u itself is often used for the fuzzy set.
A fuzzy set u on $\mathbb{R}$ is called a fuzzy number if it has the following properties:
1. u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
2. u is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$;
3. u is upper semicontinuous;
4. $supp\, u = cl\{x \in \mathbb{R} : u(x) > 0\}$, or denoted by $[u]_0$, is compact.
Let $L(\mathbb{R})$ be set of all fuzzy numbers. If $u \in L(\mathbb{R})$ and $u(t) = 0$ for $t < 0$, then u is called a non-negative fuzzy number. We write $L^*(\mathbb{R})$ by the set of all non-negative fuzzy numbers. We can say that $u \in L^*(\mathbb{R})$ iff $u_\alpha^- \geq 0$ for each $\alpha \in [0,1]$. Clearly we have $\widetilde{0} \in L(\mathbb{R})$. For

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$u\in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_{\alpha}=\begin{cases}\{x\in\mathbb{R}:u(x)\geq\alpha\}, & \text{if } \alpha\in(0,1]\\ supp\,u, & \text{if } \alpha=0.\end{cases}$$

A partial order $\preceq$ on $L(\mathbb{R})$ is defined by $u\preceq v$ if $u_{\alpha}^{-}\leq v_{\alpha}^{-}$ and $u_{\alpha}^{+}\leq v_{\alpha}^{+}$ for all $\alpha\in[0,1]$. Arithmetic operation for $t\in\mathbb{R}$, $\oplus,\ominus,\odot$ and $\oslash$ on $L(\mathbb{R})\times L(\mathbb{R})$ are defined by
$(u\oplus v)(t)=\sup_{s\in\mathbb{R}}\{u(s)\wedge v(t-s)\}$, $(u\ominus v)(t)=\sup_{s\in\mathbb{R}}\{u(s)\wedge v(s-t)\}$,
$(u\odot v)(t)=\sup_{s\in\mathbb{R},s\neq 0}\{u(s)\wedge v(t/s)\}$ and $(u\oslash v)(t)=\sup_{s\in\mathbb{R}}\{u(st)\wedge v(s)\}$.
For $k\in\mathbb{R}^{+}$, ku is defined as $ku(t)=u(t/k)$ and $0u(t)=\tilde{0}$, $t\in\mathbb{R}$.
Some arithmetic operations for $\alpha-$level sets are defined as follows:
$u,v\in L(\mathbb{R})$ and $[u]_{\alpha}=\left[u_{\alpha}^{-},u_{\alpha}^{+}\right]$ and $[v]_{\alpha}=\left[v_{\alpha}^{-},v_{\alpha}^{+}\right]$, $\alpha\in(0,1]$. Then,
$[u\oplus v]_{\alpha}=\left[u_{\alpha}^{-}+v_{\alpha}^{-},u_{\alpha}^{+}+v_{\alpha}^{+}\right]$, $[u\ominus v]_{\alpha}=\left[u_{\alpha}^{-}-v_{\alpha}^{+},u_{\alpha}^{+}-v_{\alpha}^{-}\right]$,
$[u\odot v]_{\alpha}=\left[u_{\alpha}^{-}.v_{\alpha}^{-},u_{\alpha}^{+}.v_{\alpha}^{+}\right]$ and $\left[\tilde{1}\oslash u\right]_{\alpha}=\left[\frac{1}{u_{\alpha}^{+}},\frac{1}{u_{\alpha}^{-}}\right]$, $u_{\alpha}^{-}>0$.
For $u,v\in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ defined as

$$D(u,v)=\sup_{0\leq\alpha\leq 1}\max\left\{\left|u_{\alpha}^{-}-v_{\alpha}^{-}\right|,\left|u_{\alpha}^{+}-v_{\alpha}^{+}\right|\right\}.$$

