



# On Some Matrix Representations of Bicomplex Numbers

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## Abstract

In this work, we have defined bicomplex numbers whose coefficients are from the Fibonacci sequence. We examined the matrix representations and algebraic properties of these numbers. We also computed the eigenvalues and eigenvectors of these particular matrices.

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## 1. Introduction

Quaternions and bicomplex numbers are defined as a generalization of the complex numbers.  $\mathbb{BC}$ , which is a set of bicomplex numbers, is defined as

$$\mathbb{BC} = \{z_1 + z_2\mathbf{j} \mid z_1, z_2 \in \mathbb{C}, \mathbf{j}^2 = -1\}, \tag{1.1}$$

where  $\mathbb{C}$  is the set of complex numbers with the imaginary unit  $\mathbf{i}$ , also  $\mathbf{i}$  and  $\mathbf{i} \neq \mathbf{j}$  are commuting imaginary units. Bicomplex numbers are a recent powerful mathematical tool to develop the theories of functions and play an important role in solving problems of electromagnetism. These numbers are also advantageous than quaternions due to the commutative property. In  $\mathbb{BC}$ , multiplication is commutative, associative and distributive over addition and  $\mathbb{BC}$  is a commutative algebra but not division algebra. Any element  $b$  in  $\mathbb{BC}$  is written, for  $t = 1, 2, 3, 4$  and  $a_t \in \mathbb{R}$ , as  $b = a_1\mathbf{1} + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{ij}$ . Addition, multiplication and division can be done term by term and it is helpful to understand the structure of functions of a bicomplex variable.

For the numbers  $b_1 = a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{ij}$  and  $b_2 = d_0\mathbf{1} + d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{ij}$

$$b_1 + b_2 = (a_0 + d_0)\mathbf{1} + (a_1 + d_1)\mathbf{i} + (a_2 + d_2)\mathbf{j} + (a_3 + d_3)\mathbf{ij} \tag{1.2}$$

and

$$b_1 b_2 = (a_0 d_0 - a_1 d_1 - a_2 d_2 + a_3 d_3)\mathbf{1} + (a_0 d_1 + a_1 d_0 - a_2 d_3 - a_3 d_2)\mathbf{i} \tag{1.3}$$

$$+ (a_0 d_2 - a_1 d_3 + a_2 d_0 - a_3 d_1)\mathbf{j} + (a_0 d_3 + a_1 d_2 + a_2 d_1 + a_3 d_0)\mathbf{ij},$$

is written. The multiplication operation is performed using multiplication rule of the bases elements  $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$  as follows.

$$\mathbf{i}^2 = \mathbf{j}^2 = -1, \quad \mathbf{ij} = \mathbf{ji} = \mathbf{k}, \quad \mathbf{ik} = \mathbf{ki} = -\mathbf{j}, \quad \mathbf{jk} = \mathbf{kj} = -\mathbf{i}. \tag{1.4}$$

For more knowledge on these numbers, the readers can refer to references [2], [1].

## 2. Bicomplex Numbers with coefficients are from Fibonacci Sequence

The Horadam sequence is a particular type of linear recurrence sequences [5], [6]. This sequence generalizes some of the well-known sequences such as Fibonacci, Lucas, Pell, Pell-Lucas sequences, etc. In [3], we consider quaternions with coefficients Fibonacci numbers. In [4], we give a generalization of bicomplex numbers. Also, we introduced idempotent representations for the bicomplex Fibonacci numbers worked by the author in [8]. In this section, we introduce the bicomplex numbers with coefficients from the Fibonacci sequence and examine

some characteristic properties of them. We also examine the matrix representations of the newly defined numbers. Noted that since the algebra  $\mathbb{C}$  of complex numbers is isomorphic to the set of real  $2 \times 2$  matrices of the form

$$x + iy \in \mathbb{C} \rightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

a similar reasoning applies to the ring  $\mathbb{B}\mathbb{C}$ , where the unit  $i$  can be written as  $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . The mapping

$$b = z_1 + z_2\mathbf{j} \in \mathbb{B}\mathbb{C} \rightarrow \begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{pmatrix}$$

is also an isomorphism. This representation of the number  $b$  will be used in other representations.

