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On Asymptotically I-lacunary Statistical Equivalent Functions of Order α

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Abstract

The aim of this paper is to provide a new approach to some well known summability methods. We first define asymptotically I-statistical equivalent functions of order α, asymptotically I_θ-statistical equivalent functions of order α and strongly I-lacunary equivalent functions of order α by taking two nonnegative real-valued Lebesgue measurable functions $x(t)$ and $y(t)$ in the interval $(1,\infty)$ instead of sequences and later we investigate their relationship.

Keywords: *Lacunary statistical convergence,* I*-lacunary statistical equivalence of order* α*, asymptotically equivalent functions, ideal, filter, lacunary sequence.*

2010 Mathematics Subject Classification: *40A30, 40A35, 40D25.*

1. Introduction and Preliminaries

The idea of statistical convergence was formerly introduced under the name almost convergence by Zygmund [20] in 1935. The concept was formally presented by Steinhaus [19] and Fast [3] and later was presented independently by Schoenberg [18]. In 1993, Marouf [8] gave definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003, Patterson [9] extended these definitions by presenting an asymptotically statistical equivalent analogue of these consepts. Recently, Das, Savas and Ghosal [2] provided a new approach to well-known methods of summability by using ideal, introduced new notions such as I-statistical convergence and I-lacunary statistical convergence. In [14, 10, 11, 17] several results on asymptotically I-lacunary statistical equivalent sequences are developed.

In [12, 13] Savas gave generalized summability methods of functions and also introduced statistically convergent functions via ideals. Some other works on ideals can be found in [15, 16].

We now give some definitions will be needed in the sequel.

Definition 1.1 (Marouf, [8])**.** *Two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are called asymptotically equivalent if $\lim_k \frac{x_k}{y_k}=1$ (denoted by $x\sim y$).*

Definition 1.2 (Fridy, [4])**.** *A sequence $x=(x_k)$ is called statistically convergent (or S-convergent) to L, denoted by $st-\lim x_k = L$, if for each $\varepsilon>0$,*

$$\lim_n \frac{1}{n}\ \{the\ number\ of\ k\le n\ :\ |x_k-L|\ge\varepsilon\}=0.$$

The following definition is a combination of these two definitions.

Definition 1.3 (Patterson, [9])**.** *Two nonnegative sequence $x=(x_k)$ and $y=(y_k)$ are called asymptotically statistical equivalent of multiple L, denoted by $x\overset{S_L}{\sim}y$, if for each $\varepsilon>0$,*

$$\lim_n \frac{1}{n}\left\{the\ number\ of\ k<n: \left|\frac{x_k}{y_k}-L\right|\ge\varepsilon\right\}=0.$$

This definition is called simply asymptotically statistical equivalent if $L=1$.

On the other hand, Colak [1] extend the definition of statistical convergence as follows.

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Definition 1.4 (Colak, [1]). *A sequence $x=(x_k)$ is called statistically convergent of order α where $0<\alpha\leq 1$ (or S^{α}-convergent) to L, if for each $\varepsilon>0$,*

$$\lim_{n}\frac{1}{n^{\alpha}}\ \{the\ number\ of\ k\leq n\ :\ |x_k-L|\geq\varepsilon\}=0.$$

A lacunary sequence $\theta=(p_r)_{r\in\mathbb{N}_0}$ where $p_0=0$, $p_{r-1}<p_r$ for all r and $h_r=p_r-p_{r-1}\to\infty$ as $r\to\infty$. Also let denote $q_r=\frac{p_r}{p_{r-1}}$ and $I_r=(p_{r-1},p_r]$.

Definition 1.5 (Fridy and Orhan [5]). *A sequence $x=(x_k)$ is called lacunary statistically convergent (or S_θ convergent) to L, if for each $\varepsilon>0$,*

$$\lim_{r}\frac{1}{h_r}|\{k\in I_r:|x_k-L|\geq\varepsilon\}|=0.$$

Here and in the sequel, the vertical bars indicate the cardinality of the enclosed set.