It is known that D is a metric on $L(\mathbb{R})$ and $(L(\mathbb{R}),D)$ is a complete metric space.
A sequence $x=(x_k)$ of fuzzy numbers is said to be convergent to the fuzzy number x_0, if for every $\varepsilon>0$ there exists a positive integer k_0 such that $D(x_k,x_0)<\varepsilon$ for $k>k_0$ and a sequence $x=(x_k)$ of fuzzy numbers convergent to levelwise to x_0 if and only if $\lim_{k\to\infty}[x_k]_{\alpha}=[x_0]_{\alpha}^{-}$ and $\lim_{k\to\infty}[x_k]_{\alpha}=[x_0]_{\alpha}^{+}$, where $[x_k]_{\alpha}=\left[(x_k)_{\alpha}^{-},(x_k)_{\alpha}^{+}\right]$ and $[x_0]_{\alpha}=\left[(x_0)_{\alpha}^{-},(x_0)_{\alpha}^{+}\right]$, for every $\alpha\in(0,1)$.
Let X be a vector space over $\mathbb{R}$, $\|.\|:X\to L^{*}(\mathbb{R})$ and the mappings $L;R$ (respectively, left norm and right norm) : $[0,1]\times[0,1]\to[0,1]$ be symetric, nondecreasing in both arguments and satisfy $L(0,0)=0$ and $R(1,1)=1$.
The quadruple $(X,\|.\|,L,R)$ is called fuzzy normed linear space (briefly FNS) and $\|.\|$ a fuzzy norm if the following axioms are satisfied

1. $\|x\|=\widetilde{0}$ iff $x=0$,
2. $\|rx\|=|r|\odot\|x\|$ for $x\in X$, $r\in\mathbb{R}$,
3. For all $x,y\in X$
   (a) $\|x+y\|(s+t)\geq L(\|x\|(s),\|y\|(t))$, whenever $s\leq\|x\|_{1}^{-}$, $t\leq\|y\|_{1}^{-}$ and $s+t\leq\|x+y\|_{1}^{-}$,
   (b) $\|x+y\|(s+t)\leq R(\|x\|(s),\|y\|(t))$, whenever $s\geq\|x\|_{1}^{-}$, $t\geq\|y\|_{1}^{-}$ and $s+t\geq\|x+y\|_{1}^{-}$.

Let $(X,\|.\|_{C})$ be an ordinary normed linear space. Then, a fuzzy norm $\|.\|$ on X can be obtained by

$$\|x\|(t)=\begin{cases}0, & \text{if } 0\leq t\leq a\|x\|_{C} \text{ or } t\geq b\|x\|_{C}\\ \frac{t}{(1-a)\|x\|_{C}}-\frac{a}{1-a}, & \text{if } a\|x\|_{C}\leq t\leq\|x\|_{C}\\ \frac{-t}{(b-1)\|x\|_{C}}+\frac{b}{b-1}, & \text{if } \|x\|_{C}\leq t\leq b\|x\|_{C}\end{cases}$$