**Corollary 2.1.** For any  $b$ , we have

$$b = a_0(1 * 1) + a_1(1 * i) + a_2(i * 1) + a_3(i * i). \tag{2.1}$$

*Proof.* The following equalities can be given by using Kronecker multiplication.

$$\begin{aligned} 1 * 1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}, \\ 1 * i &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbf{i}, \\ i * 1 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{j}, \\ i * i &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbf{ij}. \end{aligned}$$

The Kronecker product of the matrices corresponding to 1 and  $i$  gives the units  $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ . So, using the above  $2 \times 2$  matrices, the bicomplex numbers can be represented by following  $4 \times 4$  matrices,

$$\begin{aligned} \mathbf{1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ \mathbf{j} &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, we have

$$b = a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{ij} = a_0(1 * 1) + a_1(1 * i) + a_2(i * 1) + a_3(i * i). \tag{2.2}$$

□

Now, let us denote by  $\mathbb{B}\mathbb{C}_F$  the sequence of bicomplex numbers with coefficients from Fibonacci sequence.

$$\mathbb{B}\mathbb{C}_F = \{Q_n\}_{n \geq 0} = \{Q_0, Q_1, Q_2, \dots, Q_n, \dots\}, \tag{2.3}$$

where the  $n$ -th term of sequence  $\mathbb{B}\mathbb{C}_F$  is

$$Q_n = C_n + C_{n+2}\mathbf{j}. \tag{2.4}$$

$C_n = F_n + iF_{n+1}$  is the  $n$ -th complex Fibonacci number and  $F_n$  is the  $n$ -th Fibonacci number and defined by the recurrence relation  $F_n = F_{n-1} + F_{n-2}$ ,  $n \geq 2$ , with the initial values  $F_0 = 0$  and  $F_1 = 1$ [7].

Let us define a matrix sequence as in the following.

$$\mathbb{M}_{\mathbb{B}\mathbb{C}_F} = \left\{ \begin{pmatrix} C_n & -C_{n+2} \\ C_{n+2} & C_n \end{pmatrix} \mid C_n, C_{n+2} \in \mathbb{C}, n \geq 0 \right\}. \tag{2.5}$$

The algebraic operations in  $\mathbb{B}\mathbb{C}_F$  can be defined as

$$Q_n + Q_m = (C_n + C_m) + (C_{n+2} + C_{m+2})\mathbf{j} \tag{2.6}$$

and

$$Q_n Q_m = (C_n C_m - C_{n+2} C_{m+2}) + (C_{n+2} C_m + C_n C_{m+2})\mathbf{j} \tag{2.7}$$

or

$$Q_n Q_m = A_1 + A_2 i + A_3 j + A_4 k, \tag{2.8}$$

where the values  $A_1, A_2, A_3, A_4$  are as follows.

$$A_1 = F_n F_m - F_{n+1} F_{m+1} - F_{n+2} F_{m+2} + F_{n+3} F_{m+3}, \tag{2.9}$$

$$A_2 = F_n F_{m+1} + F_{n+1} F_m - F_{n+2} F_{m+3} - F_{n+3} F_{m+2}, \tag{2.10}$$

$$A_3 = F_n F_{m+2} - F_{n+1} F_{m+3} + F_{n+2} F_m - F_{n+3} F_{m+1}, \tag{2.11}$$

$$A_4 = F_n F_{m+3} + F_{n+1} F_{m+2} + F_{n+2} F_{m+1} + F_{n+3} F_m. \tag{2.12}$$