Definition 1.6 (Kostyrko et al., [6]). *A non-empty family $\mathrm{I}\subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if the following conditions hold:*

(i) $A,B\in\mathrm{I}$ *implies* $A\cup B\in\mathrm{I}$,
(ii) $A\in\mathrm{I}, B\subset A$ *imply* $B\in\mathrm{I}$.

Definition 1.7 (Kostyrko et al., [7]). *A non-empty family $F\subset 2^{\mathbb{N}}$ is said to be filter of $\mathbb{N}$ if the following conditions hold:*

(i) $\emptyset\notin F$,
(ii) $A,B\in F$ *implies* $A\cap B\in F$,
(iii) $A\in F, B\subset A$ *imply* $B\in F$.

If I is a proper ideal of $\mathbb{N}$ (i.e. $\mathbb{N}\notin\mathrm{I}$) then the family of sets $F(I)=\{M\subset\mathbb{N}:\exists A\in\mathrm{I}:M=\mathbb{N}/A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 1.8 (Kostyrko et al., [6]). *A proper ideal* I *is to be admissible if* $\{n\}\in\mathrm{I}$ *for each* $n\in\mathbb{N}$.

Definition 1.9 (Kostyrko et al., [6]). *Given $\mathrm{I}\subset 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. A sequence (x_k) called* I*-convergent to L if for each $\varepsilon>0$,*

$$A(\varepsilon)=\{k\in\mathbb{N}:|x_k-L|\geq\varepsilon\}\in\mathrm{I}.$$

Definition 1.10 (Das et al., [2]). *A sequence (x_k) is called* I*-statistically convergent (or $S(\mathrm{I})$-convergent) to L, if for each $\varepsilon>0$ and $\delta>0$,*

$$\left\{n\in\mathbb{N}:\frac{1}{n}|\{k\leq n:|x_k-L|\geq\varepsilon\}|\geq\delta\right\}\in\mathrm{I}.$$

In this case we write $x_k\to L(S(\mathrm{I}))$. The class of all I*-statistically convergent sequences is denoted by $S(\mathrm{I})$.*

Definition 1.11 (Das et al., [2]). *Let θ be a lacunary sequence. A sequence $x=(x_k)$ is said to be I_θ-statistically convergent (or $S_\theta(\mathrm{I})$-convergent) to L, if for all $\varepsilon>0$ and $\delta>0$,*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}|\{k\in I_r:|x_k-L|\geq\varepsilon\}|\geq\delta\right\}\in\mathrm{I}.$$

In this case, we write $(x_k)\to L(S_\theta(\mathrm{I}))$. The class of all I_θ-statistically convergent sequences is denoted by $S_\theta(\mathrm{I})$.

Definition 1.12 (Das et al., [2]). *Let θ be a lacunary sequence. A sequence $x=(x_k)$ is said to be strong* I*-lacunary convergent to (or $N_\theta(\mathrm{I})$-convergent) to L, if for any $\varepsilon>0$,*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}|x_k-L|\geq\varepsilon\right\}\in I.$$

In this case, we write $x_k\to L(N_\theta(\mathrm{I}))$. The class of all strong I*-lacunary convergent sequences is denoted by $N_\theta(\mathrm{I})$.*

Definition 1.13 (Savas, [10]). *Two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically* I*-statistical equivalent of multiple L provided that for every $\varepsilon>0$ and $\delta>0$,*

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathrm{I}\ \ (denoted\ by\ x\overset{S^L(\mathrm{I})}{\sim}y)$$

and simply asymptotically I*-statistical equivalent if $L=1$.*

Definition 1.14 (Savas, [10]). *Two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically I_θ-statistical equivalent of multiple L provided that for every $\varepsilon>0$ and $\delta>0$,*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r:\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathrm{I}\ \ (denoted\ by\ x\overset{S_\theta^L(\mathrm{I})}{\sim}y)$$

and simply asymptotically I_θ-statistical equivalent if $L=1$.