where $\|x\|_{C}$ is the ordinary norm of $x\ (\neq 0)$, $0<a<1$ and $1<b<\infty$. For $x=0$, define $\|x\|=\widetilde{0}$. Hence, $(X,\|.\|)$ is a fuzzy normed linear space.
Let us consider the topological structure of an FNS $(X,\|.\|)$. For any $\varepsilon>0,\alpha\in[0,1]$ and $x\in X$, the $(\varepsilon,\alpha)-$ neighborhood of x is the set $\mathscr{N}_{x}(\varepsilon,\alpha)=\{y\in X:\|x-y\|_{\alpha}^{+}<\varepsilon\}$.
Let $(X,\|.\|)$ be an FNS. A sequence $(x_n)_{n=1}^{\infty}$ in X is convergent to $x\in X$ with respect to the fuzzy norm on X and we denote by $x_n\overset{FN}{\to}x$, provided that $(D)-\lim_{n\to\infty}\|x_n-x\|=\widetilde{0}$; i.e., for every $\varepsilon>0$ there is an $N(\varepsilon)\in\mathbb{N}$ such that $D\left(\|x_n-x\|,\widetilde{0}\right)<\varepsilon$ for all $n\geq N(\varepsilon)$. This means that for every $\varepsilon>0$ there is an $N(\varepsilon)\in\mathbb{N}$ such that for all $n\geq N(\varepsilon)$, $\sup_{\alpha\in[0,1]}\|x_n-x\|_{\alpha}^{+}=\|x_n-x\|_{0}^{+}<\varepsilon$.
Let $(X,\|.\|)$ be an FNS. Then a double sequence $\left(x_{jk}\right)$ is said to be convergent to $x\in X$ with respect to the fuzzy norm on X if for every $\varepsilon>0$ there exist a number $N=N(\varepsilon)$ such that $D\left(\left\|x_{jk}-x\right\|,\widetilde{0}\right)<\varepsilon$, for all $j,k\geq N$.
In this case, we write $x_{jk}\xrightarrow{FN}x$. This means that, for every $\varepsilon>0$ there exist a number $N=N(\varepsilon)$ such that $\sup_{\alpha\in[0,1]}\left\|x_{jk}-x\right\|_{\alpha}^{+}=\left\|x_{jk}-x\right\|_{0}^{+}<\varepsilon$, for all $j,k\geq N$. In terms of neighnorhoods, we have $x_{jk}\xrightarrow{FN}x$ provided that for any $\varepsilon>0$, there exists a number $N=N(\varepsilon)$ such that $x_{jk}\in\mathscr{N}_{x}(\varepsilon,0)$, whenever $j,k\geq N$.
Let $X\neq\emptyset$. A class $\mathscr{I}$ of subsets of X is said to be an ideal in X provided:
(i) $\emptyset\in\mathscr{I}$, (ii) $A,B\in\mathscr{I}$ implies $A\cup B\in\mathscr{I}$, (iii) $A\in\mathscr{I}$, $B\subset A$ implies $B\in\mathscr{I}$.
$\mathscr{I}$ is called a nontrivial ideal if $X\notin\mathscr{I}$. A nontrivial ideal $\mathscr{I}$ in X is called admissible if $\{x\}\in\mathscr{I}$ for each $x\in X$.
A nontrivial ideal $\mathscr{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathscr{I}_2$ for each $i\in\mathbb{N}$. It is evident that a strongly admissible ideal is also admissible. Throughout the paper we take $\mathscr{I}_2$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.
Let $\mathscr{I}_2^0=\{A\subset\mathbb{N}\times\mathbb{N}:(\exists m(A),(i,j)\geq m(A)\Rightarrow(i,j)\notin A)\}$. Then $\mathscr{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathscr{I}_2$ is strongly admissible if and only if $\mathscr{I}_2^0\subset\mathscr{I}_2$.
Let $X\neq\emptyset$. A non empty class $\mathscr{F}$ of subsets of X is said to be a filter in X provided:
(i) $\emptyset\notin\mathscr{F}$, (ii) $A,B\in\mathscr{F}$ implies $A\cap B\in\mathscr{F}$, (iii) $A\in\mathscr{F}$, $A\subset B$ implies $B\in\mathscr{F}$.
Let $\mathscr{I}$ is a nontrivial ideal in X, $X\neq\emptyset$, then the class $\mathscr{F}(\mathscr{I})=\{M\subset X:(\exists A\in\mathscr{I})(M=X\backslash A)\}$ is a filter on X, called the filter associated with $\mathscr{I}$.

Let (X,ρ) be a linear metric space and $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x=(x_{mn})$ in X is said to be $\mathscr{I}_2$-convergent to $L\in X$, if for any $\varepsilon>0$ we have $A(\varepsilon)=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\rho(x_{mn},L)\geq\varepsilon\}\in\mathscr{I}_2$ and we write $\mathscr{I}_2-\lim_{m,n\to\infty}x_{mn}=L$.

If $\mathscr{I}_2\subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal, then usual convergence implies $\mathscr{I}_2$-convergence.