**Corollary 2.2.** *The  $n$ -th bicomplex Fibonacci number  $Q_n$  can be represented by the following form,*

$$Q_n = \begin{pmatrix} F_n & -F_{n+1} & -F_{n+2} & F_{n+3} \\ F_{n+1} & F_n & -F_{n+3} & -F_{n+2} \\ F_{n+2} & -F_{n+3} & F_n & -F_{n+1} \\ F_{n+3} & F_{n+2} & F_{n+1} & F_n \end{pmatrix}. \tag{2.13}$$

Moreover, we have

$$\det(Q_n) = \sum_{k=0}^3 F_{n+k}^4 + 8 \prod_{k=0}^3 F_{n+k} + 2(F_{n+1}^2 + F_{n+2}^2)^2 + 6F_{n+1}^2 F_{n+2}^2$$

and

$$\text{tr}(Q_n) = 4F_n. \tag{2.14}$$

*Proof.* From the matrix operations the proof is clear. □

In the following theorem we listed some properties of matrix  $Q_n$ .

**Theorem 2.3.** *For different positive integers  $n$  and  $m$ , the following equations are satisfied:*

- i)  $\text{tr}(Q_n Q_m) = \text{tr}(Q_m Q_n) = 4\text{Re}(Q_n Q_m)$
- ii)  $\text{tr}(Q_n + Q_m) = \text{tr}(Q_n) + \text{tr}(Q_m) = 4\text{Re}(Q_n + Q_m)$
- iii) For  $\alpha, \beta \in \mathbb{Z}$ ,  $\text{tr}(\alpha Q_n + \beta Q_m) = \alpha \text{tr}(Q_n) + \beta \text{tr}(Q_m)$
- iv)  $\text{tr}(Q_n Q_n^T) = 12F_{2n+3}$

and

$$\det(Q_n Q_n^T) = \det(Q_n^T Q_n) = 81F_{2n+3}^4 - 72F_{2n+3}^2 + 16$$

*Proof.* i) Due to the properties of the bicomplex numbers, the following matrix multiplication can be easily seen. Then, we have

$$Q_n Q_m = \begin{pmatrix} A_1 & -A_2 & -A_3 & A_4 \\ A_2 & A_1 & -A_4 & -A_3 \\ A_3 & -A_4 & A_1 & -A_2 \\ A_4 & A_3 & A_2 & A_1 \end{pmatrix} = Q_m Q_n$$

Thus, we obtain the following equality.

$$\text{tr}(Q_n Q_m) = \text{tr}(Q_m Q_n) = 4(F_n F_m - F_{n+1} F_{m+1} - F_{n+2} F_{m+2} + F_{n+3} F_{m+3}) = 4\text{Re}(Q_n Q_m).$$

ii) Since,

$$Q_n + Q_m = \begin{pmatrix} F_n + F_m & -(F_{n+1} + F_{m+1}) & -(F_{n+2} + F_{m+2}) & F_{n+3} + F_{m+3} \\ F_{n+1} + F_{m+1} & F_n + F_m & -(F_{n+3} + F_{m+3}) & -(F_{n+2} + F_{m+2}) \\ F_{n+2} + F_{m+2} & -(F_{n+3} + F_{m+3}) & F_n + F_m & -(F_{n+1} + F_{m+1}) \\ F_{n+3} + F_{m+3} & F_{n+2} + F_{m+2} & F_{n+1} + F_{m+1} & F_n + F_m \end{pmatrix}$$

we get

$$\text{tr}(Q_n + Q_m) = \text{tr}(Q_n) + \text{tr}(Q_m) = 4\text{Re}(Q_n + Q_m)$$

Others are easily seen in a similar way. □

It is note that the matrix  $\mathcal{Q}_n$  corresponds to the following linear transformation.

$$T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$$

$$T(F_n, F_{n+1}, F_{n+2}, F_{n+3}) = ((a, -b, -c, d), (b, a, -d, -c), (c, -d, a, -b), (d, c, b, a))$$

where  $a = F_n, b = F_{n+1}, c = F_{n+2}, d = F_{n+3}$ .