Definition 1.15 (Savas, [10]). *Two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be strong asymptotically* I*-lacunary equivalent of multiple L provided that for every $\varepsilon>0$ and $\delta>0$,*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\in I\ \ (denoted\ by\ x\overset{N_\theta^L(\mathrm{I})}{\sim}y)$$

and simply strong asymptotically I*-lacunary equivalent if $L=1$.*

2. New Definitions

Inspired by the definitions given in the previous section, we introduce new definitions on the asymptotically equivalent functions of order α.

Definition 2.1. *Let θ be a lacunary sequence, and* I *be an admissible ideal in* $\mathbb{N}$ *and $x(t)$ and $y(t)$ be two nonnegative real valued Lebesgue measurable functions in the interval $(1,\infty)$. We say that the functions $x(t)$ and $y(t)$ are asymptotically* I*-statistical equivalent of order α of multiple L if for each $\varepsilon>0$ and $\delta>0$,*

$$\left\{n\in\mathbb{N}:\frac{1}{n^{\alpha}}\left|\left\{t\le n:\left|\frac{x(t)}{y(t)}-L\right|\ge\varepsilon\right\}\right|\ge\delta\right\}\in \mathrm{I},\ \ (\textit{denoted by } x\overset{S^{L}(\mathrm{I})^{\alpha}}{\sim}y)$$

and simply asymptotically I*-statistical equivalent of order α if $L=1$.*

Definition 2.2. *Let θ be a lacunary sequence, and* I *be an admissible ideal in* $\mathbb{N}$ *and $x(t)$ and $y(t)$ be two nonnegative real valued Lebesgue measurable functions in the interval $(1,\infty)$. We say that the functions $x(t)$ and $y(t)$ are asymptotically lacunary statistical equivalent of order α to multiple L if for each $\varepsilon>0$,*

$$\lim_{r\to\infty}\frac{1}{h_r^{\alpha}}\left|\left\{t\in \mathrm{I}_r:\left|\frac{x(t)}{y(t)}-L\right|\ge\varepsilon\right\}\right|=0\ \ (\textit{denoted by } x(t)\overset{S_{\theta}^{L^{\alpha}}}{\sim}y(t))$$

and simply asymptotically lacunary statistical equivalent of order α if L=1.

Definition 2.3. *Let θ be a lacunary sequence, and* I *be an admissible ideal in* $\mathbb{N}$ *and $x(t)$ and $y(t)$ be two nonnegative real valued Lebesgue measurable functions in the interval $(1,\infty)$. We say that the functions $x(t)$ and $y(t)$ are asymptotically I_{θ}-statistical equivalent of order α of multiple L if for each $\varepsilon>0$ and $\delta>0$,*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^{\alpha}}\left|\left\{t\in \mathrm{I}_r:\left|\frac{x(t)}{y(t)}-L\right|\ge\varepsilon\right\}\right|\ge\delta\right\}\in \mathrm{I}\ \ (\textit{denoted by } x(t)\overset{S_{\theta}^{L}(\mathrm{I})^{\alpha}}{\sim}y(t))$$

and simply asymptotically I_{θ}-statistical equivalent of order α if L=1.
Note that asymptotically I_{θ}-statistical equivalent also called as asymptotically I*-lacunary statistical equivalent.*

Definition 2.4. *Let θ be a lacunary sequence, and* I *be an admissible ideal in* $\mathbb{N}$ *and $x(t)$ and $y(t)$ be two nonnegative real valued Lebesgue measurable functions in the interval $(1,\infty)$. We say that the functions $x(t)$ and $y(t)$ are strong asymptotically* I*-lacunary equivalent of order α of multiple L provided that for every $\varepsilon>0$,*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^{\alpha}}\int_{t\in \mathrm{I}_r}\left|\frac{x(t)}{y(t)}-L\right|dt\ge\varepsilon\right\}\in \mathrm{I}\ \ (\textit{denoted by } x(t)\overset{N_{\theta}^{L}(\mathrm{I})^{\alpha}}{\sim}y(t))$$