Let $(X,\|.\|)$ be a fuzzy normed space. A double sequence $x=(x_{mn})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ in X is said to be $\mathscr{I}_2-$ convergent to $L_1\in X$ with respect to fuzzy norm on X if for each $\varepsilon>0$, the set $A(\varepsilon)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|x_{mn}-L_1\|_0^+\geq\varepsilon\right\}\in\mathscr{I}_2$. In this case, we write $x_{mn}\xrightarrow{F\mathscr{I}_2}L_1$ or $x_{mn}\to L_1\,(F\mathscr{I}_2)$ or $F\mathscr{I}_2-\lim_{m,n\to\infty}x_{mn}=L_1$. The element L_1 is called the $F\mathscr{I}_2-$limit of (x_{mn}) in X. In terms of neighborhoods, we have $x_{mn}\xrightarrow{F\mathscr{I}_2}L_1$ provided that for each $\varepsilon>0$, $\{(m,n)\in\mathbb{N}\times\mathbb{N}:x_{mn}\notin\mathscr{N}_{L_1}(\varepsilon,0)\}\in\mathscr{I}_2$. A useful interpretation of the above definition is the following;

$x_{mn}\xrightarrow{F\mathscr{I}_2}L_1\Leftrightarrow F\mathscr{I}_2-\lim_{m,n\to\infty}\|x_{mn}-L_1\|_0^+=0.$

Note that $F\mathscr{I}_2-\lim_{m,n\to\infty}\|x_{mn}-L_1\|_0^+=0$ implies that

$$F\mathscr{I}_2-\lim\|x_{mn}-L_1\|_\alpha^- = F\mathscr{I}_2-\lim\|x_{mn}-L_1\|_\alpha^+=0,$$

for each $\alpha\in[0,1]$, since $0\leq\|x_{mn}-L_1\|_\alpha^-\leq\|x_{mn}-L_1\|_\alpha^+\leq\|x_{mn}-L_1\|_0^+$ holds for every $m,n\in\mathbb{N}$ and for each $\alpha\in[0,1]$.

Let $(X,\|.\|)$ be a fuzzy normed space. A double sequence $x=(x_{mn})$ in X is said to be $\mathscr{I}_2^*$-convergent to L in X with respect to the fuzzy norm on X if there exists a set $M\in\mathscr{F}(\mathscr{I}_2)$, $M=\{m_1<\cdots<m_k<\cdots;n_1<\cdots<n_l<\cdots\}\subset\mathbb{N}\times\mathbb{N}$ such that $\lim_{k,l\to\infty}\|x_{m_kn_l}-L\|$. In this case, we write $x_{mn}\xrightarrow{F\mathscr{I}_2^*}L_1$ or $x_{mn}\to L_1\left(F\mathscr{I}_2^*\right)$ or $F\mathscr{I}_2^*-\lim_{m,n\to\infty}x_{mn}=L_1$.

We say that an admissible ideal $\mathscr{I}_2\subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2), if for every countable family of mutually disjoint sets $\{A_1,A_2,...\}$ belonging to $\mathscr{I}_2$, there exists a countable family of sets $\{B_1,B_2,...\}$ such that $A_j\cap B_j\in\mathscr{I}_2^0$, i.e., $A_j\cap B_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j\in\mathbb{N}$ and $B=\bigcup_{j=1}^{\infty}B_j\in\mathscr{I}_2$ (hence $B_j\in\mathscr{I}_2$ for each $j\in\mathbb{N}$).

Lemma 1.1. *([14], Theorem 3.7) Let $(X,\|.\|)$ be fuzzy normed space, $\mathscr{I}_2$ be a admissible ideal and (x_{mn}) be a double sequence in X. Then, $x_{mn}\xrightarrow{\mathscr{I}_2^*}L_1$ implies $x_{mn}\xrightarrow{\mathscr{I}_2}L_1$.*

Lemma 1.2. *([14], Theorem 3.8) Let $(X,\|.\|)$ be fuzzy normed space, $\mathscr{I}_2$ be a admissible ideal with property $(AP2)$ and (x_{mn}) be a double sequence in X. Then, $x_{mn}\xrightarrow{\mathscr{I}_2}L_1$ if and only if $x_{mn}\xrightarrow{\mathscr{I}_2^*}L_1$.*

2. Main Results

In this section first, we investigate some properties of $\mathscr{I}_2$-convergence in fuzzy normed spaces. After, we study some relationships between $\mathscr{I}_2$-convergence and $\mathscr{I}_2^*$-convergence of double sequences in fuzzy normed spaces.