Hence,

$$\mathcal{Q}_n = \begin{pmatrix} a & -b & -c & d \\ b & a & -d & -c \\ c & -d & a & -b \\ d & c & b & a \end{pmatrix}.$$

Also, let us consider a different representation of bicomplex number  $Q_n$  and some properties with the help of this representation. Let the complex part connected to  $i$  is  $C_i(Q_n)$  and the complex part connected to  $j$  is  $C_j(Q_n)$ . And let the hyper imaginary parts connected to  $i$  and  $j$  are  $H_i(Q_n)$  and  $H_j(Q_n)$ , respectively. Then, the following corollary can be given.

**Corollary 2.4.** *We have*

$$Q_n Q_m = C_i C_i' - H_i H_i' + i(C_i H_i' + H_i C_i'). \tag{2.15}$$

Where,  $C_i$  and  $H_i$  ( $C_i'$  and  $H_i'$ ) are the complex and hyper imaginary parts of  $Q_n$  and ( $Q_m$ ), respectively.

*Proof.* From the above definition, we write

$$C_i(Q_n) = F_n + jF_{n+2}, \quad C_j(Q_n) = F_n + iF_{n+1} \tag{2.16}$$

and

$$H_i(Q_n) = F_{n+1} + jF_{n+3}, \quad H_j(Q_n) = F_{n+2} + iF_{n+3} \tag{2.17}$$

Since the numbers  $C_i(Q_n), C_j(Q_n), H_i(Q_n)$  and  $H_j(Q_n)$  are isomorphic to the complex Fibonacci sequence, we can write the following equations.

$$Q_n = C_i(Q_n) + iH_i(Q_n) = C_j(Q_n) + jH_j(Q_n) \tag{2.18}$$

So, we write

$$Q_n = (F_n + jF_{n+2}) + i(F_{n+1} + jF_{n+3}) \tag{2.19}$$

and

$$Q_n = (F_n + iF_{n+1}) + j(F_{n+2} + iF_{n+3}) = C_j(Q_n) + jH_j(Q_n). \tag{2.20}$$

Using the above representation, the desired result is obtained. □

Moreover, for the  $Q_n^k$ , hyper imaginary parts can be also defined as follows.

$$Q_n^k = (F_n - jF_{n+2}) - i(F_{n+1} - jF_{n+3}). \tag{2.21}$$

Then,

$$Q_n^k = C_i^*(Q_n) - iH_i^*(Q_n) \tag{2.22}$$

or

$$Q_n^k = C_j^*(Q_n) - jH_j^*(Q_n). \tag{2.23}$$

Here,  $a^*$  denotes the complex conjugate of any complex number  $a$ .

**Corollary 2.5.** *The following equalities are satisfied.*

$$Q_n Q_n^k = F_n^2 + F_{n+1}^2 + F_{n+2}^2 + F_{n+3}^2 + 2(F_n F_{n+3} - F_{n+1} F_{n+2})k = Q_n^k Q_n, \tag{2.24}$$

$$Q_n Q_n^j = F_n^2 - F_{n+1}^2 + F_{n+2}^2 - F_{n+3}^2 + 2(F_n F_{n+1} + F_{n+2} F_{n+3})i = Q_n^j Q_n, \tag{2.25}$$

$$Q_n Q_n^i = F_n^2 + F_{n+1}^2 - F_{n+2}^2 - F_{n+3}^2 + 2(F_n F_{n+2} - F_{n+1} F_{n+3})j = Q_n^i Q_n. \tag{2.26}$$

*Proof.* The proof can be easily by using algebraic operations and the last corollary. □

From the last corollary, for the elements  $Q_n^i, Q_n^j, Q_n^k$  the following matrix representation can be given as in the following form.