and simply strong asymptotically I*-lacunary equivalent of order α if L=1.*
If $\mathrm{I}=\mathrm{I}_{fin}=\{A\subseteq\mathbb{N}$: A is a finite subset $\}$, strong asymptotically I*-lacunary equivalent of order α becomes strong asymptotically lacunary equivalent of order α that is given as:*

$$\lim_{r\to\infty}\frac{1}{h_r^{\alpha}}\int_{t\in \mathrm{I}_r}\left|\frac{x(t)}{y(t)}-L\right|dt=0.$$

3. Main Results

In this section, we establish some implication relations.

Theorem 3.1. *Suppose that $0<\alpha\le\beta\le1$. Then, $x(t)\overset{S_{\theta}^{L}(\mathrm{I})^{\alpha}}{\sim}y(t)$ implies $x(t)\overset{S_{\theta}^{L}(\mathrm{I})^{\beta}}{\sim}y(t)$.*

Proof. For $0<\alpha\le\beta\le1$ we have

$$\frac{\left|\left\{t\in \mathrm{I}_r:\left|\frac{x(t)}{y(t)}-L\right|\ge\varepsilon\right\}\right|}{h_r^{\beta}}\le\frac{\left|\left\{t\in \mathrm{I}_r:\left|\frac{x(t)}{y(t)}-L\right|\ge\varepsilon\right\}\right|}{h_r^{\alpha}},$$

Then for some $\delta>0$,

$$\left\{r\in\mathbb{N}:\frac{\left|\left\{t\in \mathrm{I}_r:\left|\frac{x(t)}{y(t)}-L\right|\ge\varepsilon\right\}\right|}{h_r^{\beta}}\ge\delta\right\}\subset\left\{r\in\mathbb{N}:\frac{\left|\left\{t\in \mathrm{I}_r:\left|\frac{x(t)}{y(t)}-L\right|\ge\varepsilon\right\}\right|}{h_r^{\alpha}}\ge\delta\right\}.$$

Thus if the set on the right hand side is included in the ideal I then clearly the set on the left hand side included in I. This completes the proof of the theorem. □

Remark. Consider that $0<\alpha<1$. If $x(t)\overset{S_{\theta}^{L}(\mathrm{I})^{\alpha}}{\sim}y(t)$ holds, then $x(t)\overset{S_{\theta}^{L}(\mathrm{I})}{\sim}y(t)$.

We may prove the following theorem in a similar way. So, we skip the proof.

Theorem 3.2. *Suppose that $0<\alpha\le\beta\le1$. Then*

(i) $x(t)\overset{S_{\theta}^{L^{\alpha}}}{\sim}y(t)$ *implies* $x(t)\overset{S_{\theta}^{L^{\beta}}}{\sim}y(t)$.

(ii) In particularly $x(t) \overset{S_\theta^{L^\alpha}}{\sim} y(t)$ *implies* $x(t) \overset{S_\theta^L}{\sim} y(t)$.

Theorem 3.3. *Let* θ *be a lacunary sequence,*

(i) if $x(t) \overset{N_\theta^L(\mathrm{I})^\alpha}{\sim} y(t)$*, then* $x(t) \overset{S_\theta^L(\mathrm{I})^\alpha}{\sim} y(t)$*,*

(ii) if $x(t),\ y(t) \in B(X,Y)$ *and* $x(t) \overset{S_\theta^L(\mathrm{I})^\alpha}{\sim} y(t)$*, then* $x(t) \overset{N_\theta^L(\mathrm{I})^\alpha}{\sim} y(t)$*,*