Theorem 2.1. *Let $(X,\|.\|)$ be a fuzzy normed space. If a double sequence (x_{mn}) in X is $\mathscr{I}_2$-convergent to L_1, then L_1 determined uniquely.*

Proof. Let (x_{mn}) be any double sequence and suppose that

$$F\mathscr{I}_2-\lim_{m,n\to\infty}x_{mn}=L_1 \ \ and\ \ F\mathscr{I}_2-\lim_{m,n\to\infty}x_{mn}=L_2,$$

where $L_1\neq L_2$. Since $L_1\neq L_2$, we may suppose that $L_1>L_2$. Select $\varepsilon=\frac{L_1-L_2}{3}$, so that the neighborhoods $(L_1-\varepsilon,L_1+\varepsilon)$ and $(L_2-\varepsilon,L_2+\varepsilon)$ of L_1 and L_2 respectively are disjoints. Since L_1 and L_2 both are $\mathscr{I}_2-$limit of the sequence (x_{mn}). Therefore, both the sets

$A(\varepsilon)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|x_{mn}-L_1\|_0^+\geq\varepsilon\right\}$ and $B(\varepsilon)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|x_{mn}-L_2\|_0^+\geq\varepsilon\right\}$

belongs to $\mathscr{I}_2$. This implies that the sets

$A^c(\varepsilon)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|x_{mn}-L_1\|_0^+<\varepsilon\right\}$ and $B^c(\varepsilon)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|x_{mn}-L_2\|_0^+<\varepsilon\right\}$

belongs to $\mathscr{F}(\mathscr{I}_2)$. Since $\mathscr{F}(\mathscr{I}_2)$ is a filter on $\mathbb{N}\times\mathbb{N}$ therefore $A^c(\varepsilon)\cap B^c(\varepsilon)$ is a non-empty set in $\mathscr{F}(\mathscr{I}_2)$. In this way we obtain a contradiction to the fact that the neighborhoods $(L_1-\varepsilon,L_1+\varepsilon)$ and $(L_2-\varepsilon,L_2+\varepsilon)$ of L_1 and L_2, respectively, are disjoints. Hence, we have $L_1=L_2$. □

Theorem 2.2. *Let $(X,\|.\|)$ be a fuzzy normed space, (x_{mn}) be a double sequence in X and $L_1\in X$. Then, $FP-\lim_{m,n\to\infty}x_{mn}=L_1\Rightarrow F\mathscr{I}_2-\lim_{m,n\to\infty}x_{mn}=L_1$.*

Proof. Let $FP-\lim_{m,n\to\infty}x_{mn}=L_1$. For $\varepsilon>0$ there exists a positive integer $k_0=k_0(\varepsilon)$ such that $\|x_{mn}-L_1\|_0^+<\varepsilon$, whenever $m,n>k_0$. This implies that the set

$$A(\varepsilon)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|x_{mn}-L_1\|_0^+\geq\varepsilon\right\}\subset(\mathbb{N}\times\{1,2,...,k_0\})\cup(\{1,2,...,k_0\}\times\mathbb{N}).$$