$$Q_n^i = \begin{pmatrix} F_n & F_{n+1} & -F_{n+2} & -F_{n+3} \\ -F_{n+1} & F_n & F_{n+3} & -F_{n+2} \\ F_{n+2} & F_{n+3} & F_n & F_{n+1} \\ -F_{n+3} & F_{n+2} & -F_{n+1} & F_n \end{pmatrix},$$

$$Q_n^j = \begin{pmatrix} F_n & -F_{n+1} & F_{n+2} & -F_{n+3} \\ F_{n+1} & F_n & F_{n+3} & F_{n+2} \\ -F_{n+2} & F_{n+3} & F_n & -F_{n+1} \\ -F_{n+3} & -F_{n+2} & F_{n+1} & F_n \end{pmatrix},$$

$$Q_n^k = \begin{pmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} \\ -F_{n+1} & F_n & -F_{n+3} & F_{n+2} \\ -F_{n+2} & -F_{n+3} & F_n & F_{n+1} \\ F_{n+3} & -F_{n+2} & -F_{n+1} & F_n \end{pmatrix}$$

respectively. Also, from the above equalities, we have

$$Q_n^T = \begin{pmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} \\ -F_{n+1} & F_n & -F_{n+3} & F_{n+2} \\ -F_{n+2} & -F_{n+3} & F_n & F_{n+1} \\ F_{n+3} & -F_{n+2} & -F_{n+1} & F_n \end{pmatrix} = Q_n^k.$$

So, we write  $\det(Q_n^T) = \det(Q_n^k)$ .

When  $\lambda$  holds the equation  $Ax = \lambda x$ ,  $\lambda$  is called an eigenvalue of the matrix  $A$ . Where  $x$  is bicomplex column vector and  $A$  is a  $n \times n$  bicomplex matrix. Hence, the set of eigenvalues is denoted by  $\sigma(A)$  and it is called spectrum of  $A$ .

Now, let us investigate the eigenvalues and eigenvectors of the matrix  $Q_n$  corresponding to bicomplex Fibonacci number. In the next theorem, we give the spectrum of the matrix  $Q_n$ .

**Theorem 2.6.** For the matrix  $Q_n$ , we have

$$\sigma(Q_n) = \{\lambda_i | Q_n x = \lambda_i x, i = 1, 2, 3, 4\} \tag{2.27}$$

where  $n$  is a positive integer and

$$\lambda_1 = -2F_{n+1} - iF_{n+3}, \lambda_2 = -2F_{n+1} + iF_{n+3}, \tag{2.28}$$

$$\lambda_3 = 2F_{n+2} + iF_n, \lambda_4 = 2F_{n+2} - iF_n. \tag{2.29}$$

*Proof.* For the characteristic equation of the matrix  $Q_n$ , we write

$$D_4(\lambda) = \begin{vmatrix} F_n - \lambda & -F_{n+1} & -F_{n+2} & F_{n+3} \\ F_{n+1} & F_n - \lambda & -F_{n+3} & -F_{n+2} \\ F_{n+2} & -F_{n+3} & F_n - \lambda & -F_{n+1} \\ F_{n+3} & F_{n+2} & F_{n+1} & F_n - \lambda \end{vmatrix}.$$

If we calculate this determinant, then we get

$$D_4(\lambda) = (F_n - \lambda)^4 + F_{n+1}^4 + F_{n+2}^4 + F_{n+3}^4 + 2(F_n - \lambda)^2(F_{n+1}^2 + F_{n+2}^2 - F_{n+3}^2) + 2(F_{n+3}^2 F_{n+1}^2 - F_{n+2}^2 F_{n+1}^2 + F_{n+2}^2 F_{n+3}^2) + 8(F_n - \lambda)F_{n+1}F_{n+2}F_{n+3}.$$