(iii) $x(t) \overset{S_\theta^L(\mathrm{I})^\alpha}{\sim} y(t) \cap B(X,Y) = x(t) \overset{N_\theta^L(\mathrm{I})^\alpha}{\sim} y(t) \cap B(X,Y)$*,*

where $B(X,Y)$ *is the set of bounded functions.*

Proof. *(i)* Let $\varepsilon > 0$ and $x(t) \overset{N_\theta^L(\mathrm{I})^\alpha}{\sim} y(t)$. We obtain

$$\int_{t\in \mathrm{I}_r} \left|\frac{x(t)}{y(t)} - L\right| dt \geq \int_{t\in I_r \& \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon} \left|\frac{x(t)}{y(t)} - L\right| dt$$
$$\geq \varepsilon \left|\left\{t \in \mathrm{I}_r : \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right|$$

and then,

$$\frac{1}{\varepsilon h_r^\alpha} \int_{t\in \mathrm{I}_r} \left|\frac{x(t)}{y(t)} - L\right| dt \geq \frac{1}{h_r^\alpha} \left|\left\{t \in \mathrm{I}_r : \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right|.$$

Hence, for some $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left|\left\{t \in \mathrm{I}_r : \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \int_{t\in \mathrm{I}_r} \left|\frac{x(t)}{y(t)} - L\right| dt \geq \varepsilon.\delta\right\} \in \mathrm{I}.$$

Therefore we have $x(t) \overset{S_\theta^L(\mathrm{I})^\alpha}{\sim} y(t)$.

(ii) Assume $x(t)$ and $y(t)$ are in $B(X,Y)$ and $x(t) \overset{S_\theta^L(\mathrm{I})^\alpha}{\sim} y(t)$. Then there exists a positive K satisfying $\left|\frac{x(t)}{y(t)} - L\right| \leq K$ for every t. Given $\varepsilon > 0$, we get

$$\frac{1}{h_r^\alpha} \int_{t\in \mathrm{I}_r} \left|\frac{x(t)}{y(t)} - L\right| dt = \frac{1}{h_r^\alpha} \int_{t\in \mathrm{I}_r \& \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon} \left|\frac{x(t)}{y(t)} - L\right| dt + \frac{1}{h_r^\alpha} \int_{t\in \mathrm{I}_r \& \left|\frac{x(t)}{y(t)} - L\right| < \varepsilon} \left|\frac{x(t)}{y(t)} - L\right| dt$$
$$\leq \frac{K}{h_r^\alpha} \left|\left\{t \in \mathrm{I}_r : \left|\frac{x(t)}{y(t)} - L\right| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2}.$$

Thus, we observe

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \int_{t\in \mathrm{I}_r} \left|\frac{x(t)}{y(t)} - L\right| dt \geq \varepsilon\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left|\left\{t \in \mathrm{I}_r : \left|\frac{x(t)}{y(t)} - L\right| \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\varepsilon}{2K}\right\} \in \mathrm{I}.$$

Consequently, we obtain $x(t) \overset{N_\theta^L(\mathrm{I})^\alpha}{\sim} y(t)$.

(iii) It can be proved by using (i) and (ii). □

Theorem 3.4. *Let* θ *be a lacunary sequence with* $\liminf_r q_r^\alpha > 1$*, then* $x(t) \overset{S^L(\mathrm{I})^\alpha}{\sim} y(t)$ *implies* $x(t) \overset{S_\theta^L(\mathrm{I})^\alpha}{\sim} y(t)$.

Proof. Assume that $\liminf q_r^\alpha > 1$. Hence, there exist a $\beta > 0$ such that $q_r^\alpha \geq 1 + \beta$ for large enough r, that implies

$$\frac{h_r^\alpha}{p_r^\alpha} \geq \frac{\beta}{1+\beta}.$$

Provided that $x(t) \overset{S^L(\mathrm{I})^\alpha}{\sim} y(t)$, then for all $\varepsilon > 0$ and large enough r, we find

$$\frac{1}{p_r^\alpha} \left|\left\{t \leq p_r : \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right| \geq \frac{1}{p_r^\alpha} \left|\left\{t \in \mathrm{I}_r : \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right|$$
$$\geq \frac{\beta}{1+\beta} \frac{1}{h_r^\alpha} \left|\left\{t \in \mathrm{I}_r : \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right|.$$

Thus, for $\delta > 0$, we obtain

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left|\left\{t \in \mathrm{I}_r : \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{p_r^\alpha} \left|\left\{t \leq p_r : \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right| \geq \frac{\delta\beta}{(1+\beta)}\right\} \in \mathrm{I},$$

which completes the proof. □

For the next theorem we suppose that the lacunary sequence θ satisfies that for any set $C \in F(\mathrm{I})$, $\bigcup\{n : p_{r-1} < n < p_r, r \in C\} \in F(\mathrm{I})$.