Since $\mathscr{I}_2$ is a admissible ideal, then

$$(\mathbb{N}\times\{1,2,...,k_0\})\cup(\{1,2,...,k_0\}\times\mathbb{N})\in\mathscr{I}_2$$

and so $A(\varepsilon)\in\mathscr{I}_2$. Hence, we have

$$F\mathscr{I}_2-\lim_{m,n\to\infty}x_{mn}=L_1.$$

□

Theorem 2.3. *Let $(X,\|.\|)$ be a fuzzy normed space.*
(i) *If X has no accumulation point, then $F\mathscr{I}_2$-convergence and $F\mathscr{I}_2^*$-convergence coincide for each strongly admissible ideal $\mathscr{I}_2$.*
(ii) *If X has an accumulation point L, then there exists a strongly admissible ideal $\mathscr{I}_2$ and a double sequence (x_{mn}) for which $F\mathscr{I}_2$-$\lim_{m,n\to\infty} x_{mn} = L$ but $F\mathscr{I}_2^*$-$\lim_{m,n\to\infty} x_{mn}$ does not exist.*

Proof. **(i)** Let $x=(x_{mn})$ be a double sequence in X and $L\in X$. By Lemma 1.1, $x_{mn} \overset{F\mathscr{I}_2^*}{\to} L_1$ implies $x_{mn} \overset{F\mathscr{I}_2}{\to} L_1$. Assume that $F\mathscr{I}_2$-$\lim_{m,n\to\infty} x_{mn} = L$. Since X has no accumulation point, so there exists $\varepsilon>0$ such that

$$B_L(\varepsilon,0)=\{x\in X: \|x-L\|_0^+<\varepsilon\}=\{L\}.$$

Since $F\mathscr{I}_2$-$\lim_{m,n\to\infty} x_{mn} = L$, so

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}: \|x_{mn}-L\|_0^+\geq\varepsilon\right\}\in\mathscr{I}_2.$$

Hence, we have

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}: \|x_{mn}-L\|_0^+<\varepsilon\right\}=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}: \|x_{mn}-L\|_0^+=0\right\}\in\mathscr{F}(\mathscr{I}_2).$$

Therefore, $F\mathscr{I}_2^*$-$\lim_{m,n\to\infty} x_{mn} = L$.
(ii) Since L is an accumulation point of X, so there exists a sequence $(t_i)_{i\in\mathbb{N}}$ of distinct points all different from L in X which is convergent to L such that the sequence $\{\|t_i-L\|_0^+\}_{i\in\mathbb{N}}$ is decreasing to 0. Let $\{T_i\}_{i\in\mathbb{N}}$ be a decomposition of $\mathbb{N}$ onto infinite sets and put $\Delta_i=\{(m,n):\min\{m,n\}\in T_i\}$. Then, $\{\Delta_i\}_{i\in\mathbb{N}}$ is a decomposition of $\mathbb{N}\times\mathbb{N}$ and the ideal

$$\mathscr{I}_2=\{A: \text{ A is included in a finite union of } \Delta_i's\}$$

is a strongly admissible ideal. Put $x_{mn}=t_i$ if and only if $(m,n)\in\Delta_i$. Put $s_n=\{\|t_n-L\|_0^+\}$, for $n\in\mathbb{N}$. Let $\delta>0$ be given. Select $\gamma\in\mathbb{N}$ such that $s_\gamma<\delta$. Then,

$$A(\delta)=\{(m,n)\in\mathbb{N}\times\mathbb{N}: \|x_{mn}-L\|_0^+\geq\delta\}\subset\Delta_1\cup\Delta_2\cup\cdots\cup\Delta_\gamma.$$

Hence, $A(\delta)\in\mathscr{I}_2$ and $F\mathscr{I}_2$-$\lim_{m,n\to\infty} x_{mn} = L$.
Now suppose that $F\mathscr{I}_2^*$-$\lim_{m,n\to\infty} x_{mn} = L$. Then, there exists $H\in\mathscr{I}_2$ such that for $M=\mathbb{N}\times\mathbb{N}\backslash H$ we have $FP-\lim_{m,n\to\infty} x_{mn}=L$, for $(m,n)\in M$. By definition of the ideal $\mathscr{I}_2$, there exists $k\in\mathbb{N}$ such that