From the the equality  $D_4(\lambda) = 0$ , the roots are as follows:

$$\lambda_1 = -2F_{n+1} - iF_{n+3}, \lambda_2 = -2F_{n+1} + iF_{n+3},$$

$$\lambda_3 = 2F_{n+2} + iF_n, \lambda_4 = 2F_{n+2} - iF_n.$$

Thus, the eigenvectors corresponding to these eigenvalues are

$$v_1 = \begin{pmatrix} -1 \\ -i \\ -i \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ i \\ i \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ i \\ -i \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ -i \\ i \\ 1 \end{pmatrix},$$

respectively. Then, a regular matrix  $P$  is

$$P = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -i & i & i & -i \\ -i & i & -i & i \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

So, the matrix  $\mathcal{Q}_n$  can be diagonalized. Then  $D = P^{-1}\mathcal{Q}_nP$

$$P^{-1}\mathcal{Q}_nP = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4}i & \frac{1}{4}i & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4}i & -\frac{1}{4}i & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4}i & \frac{1}{4}i & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4}i & -\frac{1}{4}i & \frac{1}{4} \end{pmatrix} \begin{pmatrix} F_n & -F_{n+1} & -F_{n+2} & F_{n+3} \\ F_{n+1} & F_n & -F_{n+3} & -F_{n+2} \\ F_{n+2} & -F_{n+3} & F_n & -F_{n+1} \\ F_{n+3} & F_{n+2} & F_{n+1} & F_n \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -i & i & i & -i \\ -i & i & -i & i \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Thus, it follows that

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \bar{\lambda}_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \bar{\lambda}_3 \end{pmatrix}$$

where

$$\lambda_1 = -2F_{n+1} - iF_{n+3}, \quad \lambda_3 = 2F_{n+2} + iF_n$$

Therefore, the proof is completed. □

**Theorem 2.7.** For the matrix  $\mathcal{Q}_n$ , we have

$$\mathcal{Q}_n^m = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -i & i & i & -i \\ -i & i & -i & i \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^m & 0 & 0 & 0 \\ 0 & \bar{\lambda}_1^m & 0 & 0 \\ 0 & 0 & \lambda_3^m & 0 \\ 0 & 0 & 0 & \bar{\lambda}_3^m \end{pmatrix} \begin{pmatrix} -\frac{1}{4} & \frac{1}{4}i & \frac{1}{4}i & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4}i & -\frac{1}{4}i & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4}i & \frac{1}{4}i & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4}i & -\frac{1}{4}i & \frac{1}{4} \end{pmatrix}, \tag{2.30}$$

where  $m$  is an positive integer.

*Proof.* The product  $\mathcal{Q}_n^m = PD^mP^{-1}$  is easy to compute since  $D^m$  is simply the diagonalized matrix with entries equal to the  $m$ -th power of those of  $D$ . □

Let's complete the study by giving an example for the above theorem. For this purpose, we take  $n = 1, m = 2$  and let us calculate the following matrices.

$$\mathcal{Q}_1^2 = \begin{pmatrix} 1 & -1 & -2 & 3 \\ 1 & 1 & -3 & -2 \\ 2 & -3 & 1 & -1 \\ 3 & 2 & 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 5 & 10 & 2 & 10 \\ -10 & 5 & -10 & 2 \\ -2 & -10 & 5 & 10 \\ 10 & -2 & -10 & 5 \end{pmatrix}. \tag{2.31}$$

The characteristic polynomial of the matrix  $\mathcal{Q}_n$  is,  $D_1(\lambda)$ ,

$$D_1(\lambda) = \lambda^4 - 4\lambda^3 - 2\lambda^2 - 36\lambda + 221 = 0.$$

Then, the roots of equation  $D_1(\lambda)$  are

$$\lambda_1 = -2 - 3i, \quad \lambda_2 = -2 + 3i, \quad \lambda_3 = 4 + i, \quad \lambda_4 = 4 - i$$