Theorem 3.5. *Let θ be a lacunary sequence that satisfy the condition above, then $x(t)\overset{S_{\theta}^{L}(\mathrm{I})^{\alpha}}{\sim}y(t)$ implies $x(t)\overset{S^{L}(\mathrm{I})^{\alpha}}{\sim}y(t)$ provided that* $B:=\sup_{r}\sum_{i=0}^{r-1}\frac{h_{i+1}^{\alpha}}{p_{r-1}^{\alpha}}<\infty$.

Proof. Assume $x(t)\overset{S_{\theta}^{L}(\mathrm{I})^{\alpha}}{\sim}y(t)$. For $\delta,\delta_1,\varepsilon>0$ introduce the sets

$$C=\left\{r\in\mathbb{N}:\frac{1}{h_r^{\alpha}}\left|\left\{t\in \mathrm{I}_r:\left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|<\delta\right\}$$

and

$$T=\left\{n\in\mathbb{N}:\frac{1}{n^{\alpha}}\left|\left\{t\leq n:\left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|<\delta_1\right\}$$

It is clear that $C\in F(\mathrm{I})$, the filter associated with the ideal I. Besides see that, for every $j\in C$ $A_j=\frac{1}{h_j^{\alpha}}\left|\left\{t\in \mathrm{I}_j:\left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|<\delta$. Assume $n\in\mathbb{N}$ be such that $p_{r-1}<n<p_r$ for some $r\in C$. Next,

$$\begin{aligned}
\frac{1}{n^{\alpha}}\left|\left\{t\leq n:\left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right| &\leq \frac{1}{p_{r-1}^{\alpha}}\left|\left\{t\leq p_r:\left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\\
&=\frac{1}{p_{r-1}^{\alpha}}\left\{t\in \mathrm{I}_1:\left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}+\ldots+\frac{1}{p_{r-1}^{\alpha}}\left\{t\in \mathrm{I}_r:\left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\\
&=\frac{p_1^{\alpha}}{p_{r-1}^{\alpha}}.\frac{1}{h_1^{\alpha}}\left|\left\{t\in \mathrm{I}_1:\left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|+\frac{(p_2-p_1)^{\alpha}}{p_{r-1}^{\alpha}}.\frac{1}{h_2^{\alpha}}\left|\left\{t\in \mathrm{I}_2:\left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\\
&\quad+\ldots+\frac{(p_r-p_{r-1})^{\alpha}}{p_{r-1}^{\alpha}}.\frac{1}{h_r^{\alpha}}\left|\left\{t\in \mathrm{I}_r:\left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\\
&=\frac{p_1^{\alpha}}{p_{r-1}^{\alpha}}.A_1+\frac{(p_2-p_1)^{\alpha}}{p_{r-1}^{\alpha}}.A_2+\ldots+\frac{(p_r-p_{r-1})^{\alpha}}{p_{r-1}^{\alpha}}.A_r\\
&\leq \sup_{j\in C}A_j.\sup_{r}\sum_{i=0}^{r-1}\frac{(p_{i+1}-p_i)^{\alpha}}{p_{r-1}^{\alpha}}<B.\delta.
\end{aligned}$$

Considering $\delta_1=\frac{\delta}{B}$ and since $\bigcup\{n:p_{r-1}<n<p_r,r\in C\}\subset T$ where $C\in F(\mathrm{I})$, it is concluded from the condition on θ that also $T\in F(\mathrm{I})$. This completes the proof. □

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