$$H\subset\Delta_1\cup\Delta_2\cup\cdots\cup\Delta_k.$$

But then, $\Delta_{k+1}\subset\mathbb{N}\times\mathbb{N}\backslash H=M$. Then, from the construction of Δ_{k+1} it follows that for any $n_0\in\mathbb{N}$,

$$\|x_{mn}-L\|_0^+=s_{k+1}>0$$

hold for infinitely many (m,n)'s with $(m,n)\in M$ and $m,n\geq n_0$. This contradicts that $FP-\lim_{m,n\to\infty} x_{mn}=L$, for $(m,n)\in M$. Also the assumption $F\mathscr{I}_2^*$-$\lim_{m,n\to\infty} x_{mn}=q$, for $q\neq L$ leads to the contradiction. □

Theorem 2.4. *Let $(X,\|.\|)$ be a fuzzy normed space. If X has at least one accumulation point and for any arbitrary double sequence (x_{mn}) of elements of X and for each $L\in X$, $F\mathscr{I}_2$-$\lim_{m,n\to\infty} x_{mn}=L$ implies $F\mathscr{I}_2^*$-$\lim_{m,n\to\infty} x_{mn}=L$, then $\mathscr{I}_2$ has the property $(AP2)$.*

Proof. Assume that $L\in X$ is an accumulation point of X. There exists a sequence $(t_k)_{k\in\mathbb{N}}$ of distinct elements of X such that $t_k\neq L$ for any k, $\lim_{k\to\infty} t_k=L$ and the sequence $\{\|t_k-L\|_0^+\}_{k\in\mathbb{N}}$ is decreasing to 0. Put

$$s_k=\{\|t_k-L\|_0^+\},$$

for $k\in\mathbb{N}$. Let $\{A_i\}_{i\in\mathbb{N}}$ be a disjoint family of nonempty sets from $\mathscr{I}_2$.
Define a sequence (x_{mn}) as following:

$$x_{mn}=\begin{cases} t_i, & \text{if } (m,n)\in A_i \\ L, & \text{if } (m,n)\notin A_i, \end{cases}$$

for any i. Let $\delta>0$. Select $k\in\mathbb{N}$ such that $s_k<\delta$. Then,

$$A(\delta)=\{(m,n)\in\mathbb{N}\times\mathbb{N}: \|x_{mn}-L\|_0^+\geq\delta\}\subset A_1\cup A_2\cup\cdots\cup A_k.$$

Hence, $A(\delta)\in\mathscr{I}_2$ and so,

$$F\mathscr{I}_2-\lim_{m,n\to\infty} x_{mn}=L.$$

By virtue of our assumption, we have

$$F\mathscr{I}_2^*-\lim_{m,n\to\infty} x_{mn}=L.$$

So, there exists a set $H\in\mathscr{I}_2$ such that $M=\mathbb{N}\times\mathbb{N}\backslash H\in\mathscr{F}(\mathscr{I}_2)$ and

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} x_{mn}=L. \tag{2.1}$$

Now, put $H_i = A_i \cap H$, for $i \in \mathbb{N}$. Then, $H_i \in \mathscr{I}_2$, for each $i \in \mathbb{N}$. Also,

$$\bigcup_{i=1}^{\infty} H_i = H \cap \bigcup_{i=1}^{\infty} A_i \subset H \text{ and so } \bigcup_{i=1}^{\infty} H_i \in \mathscr{I}_2.$$

Fix $i \in \mathbb{N}$. If $A_i \cap M$ is not included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$, then M must contain an infinite sequence of elements $\{(m_k, n_k)\}$, where both $m_k, n_k \to \infty$ and $x_{m_k n_k} = t_k \neq L$, for all $k \in \mathbb{N}$ which contradicts (2.1). Hence, $A_i \cap M$ must be contained in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$. Therefore,

$$A_i \Delta H_i = A_i \backslash H_i = A_i \backslash H = A_i \cap M$$

is also included in the finite union of rows and columns. Thus, $\mathscr{I}_2$ has the property $(AP2)$. □

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