The eigenvectors corresponding to these eigenvalues can be found as follows:

$$v_1 = (-1, -i, -i, 1), \quad v_2 = (-1, i, i, 1), \quad v_3 = (1, i, -i, 1), \quad v_4 = (1, -i, i, 1).$$

So, we get

$$PD^2P^{-1} = ABC$$

where

$$A = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -i & i & i & -i \\ -i & i & -i & i \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4}i & \frac{1}{4}i & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4}i & -\frac{1}{4}i & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4}i & \frac{1}{4}i & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4}i & -\frac{1}{4}i & \frac{1}{4} \end{pmatrix}$$

and

$$B = \begin{pmatrix} -5 + 12i & 0 & 0 & 0 \\ 0 & -5 - 12i & 0 & 0 \\ 0 & 0 & 15 + 8i & 0 \\ 0 & 0 & 0 & 15 - 8i \end{pmatrix}.$$

Then, we have

$$PD^2P^{-1} = \begin{pmatrix} 5 & 10 & 2 & 10 \\ -10 & 5 & -10 & 2 \\ -2 & -10 & 5 & 10 \\ 10 & -2 & -10 & 5 \end{pmatrix}.$$

Hence, we write  $\mathcal{Q}_1^2 = PD^2P^{-1}$ . For  $n = 2$  and  $m = 2$ , we get

$$\mathcal{Q}_2^2 = \begin{pmatrix} 1 & -2 & -3 & 5 \\ 2 & 1 & -5 & -3 \\ 3 & -5 & 1 & -2 \\ 5 & 3 & 2 & 1 \end{pmatrix}^2 = \begin{pmatrix} 13 & 26 & 14 & 22 \\ -26 & 13 & -22 & 14 \\ -14 & -22 & 13 & 26 \\ 22 & -14 & -26 & 13 \end{pmatrix}.$$

The characteristic polynomial of this matrix is,

$$D_2(\lambda) = \lambda^4 - 4\lambda^3 - 18\lambda^2 - 196\lambda + 1517 = 0.$$

The roots of the equation  $D_2(\lambda)$  are as follows.

$$\lambda_1 = -4 - 5i, \lambda_2 = -4 + 5i, \lambda_3 = 6 + i, \lambda_4 = 6 - i$$

Then, we have

$$v_1 = (-1, -i, -i, 1), v_2 = (-1, i, i, 1), v_3 = (1, i, -i, 1), v_4 = (1, -i, i, 1).$$

Thus, we write

$$P \begin{pmatrix} -9+40i & 0 & 0 & 0 \\ 0 & -9-40i & 0 & 0 \\ 0 & 0 & 35+12i & 0 \\ 0 & 0 & 0 & 35-12i \end{pmatrix} P^{-1} \tag{2.32}$$

where

$$P = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -i & i & i & -i \\ -i & i & -i & i \\ 1 & 1 & 1 & 1 \end{pmatrix}, P^{-1} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4}i & \frac{1}{4}i & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4}i & -\frac{1}{4}i & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4}i & \frac{1}{4}i & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4}i & -\frac{1}{4}i & \frac{1}{4} \end{pmatrix}$$

So,

$$PD^2P^{-1} = \begin{pmatrix} 13 & 26 & 14 & 22 \\ -26 & 13 & -22 & 14 \\ -14 & -22 & 13 & 26 \\ 22 & -14 & -26 & 13 \end{pmatrix} = \mathcal{Q}_2^2 = PD^2P^{-1}.$$

Thus, by the aid of all these calculations, we can conclude that  $\mathcal{Q}_n^m = PD^mP^{-1}$ .

### 3. Conclusion

In this work, we have defined a new set of numbers whose coefficients are from a special number sequence, and examined the properties of this new set. We obtained the matrix representations of these numbers and calculated the eigenvalues and eigenvectors of matrices related with these numbers. We diagonalized this matrix and examined its properties with the help of eigenvectors. Hence, one can calculate all the powers of the matrix  $\mathcal{Q}_n$ .

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